Dynamics of Quadratic Polynomials over Quadratic Fields

by

John Robert Doyle

(Under the direction of Robert Rumely)

Abstract

In 1998, Poonen gave a conjecturally complete classification of the possible preperiodic structures for quadratic polynomials defined over $\mathbb{Q}$. In this thesis, we prove a number of results toward a similar classification, but over quadratic extensions of $\mathbb{Q}$. In order to do so, we formalize a more general notion of *dynamical modular curve* in the setting of quadratic polynomial dynamics. We also discuss the geometry of these curves, proving in some cases that they are irreducible over $\mathbb{C}$, extending a result of Bousch.

Index words: Arithmetic dynamics, preperiodic points, uniform boundedness conjecture, quadratic polynomials, dynamical modular curves, quadratic fields

Dynamics of Quadratic Polynomials over Quadratic Fields

by

John Robert Doyle

B.S., University of Georgia, 2008

M.A., University of Georgia, 2010

A Dissertation Submitted to the Graduate Faculty

of The University of Georgia in Partial Fulfillment

of the

Requirements for the Degree

Doctor of Philosophy

Athens, Georgia

2014

Dynamics of Quadratic Polynomials over Quadratic Fields

by

John Robert Doyle

Approved:

Major Professor: Robert Rumely

Committee: Pete L. Clark
Dino J. Lorenzini
Robert Varley

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
May 2014

Dynamics of Quadratic Polynomials over Quadratic Fields

John Robert Doyle

April 28, 2014

Acknowledgments

There are a number of people without whom this thesis could not have been written. First, I'd like to express my sincerest gratitude to my advisor, Robert Rumely, for his invaluable guidance, insight, and encouragement. I thank Pete Clark, Robert Varley, and Dino Lorenzini for carefully reading an earlier draft of my thesis, and for taking the time to answer my countless questions. I have also benefited greatly from many helpful discussions with my fellow students at UGA, especially David Krumm, Kenny Jacobs, and Patrick McFaddin. I would also like to thank Ted Shifrin, an excellent mentor and teacher, who convinced me that I could study mathematics at the graduate level. Without his support, I would not have set out on this path in the first place, and I thank him for pointing me in the right direction.

Of course, none of this would be possible without my very first math teachers — my mother and father, Suzanne and Henry, who instilled in me the importance of education and encouraged me to do what I enjoy. Finally, I'd like to thank my amazing and beautiful wife, Jessica, whose love and support over the past few years has been an incredible blessing. She has been my best friend throughout this entire journey, and I am happy to share this achievement with her.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The main motivation for this thesis was a 1998 paper of Bjorn Poonen [32], in which he studied the dynamics of quadratic polynomials over $\mathbb{Q}$. Specifically, Poonen was interested in finding an upper bound for the number of rational preperiodic points such a polynomial might have. Poonen proves an upper bound of nine, conditional on a certain hypothesis concerning the possible periods of rational points under quadratic iteration.

Our goal in this thesis is to extend Poonen's result to the case of quadratic extensions of $\mathbb{Q}$. In particular, we are interested in the following question:

Question 1.1. *Given a quadratic extension $K/\mathbb{Q}$ and a quadratic polynomial $f(z) \in K[z]$, how large can the set of K-rational preperiodic points for f be?*

Combining the results of Chapters 4 and 5 with the results from joint work with Xander Faber and David Krumm [8], we arrive at a partial result toward answering Question 1.1, which we state in Theorem 6.1. Given the evidence from [8] as well as this thesis, we conjecture that the answer to Question 1.1 is fifteen — see Conjecture 1.13.

1.1 Background and notation

Let K be a number field, and let $f \in K(z)$ be a rational function of degree $d \geq 2$, which we will think of as a self-map of $\mathbb{P}^1(K)$. For each nonnegative integer n, denote by f^n the n-fold composition of f; that is, take f^0 to be the identity map on $\mathbb{P}^1$, and for each $n \geq 1$ take $f^n = f \circ f^{n-1}$. The **orbit** of a point $x \in \mathbb{P}^1(K)$ is the set $\{f^n(x) : n \geq 0\}$. A point x is called **periodic** for f if $f^n(P) = P$ for some $n \geq 1$. A point x is called **preperiodic** for f if there exist $0 \leq m < n$ for which $f^m(x) = f^n(x)$. Equivalently, a point is preperiodic if its orbit is finite.

If x is a periodic point, then the **(exact) period** of x is the least positive integer n for which $f^n(x) = x$. In this case, the set $\{x, f(x), f^2(x), \ldots, f^{n-1}(x)\}$ is called an n**-cycle**. If x is a preperiodic point for which $f^m(x)$ has period n, but $f^{m-1}(x)$ is not periodic, then we say that x is a point of **preperiodic type** n_m (or just **type** n_m). In other words, x is a point of type n_m if x enters into an n-cycle after exactly m iterations. When necessary, we will say that a point of period n is of type n_0.

We define

$$\mathrm{PrePer}(f, K) := \{x \in \mathbb{P}^1(K) : x \text{ is preperiodic for } f\}.$$

The set $\mathrm{PrePer}(f, K)$ comes naturally equipped with the structure of a directed graph, which we denote $G(f, K)$. The vertices of $G(f, K)$ are the elements of $\mathrm{PrePer}(f, K)$, and there is a directed edge from x to x' if and only if $x' = f(x)$.

Remark 1.2. For any polynomial $p(z) \in K[z]$, the point $\infty \in \mathbb{P}^1(K)$ is a fixed point. As we will henceforth be concerned exclusively with polynomials, we follow the convention of [32] and omit ∞ from the graph $G(p, K)$.

As an example, consider the rational map $f(z) = z^2 - 1$ defined over $\mathbb{Q}$. The points $x = 0$ and $x = -1$ are points of period 2, since $f(0) = -1$ and $f(-1) = 0$. The point $x = 1$ is a point of type 2_1, since 1 itself is not periodic, but $f(1) = 0$ is a point of period 2. Moreover,

f admits no other finite rational preperiodic points, so $\mathrm{PrePer}(f, \mathbb{Q}) = \{-1, 0, 1, \infty\}$ and the associated graph $G(f, \mathbb{Q})$ is shown in Figure 1.1.

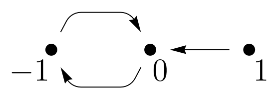


Figure 1.1: The graph $G(f, \mathbb{Q})$ for $f(z) = z^2 - 1$

If L/K is any field extension, then we can also consider f as a rational map over L, and we have inclusions $\mathrm{PrePer}(f, K) \subseteq \mathrm{PrePer}(f, L)$ and $G(f, K) \subseteq G(f, L)$.

We say that two maps $f, g \in K(z)$ are **conjugate** if there exists an automorphism γ of $\mathbb{P}^1$ defined over K (i.e., an element $\gamma \in \mathrm{PGL}_2(K)$) such that $f = g^\gamma := \gamma^{-1} \circ g \circ \gamma$. This is the appropriate dynamical notion of equivalence, since a point x is preperiodic for f if and only if $\gamma(x)$ is preperiodic for g; in fact, γ induces a graph isomorphism $G(f, K) \xrightarrow{\sim} G(g, K)$.

1.2 Dynamical Uniform Boundedness Conjecture

Much of the current work in arithmetic dynamics is driven by an analogy between *preperiodic points* for rational maps and *torsion points* for elliptic curves (or, more generally, for abelian varieties). If E is an elliptic curve defined over a number field K and $m \in \mathbb{N}$, then the set $E(K)_{\mathrm{tors}}$ of K-rational torsion points on E is precisely the set of preperiodic points for the multiplication-by-m map on $E(K)$. If one would rather consider maps on $\mathbb{P}^1(K)$ rather than on $E(K)$, we can replace the map $P \mapsto mP$ on E with the map $x(P) \mapsto x(mP)$ on $\mathbb{P}^1$, where $x(P)$ denotes the x-coordinate of the point $P \in E(K)$. We will discuss maps of this form later.

Because of this analogy, it is expected that certain results about torsion points on elliptic curves should have dynamical analogues in terms of preperiodic points for rational maps. For example, it is well known that for any elliptic curve E defined over a number field K, the torsion subgroup $E(K)_{\mathrm{tors}}$ is finite. (See [35, §VII.7], for example.) In 1950, Northcott used the theory of heights to prove the analogous statement for rational maps.

Theorem 1.3 (Northcott [30], 1950)**.** *Let K be a number field, and let $f \in K(z)$ be a rational map of degree $d \geq 2$. Then the set $\mathrm{PrePer}(f, K)$ is finite.*

Since we have finiteness results for torsion points on elliptic curves and for preperiodic points of rational maps, a natural question is whether there exist uniform bounds on the number of such points. This question was answered for elliptic curves defined over $\mathbb{Q}$ by Mazur in 1977.

Theorem 1.4 (Mazur [17], 1977)**.** *Let E be an elliptic curve defined over $\mathbb{Q}$. Then $E(\mathbb{Q})_{\mathrm{tors}}$ is isomorphic to one of the following 15 groups:*

$$\mathbb{Z}/m\mathbb{Z}, \textit{ for } 1 \leq m \leq 10 \textit{ or } m = 12; \textit{ or}$$
$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2n\mathbb{Z}, \textit{ for } 1 \leq n \leq 4.$$

In particular, $\#E(\mathbb{Q})_{\mathrm{tors}} \leq 16$.

For general number fields $K/\mathbb{Q}$, it had been conjectured that the order of the torsion subgroup of an elliptic curve over K should be bounded uniformly in terms only of the degree $[K : \mathbb{Q}]$. This conjecture, known as the (Strong) Uniform Boundedness Conjecture, was proven in 1996 by Merel.

Theorem 1.5 (Merel [19], 1996)**.** *There is a constant $C(n)$ such that for any number field $K/\mathbb{Q}$ of degree n, and for any elliptic curve E defined over K,*

$$\#E(K)_{\mathrm{tors}} \leq C(n).$$

Motivated by Merel's theorem, Morton and Silverman have made the following conjecture, a dynamical analogue of Theorem 1.5.

Conjecture 1.6 (Dynamical Uniform Boundedness Conjecture [25]). *There is a constant $C(n,d)$ such that for any number field $K/\mathbb{Q}$ of degree n, and for any rational map $f \in K(z)$ of degree $d \geq 2$,*

$$\#\operatorname{PrePer}(f,K) \leq C(n,d).$$

One indication of the difficulty of proving Conjecture 1.6 is the fact that Theorem 1.5 follows from a special case of Conjecture 1.6. Let E be an elliptic curve defined over a number field K, given by the equation $y^2 = f(x)$, where $f(x) \in K[x]$ has degree 3 and has no repeated roots. For a point $P \in E(K)$, denote by $x(P) \in \mathbb{P}^1(K)$ the x-coordinate of the point Q. (We consider the x-coordinate of the point at infinity on E to be ∞.) A point $P \in E(K)$ is a torsion point if and only if the set $\{x(2^n P) : n \in \mathbb{N}\}$ is finite. The *duplication formula* (see [35, p. 54], for example) shows that $x(2P)$ can be expressed as a degree four rational function $\Phi_{E,2}(x) \in K(x)$:

$$\begin{tikzcd} E \arrow[r, "\cdot 2"] \arrow[d, "x"'] & E \arrow[d, "x"] \\ \mathbb{P}^1 \arrow[r, "\Phi_{E,2}"] & \mathbb{P}^1 \end{tikzcd}$$

One can see that if $P \in E(K)$, then

$$\Phi_{E,2}^n(x(P)) = x(2^n P).$$

It follows, then, that P is a torsion point on E if and only if $x(P)$ has a finite orbit under $\Phi_{E,2}$, which is equivalent to saying that $x(P)$ is preperiodic for $\Phi_{E,2}$. Therefore Theorem 1.5 follows from the $d = 4$ case of Conjecture 1.6.

In fact, the only nontrivial family of rational maps for which Conjecture 1.6 is known to hold is the family of *Lattès maps*, which are maps on $\mathbb{P}^1$ corresponding to endomorphisms of elliptic curves. For example, for each $m \geq 2$ one can define the Lattès map $\Phi_{E,m}$ for which $\Phi_{E,m}(x(P)) = x(mP)$. Uniform boundedness for Lattès maps follows from Theorem 1.5.

1.3 Quadratic polynomials over number fields

The simplest case of Conjecture 1.6 is the case $d = 2$, $n = 1$. We restrict ourselves even further to the case[1] where f is a *polynomial* of degree 2 defined over $\mathbb{Q}$. In this situation, the conjecture becomes the following:

Conjecture 1.7. *There exists an absolute constant C such that for any quadratic polynomial $f(z) \in \mathbb{Q}[z]$,*

$$\# \operatorname{PrePer}(f, \mathbb{Q}) \leq C.$$

Even this simplest case of Conjecture 1.6 is currently far from being solved, though significant progress has been made. We describe here the current status of Conjecture 1.7.

In order to study the dynamics of quadratic polynomials, it suffices to consider a single quadratic polynomial from each conjugacy class. If K is a number field (in fact, any field of characteristic different from two), then for any quadratic polynomial $f \in K[z]$, there is a unique $c \in K$ for which f is conjugate to the map

$$f_c(z) := z^2 + c;$$

see [34, p. 156], for example. We therefore restrict our attention to the one-parameter family of maps of the form f_c.

The main result in the direction of Conjecture 1.7 is a theorem of Poonen that was the major motivation for much of this thesis. Just as the Dynamical Uniform Boundedness Conjecture is a dynamical analogue of Merel's Theorem, Poonen's result is a dynamical analogue (in the quadratic polynomial case) of Mazur's Theorem, which provides not only a bound on the size of the torsion subgroup $E(\mathbb{Q})_{\text{tors}}$ for an elliptic curve $E/\mathbb{Q}$, but also completely classifies all such torsion subgroups.

[1]Morton and Silverman [25] remark that, "the proof of even this special case would be a significant achievement."

Theorem 1.8 (Poonen [32], 1998). *Let $c \in \mathbb{Q}$. If f_c does not admit rational points of period greater than 3, then $G(f_c, \mathbb{Q})$ is isomorphic to one of the following twelve directed graphs:*

$$0,\ 2(1),\ 3(1,1),\ 3(2),\ 4(1,1),\ 4(2),\ 5(1,1)a,\ 6(1,1),\ 6(2),\ 6(3),\ 8(2,1,1),\ 8(3),$$

where the labels are as in Appendix A. In particular, $\#\operatorname{PrePer}(f_c, \mathbb{Q}) \leq 9$.

Note that the given upper bound is nine, despite the fact that the largest graph listed in the theorem has eight vertices. This is due to the fact that, as mentioned in Remark 1.2, we exclude the fixed point at infinity from the preperiodic graph.

It was shown by Walde and Russo [37] that, for each $n \in \{1, 2, 3\}$, there are infinitely many $c \in \mathbb{Q}$ for which f_c admits rational points of period n; see also Propositions 4.1 and 4.2. For points of period greater than 3, we have the following results.

Theorem 1.9 (Morton [22], 1998). *If $c \in \mathbb{Q}$, then f_c admits no rational points of period 4.*

Theorem 1.10 (Flynn–Poonen–Schaefer [10], 1997). *If $c \in \mathbb{Q}$, then f_c admits no rational points of period 5.*

Theorem 1.11 (Stoll [36], 2006). *Suppose that the Birch and Swinnerton-Dyer Conjecture holds for the Jacobian of a certain genus 4 curve over $\mathbb{Q}$. If $c \in \mathbb{Q}$, then f_c admits no rational points of period 6.*

Currently very little is known for cycles of length greater than six. However, it is conjectured in [10] that a quadratic polynomial over $\mathbb{Q}$ cannot have rational points of period larger than three, and Hutz and Ingram [12] have provided substantial computational evidence to support this conjecture. Should this conjecture prove correct, then Poonen's theorem would provide a complete classification of preperiodic graphs for quadratic polynomials over $\mathbb{Q}$ and would therefore settle one special case of Conjecture 1.6.

The main results of this thesis are intended to provide evidence in support of a statement similar to that of Theorem 1.8, but with $\mathbb{Q}$ replaced by an arbitrary quadratic extension $K/\mathbb{Q}$. In this setting, experimental evidence in [12] suggests the following conjecture:

Conjecture 1.12. *Let $K/\mathbb{Q}$ be a quadratic field, and let $c \in K$. Then f_c admits no rational points of period greater than 6.*

Over quadratic fields, cycles of length $n \in \{1, 2, 3, 4\}$ occur infinitely often. However, only one example of a 6-cycle is known ([10], [36], [12], [8]), and there are no known examples of 5-cycles.

The computations in [12] dealt specifically with the possible periods of periodic points. In order to formulate an appropriate version of Theorem 1.8 for quadratic extensions of $\mathbb{Q}$, however, we require the data of not just cycle lengths, but full preperiodic graphs $G(f_c, K)$. In joint work with Faber and Krumm [8], we gathered a substantial amount of data to this end. For every pair (K, c) of a quadratic field K and an element $c \in K$ that we tested, we found that f_c had at most 15 K-rational preperiodic points. We therefore make the following conjecture:

Conjecture 1.13. *Let $K/\mathbb{Q}$ be a quadratic field, and let $c \in K$. Then*

$$\# \operatorname{PrePer}(f_c, K) \leq 15.$$

Moreover, the only graphs which may occur (up to isomorphism) as $G(f_c, K)$ for some quadratic field K and element $c \in K$ are the 46 graphs appearing in Appendix A.

Our method of attacking Conjecture 1.13 is twofold and follows the basic structure of Poonen's proof of Theorem 1.8.

First, for each of the 46 graphs G appearing in Appendix A, we try to explicitly determine all realizations of G as the graph $G(f_c, K)$ for some quadratic field K and element $c \in K$. If there are only finitely many such instances for a particular graph G, we would like to find all of them (or show that we already have). This is the direction we pursue in [8].

Second, we consider a number of graphs G which are "minimal" among graphs that were *not* found in the search conducted in [8], and we attempt to prove that such graphs G cannot be realized as $G(f_c, K)$ for any quadratic field K and $c \in K$. This is the analysis we carry out in this thesis.

We are therefore interested in the following question:

Question 1.14. *Given a finite directed graph G, does there exist a quadratic field K and an element $c \in K$ for which $G(f_c, K)$ contains G? If so, can we determine all such occurrences?*

The goal of the two lines of attack described above is to narrow down the possible preperiodic graphs $G(f_c, K)$ until all that remains are the 46 graphs appearing in Appendix A. This goal has not yet been attained, but the combined work of [8] and this thesis make significant progress toward this aim. We now state the partial result obtained as a result of our analysis:

Theorem 6.1. *Let (K, c) be a pair consisting of a quadratic field K and an element $c \in K$. Suppose f_c does not admit K-rational points of period greater than four, and suppose that $G(f_c, K)$ is not isomorphic to one of the 46 graphs shown in Appendix A. Then one of the following must be true:*

(A) $G(f_c, K)$ properly contains one of the following graphs: 10(1,1)b, 10(2), 10(3)a, 10(3)b, 12(2,1,1)b, 12(4), 12(4,2);

(B) $G(f_c, K)$ contains the graph G_2;

(C) $G(f_c, K)$ contains the graph G_4; or

(D) $G(f_c, K)$ contains the graph G_{11}.

Moreover,

(a) there are at most 4 pairs (K, c) *for which* $G(f_c, K)$ *properly contains the graph 12(2,1,1)b, at most 5 pairs for which* $G(f_c, K)$ *properly contains 12(4), and at most 1 pair for which* $G(f_c, K)$ *properly contains 12(4,2);*

(b) there are at most 4 pairs (K, c) *for which* $G(f_c, K)$ *contains* G_2*;*

(c) there are at most 3 pairs (K, c) *for which* $G(f_c, K)$ *contains* G_4*; and*

(d) there is at most 1 pair (K, c) *for which* $G(f_c, K)$ *contains* G_{11}*.*

The graphs given in the form $N(\ell_1, \ell_2, \ldots)$ are graphs appearing in Appendix A, and the graphs given in the form G_n appear in Figure 6.1.

Finally, we introduce some terminology to be used throughout the remainder of this thesis. If K is a quadratic field and $c \in K$, then we refer to the pair (K, c) as a **quadratic pair**. For the sake of brevity, we will often ascribe to a quadratic pair (K, c) the properties of the dynamical system f_c over K. For example, we may say "quadratic pairs (K, c) that admit points of period N" rather than "quadratic fields K and elements $c \in K$ for which f_c admits K-rational points of period N."

We end this chapter by giving a brief outline of the rest of this thesis. In Chapter 2, we define *dynamical modular curves* for quadratic polynomials. These curves parametrize maps f_c together with a collection of preperiodic points for f_c that *generate* a given preperiodic graph G. We give in Chapter 3 a collection of results about algebraic curves that will be used to study the dynamical modular curves defined in Chapter 2. Chapters 4 and 5 provide the main results concerning Question 1.1, and we use the results of these chapters to prove Theorem 6.1 in Chapter 6. In Chapter 7 we discuss dynamical modular curves of small genus, and we use the results from that chapter to give conditional answers to Question 1.1 over certain individual quadratic fields $K/\mathbb{Q}$ in Chapter 8.

Chapter 2

Dynamical modular curves

We now aim to rephrase Question 1.14 in a more computationally useful form. Given a finite directed graph G, we define a *dynamical modular curve* $Y_1^{\mathrm{dyn}}(G)$ which parametrizes maps f_c together with a collection of marked points that "generate" a preperiodic graph isomorphic to G. Question 1.14 then becomes a question of finding quadratic points (i.e., points whose field of definition is a quadratic extension of $\mathbb{Q}$) on certain algebraic curves. We will therefore restrict our discussion to fields of characteristic zero. For the remainder of this chapter, K will always denote such a field, and $\overline{K}$ will denote an algebraic closure of K.

2.1 Definitions and basic properties

We will give a precise definition of $Y_1^{\mathrm{dyn}}(G)$ shortly, but we first discuss a special case. Let N be any positive integer. If c is an element of K and $x \in K$ is a point of period N for f_c, then $f_c^N(x) - x = 0$. However, this equation is also satisfied by points of period dividing N. We therefore define the N**th dynatomic polynomial** to be the polynomial

$$\Phi_N(X, C) := \prod_{n|N} (f_C^n(X) - X)^{\mu(N/n)} \in \mathbb{Z}[X, C],$$

where μ is the Möbius function, which has the property that

$$f_C^N(X) - X = \prod_{n|N} \Phi_n(X, C) \tag{2.1}$$

for all $N \in \mathbb{N}$ — see [26, p. 571]. If $(x, c) \in K^2$ is such that $\Phi_N(x, c) = 0$, we say that x is a point of **formal period** N for f_c. Every point of exact period N has formal period N, but in some cases a point of formal period N may have exact period n a proper divisor of N.

The fact that $\Phi_N(X, C)$ is a polynomial is shown in [34, Thm. 4.5]. We now record two useful facts about the polynomials $\Phi_N(X, C)$.

Lemma 2.1. *For all $N \in \mathbb{N}$, the polynomial $\Phi_N(X, C)$ is monic in both X and C.*

Proof. First, we define

$$\begin{aligned} \Phi_N^+(X, C) &:= \prod_{\substack{n|N \\ \mu(N/n)=1}} (f_C^n(X) - X); \\ \Phi_N^-(X, C) &:= \prod_{\substack{n|N \\ \mu(N/n)=-1}} (f_C^n(X) - X), \end{aligned}$$

so that $\Phi_N = \Phi_N^+/\Phi_N^-$. It is not difficult to see that $f_C^n(X) - X$ is monic in both X and C for all $n \in \mathbb{N}$, so the same must be true for Φ_N^+ and Φ_N^-. Finally, since $\Phi_N^+ = \Phi_N \cdot \Phi_N^-$, it follows that Φ_N must be monic in both X and C. □

Theorem 2.2 (Bousch [2, Thm. 3.1])**.** *For all $N \in \mathbb{N}$, the polynomial $\Phi_N(X, C)$ is irreducible over $\mathbb{C}[X, C]$.*

Since $\Phi_N(X, C) \in \mathbb{Z}[X, C]$, the equation $\Phi_N(X, C) = 0$ defines an affine curve $Y_N \subseteq \mathbb{A}^2$ over K. By applying the Lefschetz principle, it follows from Theorem 2.2 that Y_N is irreducible over any field K of characteristic zero. We define $Y_1^{\text{dyn}}(N)$ to be the Zariski open subset of Y_N on which $\Phi_n(X, C) \neq 0$ for each $n < N$ with $n \mid N$. In other words, (x, c) lies on $Y_N(K)$ (resp., $Y_1^{\mathrm{dyn}}(N)(K)$) if and only if x has formal (resp., exact) period N for f_c. We denote by $X_1^{\mathrm{dyn}}(N)$ the normalization of the projective closure of $Y_1^{\mathrm{dyn}}(N)$. Note that when we take the projective closure of $Y_1^{\mathrm{dyn}}(N)$, the points in $Y_N \setminus Y_1^{\mathrm{dyn}}(N)$ are necessarily reintroduced.

Remark 2.3. Our definition of $Y_1^{\mathrm{dyn}}(N)$ differs from the definition given in [34], which defines $Y_1^{\mathrm{dyn}}(N)$ to be the curve we call Y_N. Our definition excludes the finitely many points (x, c) on Y_N for which x has period strictly less than N for f_c. In other words, we are removing the points (x, c) for which x has *formal* period N but not *exact* period N for f_c. Our reason for doing so is that this more restrictive definition fits better into the more general framework that we develop below. The curve $X_1^{\mathrm{dyn}}(N)$, however, is the same as in [34].

There is a natural automorphism σ on $Y_1^{\mathrm{dyn}}(N)$ — and hence on $X_1^{\mathrm{dyn}}(N)$ — given by $(x, c) \mapsto (f_c(x), c)$. Indeed, if x has period N for f_c, then so does $f_c(x)$. Moreover, if $(x, c) \in Y_1^{\mathrm{dyn}}(N)(K)$, then $f_c^N(x) = x$ and $f_c^n(x) \neq x$ for all $n < N$. It follows that σ is an automorphism of order N, with inverse given by $(x, c) \mapsto (f_c^{N-1}(x), c)$. We denote by $Y_0^{\mathrm{dyn}}(N)$ (resp., $X_0^{\mathrm{dyn}}(N)$) the quotient of $Y_1^{\mathrm{dyn}}(N)$ (resp., $X_1^{\mathrm{dyn}}(N)$) by σ. The K-rational points on the curve $Y_0^{\mathrm{dyn}}(N)$ parametrize quadratic maps $f_c \in K[z]$ together with K-rational *cycles* of length N for f_c.

Finally, we define

$$\begin{aligned} d(N) &:= \deg_X \Phi_N(X, C) = \sum_{n \mid N} \mu(N/n) 2^n; \\ r(N) &:= \frac{d(N)}{N}. \end{aligned}$$

For a given $N \in \mathbb{N}$, the number $d(N)$ (resp., $r(N)$) represents the number of points of period N (resp., periodic cycles of length N) for a general quadratic polynomial f_c over

$\overline{K}$, excluding the fixed point at infinity in the case $N = 1$. The maps $X_1^{\mathrm{dyn}}(N) \to \mathbb{P}^1$ and $X_0^{\mathrm{dyn}}(N) \to \mathbb{P}^1$ obtained by projection onto the C-coordinate have degrees $d(N)$ and $r(N)$, respectively. The first few values of $d(N)$ and $r(N)$ are shown in Table 2.1.

Table 2.1: Values of $d(N)$ and $r(N)$ for small values of N

N	1	2	3	4	5	6	7	8
$d(N)$	2	2	6	12	30	54	126	240
$r(N)$	2	1	2	3	6	9	18	30

For a pair of integers $M, N \in \mathbb{N}$, we define the **generalized dynatomic polynomial**

$$\Phi_{N,M}(X, C) := \frac{\Phi_N(f_C^M(X), C)}{\Phi_N(f_C^{M-1}(X), C)} \in \mathbb{Z}[X, C],$$

and we make the convention that $\Phi_{N,0} = \Phi_N$. That this defines a polynomial is proven in [11, Thm. 1]. A general solution $(x, c) \in K^2$ to the equation $\Phi_{N,M}(x, c) = 0$ has the property that x is of preperiodic type N_M for f_c. We now record a series of useful results about the polynomial $\Phi_{N,M}$.

Lemma 2.4. *For $M, N \geq 1$, the polynomial $\Phi_{N,M}(X, C)$ has degree $2^{M-1} \cdot d(N)$ in X.*

Proof. For any $m \in \mathbb{Z}_{\geq 0}$, we have

$$\deg_X \Phi_N(f_C^m(X), C) = (\deg_X f_C^m(X)) \cdot (\deg_X \Phi_N(X, C)) = 2^m \cdot d(N).$$

Therefore

$$\deg_X \Phi_{N,M}(X, C) = \deg_X \frac{\Phi_N(f_c^M(X), C)}{\Phi_N(f_c^{M-1}(X), C)} = 2^M \cdot d(N) - 2^{M-1} \cdot d(N) = 2^{M-1} \cdot d(N).$$

$\square$

Lemma 2.5. *For each $M, N \in \mathbb{N}$, the polynomial $\Phi_{N,M}(X, C)$ is monic in both X and C.*

Proof. Since $f_c^m(X)$ is monic in both X and C for all $m \in \mathbb{N}$, it follows from Lemma 2.1 that $\Phi_N(f_c^M(X), C)$ and $\Phi_N(f_c^{M-1}(X), C)$ are monic in X and C. Since $\Phi_N(f_c^M(X), C) = \Phi_{N,M}(X, C) \cdot \Phi_N(f_c^{M-1}(X), C)$, the result follows. □

Lemma 2.6. *For all $N \in \mathbb{N}$,*

$$\Phi_N(f_c(X), C) = \Phi_N(X, C) \cdot \Phi_N(-X, C).$$

Proof. That $\Phi_N(X, C)$ divides $\Phi_N(f_c(X), C)$ is proven in [24, Thm. 3.3], and therefore we also have that $\Phi_N(-X, C)$ divides $\Phi_N(f_c(-X), C) = \Phi_N(f_c(X), C)$. By Theorem 2.2, the polynomial $\Phi_N(X, C)$ (hence also $\Phi_N(-X, C)$) is irreducible over $\mathbb{C}$. Since f_c is quadratic, we have

$$\deg_X \Phi_N(f_c(X), C) = 2 \cdot \deg_X \Phi_N(X, C) = \deg_X \Phi_N(X, C) \cdot \Phi_N(-X, C).$$

By applying Lemma 2.1, one can see that both $\Phi_N(f_c(X), C)$ and $\Phi_N(X, C) \cdot \Phi_N(-X, C)$ are monic, so it suffices to show that $\Phi_N(X, C) \neq \Phi_N(-X, C)$. Let $(x, c) \in \mathbb{C}^2$ be such that $x \neq 0$ and $\Phi_N(x, c) = 0$. Then x is a periodic point for f_c, which means that $-x$ cannot be a periodic point for f_c, since $f_c(x)$ is periodic and can have only a single periodic preimage. Therefore $\Phi_N(-x, c) \neq 0$, and it follows that $\Phi_N(X, C)$ cannot equal $\Phi_N(-X, C)$. □

Corollary 2.7. *For all $N \in \mathbb{N}$,*

$$\Phi_{N,1}(X, C) = \Phi_N(-X, C).$$

Proof. By definition, we have

$$\Phi_{N,1}(X, C) = \frac{\Phi_N(f_c(X), C)}{\Phi_N(X, C)},$$

which equals $\Phi_N(-X, C)$ by Lemma 2.6. $\square$

We are now interested in defining $Y_1^{\text{dyn}}(G)$ for general finite directed graphs G. However, certain graphs G must immediately be excluded from consideration. For example, if G is a directed graph that contains a vertex with out-degree two, then G can never be contained in a graph of the form $G(f_c, K)$, since every point $x \in \text{PrePer}(f_c, K)$ maps to precisely one preperiodic point under f_c. We must therefore impose certain restrictions on the directed graphs we should allow.

Definition 2.8. We say that a finite directed graph is **admissible** if it satisfies the following properties:

(a) Every vertex of G has out-degree 1 and in-degree either 0 or 2.

(b) If G contains a vertex with a self-loop (i.e., a 1-cycle), then it contains exactly two such vertices.

(c) For each $N \geq 2$, G contains at most $r(N)$ N-cycles.

Our definition of admissibility is based on the following observations regarding the set of K-rational preperiodic points for a quadratic polynomial $f_c \in K[z]$:

(a) Every point $x \in \text{PrePer}(f_c, K)$ maps to a unique point $f_c(x) \in \text{PrePer}(f_c, K)$, and if one preimage x' of x lies in K, then the two preimages $\pm x'$ of x are distinct elements of K — unless $x' = 0$.

(b) If one fixed point x for f_c lies in K, then *both* fixed points x and $1 - x$ lie in K — unless $c = 1/4$ and $x = 1 - x = 1/2$.

(c) For each $N \geq 2$, the map f_c admits at most $d(N)$ points of period N, partitioned into $r(N) = d(N)/N$ cycles.

If G is an admissible graph, define $F : G \to G$ to be the map taking a vertex P to the unique vertex Q for which there is an edge from P to Q. For each vertex P, let $-P$ denote the unique vertex different from P with $F(P) = F(-P)$. We define the **orbit** of a vertex P to be the set of vertices

$$\mathcal{O}(P) := \{F^k(P) : k \in \mathbb{Z}_{\geq 0}\}.$$

We define the **period** or **preperiodic type** of a vertex P just as we did in §1.1 for the preperiodic points of a rational map. If P is a vertex with in-degree zero, we call P an **endpoint** for the graph G.

Returning to preperiodic points for quadratic maps, observe that if $x \in \mathrm{PrePer}(f_c, K)$, then all points of the form $\pm f_c^n(x)$ with $n \geq 0$ must also lie in $\mathrm{PrePer}(f_c, K)$. It therefore makes sense to say that x *generates* the subgraph of $G(f_c, K)$ containing all of the points $\{\pm f_c^n(x) : n \geq 0\}$. This idea motivates the following definition:

Definition 2.9. Let G be an admissible graph, and let $\{P_1, \ldots, P_n\}$ be a set of vertices of G. We say that $\{P_1, \ldots, P_n\}$ is a **generating set** for the graph G if the following conditions are satisfied:

(a) For every vertex P such that neither P nor $-P$ is a fixed point, either P or $-P$ lies in $\mathcal{O}(P_i)$ for some $i \in \{1, \ldots, n\}$.

(b) If P is a fixed point, and if P' is the unique fixed point different from P, then either P or P' lies in $\mathcal{O}(P_i)$ for some $i \in \{1, \ldots, n\}$.

(c) None of the vertices $P_1, \ldots, P_n$ have preperiodic type N_1 for any N.

We say that $\{P_1, \ldots, P_n\}$ is a **minimal generating set** for G if any other generating set contains at least n vertices.

Remark 2.10. To say that $\{P_1, \ldots, P_n\}$ generates the graph G is equivalent to saying that G is the minimal admissible graph containing the vertices $P_1, \ldots, P_n$ and their orbits. Condition (c) from Definition 2.9 is for convenience, coming from the fact that if P is a type N_1 generator for G, then P may be replaced by $-P$, which necessarily has period N.

In order to provide a good definition of the dynamical modular curves $Y_1^{\text{dyn}}(G)$, we require a basic lemma giving a minimal amount of information required to uniquely determine an admissible graph. First, we set some notation.

Suppose G is an admissible graph *minimally* generated by $\{P_1, \ldots, P_n\}$. Let G_0 denote the empty graph. For each $i \in \{1, \ldots, n\}$ let G_i be the subgraph of G generated by $\{P_1, \ldots, P_i\}$, and let H_i be the set of vertices in G_i that do not lie in G_{i-1}. It follows from minimality of the generating set $\{P_1, \ldots, P_n\}$ that H_i is nonempty for all $i \in \{1, \ldots, n\}$.

For a given $i \in \{1, \ldots, n\}$, it may be the case that $\mathcal{O}(P_i)$ is disjoint from G_{i-1}. In this case, H_i is itself an admissible graph. For example, this is necessarily the case for $i = 1$. If, on the other hand, $\mathcal{O}(P_i)$ intersects G_{i-1}, we set

$$\ell_i := \min\{\ell \in \mathbb{N} : F^{\ell}(P_i) \in G_{i-1}\}.$$

Since $\{P_1, \ldots, P_n\}$ is a *minimal* generating set for G, we must have $\ell_i \geq 1$. Since $F^{\ell_i}(P_i) \in G_{i-1}$, we let $j_i < i$ denote the unique index for which $F^{\ell_i}(P_i) \in H_{j_i}$. Then one of the vertices $\pm F^{\ell_i}(P_i)$ lies in $\mathcal{O}(P_{j_i})$. We claim that $F^{\ell_i}(P_i) \notin \mathcal{O}(P_{j_i})$, so that $-F^{\ell_i}(P_i) \in \mathcal{O}(P_{j_i})$. Suppose to the contrary that $F^{\ell_i}(P_i) = F^k(P_{j_i})$ for some $k \geq 0$. We cannot have $k = 0$, since this would imply that P_{j_i} is in the orbit of P_i, contradicting minimality of the generating set for G. If $k > 0$, then we have $F^{\ell_i - 1}(P_i) = \pm F^{k-1}(P_{j_i})$, and therefore $F^{\ell_i - 1}(P_i) \in G_{i-1}$, contradicting minimality of ℓ_i. Therefore $F^k(P_{j_i}) = -F^{\ell_i}(P_i)$ for some $k \in \mathbb{N}$, and we set

$$\overline{\ell}_i := \min\{k \in \mathbb{N} : F^k(P_{j_i}) = -F^{\ell_i}(P_i)\}.$$

Finally, we remark that an isomorphism of directed graphs $\varphi : G \to G'$, with G and G' admissible, necessarily has the property that $F(\varphi(P)) = \varphi(F(P))$ for every vertex P of G.

We now give a necessary and sufficient condition for two admissible graphs to be isomorphic.

Lemma 2.11. *Let G and G' be admissible graphs, minimally generated by $\{P_1, \dots, P_n\}$ and $\{P'_1, \dots, P'_n\}$, respectively. Then there is an isomorphism of directed graphs $G \xrightarrow{\sim} G'$ mapping $P_i \mapsto P'_i$ if and only if the following conditions are satisfied for all $i \in \{1, \dots, n\}$:*

(A) If $\mathcal{O}(P_i)$ is disjoint from G_{i-1}, then $\mathcal{O}(P'_i)$ is also disjoint from G'_{i-1}, and the preperiodic type $(N_i)_{M_i}$ of P_i equals the preperiodic type $(N'_i)_{M'_i}$ of P'_i.

(B) If $\mathcal{O}(P_i)$ intersects G_{i-1}, then

$$F^{\ell_i}(P'_i) = -F^{\bar{\ell}_i}(P'_{j_i}).$$

In particular, $\mathcal{O}(P'_i)$ intersects G'_{i-1}.

Proof. The forward implication is clear, so we suppose that conditions (A) and (B) are satisfied and show that there is a graph isomorphism $G \to G'$ mapping $P_i \mapsto P'_i$ for each $i \in \{1, \dots, n\}$. We proceed by induction on n.

If $n = 1$, then the lemma reduces to the statement that there is a unique graph, up to isomorphism, generated by a point of preperiodic type N_M. This follows immediately from Definition 2.9.

Now suppose that $n > 1$. By the induction hypothesis, there is a graph isomorphism $\varphi : G_{n-1} \xrightarrow{\sim} G'_{n-1}$ that maps $P_i \mapsto P'_i$ for each $i \in \{1, \dots, n-1\}$. We must handle separately the two cases depending on whether $\mathcal{O}(P_n)$ is disjoint from G_{n-1}.

First, suppose that $\mathcal{O}(P_n)$ is disjoint from G_{n-1}. Then G is the disjoint union $G_{n-1} \sqcup H_n$ of two admissible graphs. By condition (A), the same must be true of G'; that is, we have that $G' = G'_{n-1} \sqcup H'_n$ is a disjoint union of two admissible graphs. By the $n = 1$ case proven above, there is an isomorphism $H_n \xrightarrow{\sim} H'_n$ mapping $P_n \mapsto P'_n$. Combining this with the isomorphism φ clearly induces the desired isomorphism $G \xrightarrow{\sim} G'$.

Finally, suppose that $\mathcal{O}(P_n)$ intersects G_{n-1}, in which case

$$H_n = \{\pm F^{\ell}(P_n) : 0 \leq \ell \leq \ell_n - 1\},$$

and G is the disjoint union $G = G_{n-1} \sqcup H_n$. To show that φ extends to an isomorphism $G \xrightarrow{\sim} G'$ that takes $P_n \mapsto P'_n$, it suffices to show that $F^{\ell_n}(P'_n)$ is the *first* vertex in the orbit of P'_n to lie in G'_{n-1}. Indeed, this would imply that

$$H'_n = \{\pm F^{\ell}(P'_n) : 0 \leq \ell \leq \ell_n - 1\},$$

and $G' = G'_{n-1} \sqcup H'_n$. The isomorphism $\varphi : G_{n-1} \to G'_{n-1}$ could therefore be easily extended to an isomorphism $G \to G'$ since we have $F^{\ell_n}(P'_n) = -F^{\bar{\ell}_n}(P'_{j_n})$ by condition (B).

Suppose to the contrary that $F^{\ell_n - 1}(P'_n) \in G'_{n-1}$. Then $F^{\ell_n - 1}(P'_n) = \pm F^k(P'_j)$ for some $j \in \{1, \ldots, n-1\}$ and $k \in \mathbb{N}$, which implies that $F^{\ell_n}(P'_n) = F^{k+1}(P'_j)$. By condition (B), we have

$$-F^{\bar{\ell}_n}(P'_{j_n}) = F^{\ell_n}(P'_n) = F^{k+1}(P'_j).$$

Now, since φ^{-1} takes $P'_j \mapsto P_j$ and $P'_{j_n} \mapsto P_{j_n}$, we see that we must have

$$-F^{\bar{\ell}_n}(P_{j_n}) = F^{k+1}(P'_j),$$

and therefore $F^{\ell_n}(P_n) = F^{k+1}(P'_j)$. This implies that $F^{\ell_n - 1}(P_n) = \pm F^k(P_j) \in G_j \subseteq G_{n-1}$, contradicting the fact that $F^{\ell_n}(P_n)$ is the first element of $\mathcal{O}(P_n)$ to lie in G_{n-1}. $\square$

Now suppose $c \in K$ is such that $\mathrm{PrePer}(f_c, K)$ is finite. For example, by Northcott's theorem (Theorem 1.3) this is always the case when K is a number field. As mentioned previously, if $c \neq 1/4$ (so that f_c has two distinct fixed points over $\overline{K}$), then the only reason $G(f_c, K)$ might fail to be admissible is that the critical point 0 might be preperiodic for f_c, in which case exactly one vertex in $G(f_c, K)$ (namely, c) has a single preimage (namely, 0). If $c = 1/4$, then $0 \notin \mathrm{PrePer}(f_c, K)$, so the only reason $G(f_c, K)$ is inadmissible is that $G(f_c, K)$ contains a single fixed point.

With this in mind, we will say that a finite directed graph G is **critically degenerate** (resp., **fixed-point degenerate**) if it satisfies all of the conditions for admissibility, except that precisely one vertex has a single preimage (resp., except that there is precisely one fixed point). We define a **generating set** for G in exactly the same way that we did for admissible graphs, with the understanding that if G is critically degenerate and Q is the unique vertex with a single preimage P, then $P = -P$. If G is admissible and G' is critically degenerate (resp., fixed-point degenerate), then we say that G' is a **critical degeneration of** G (resp., **fixed-point degeneration of** G) if there is a generating set $\{P_1, \dots, P_n\}$ for G and a generating set $\{P'_1, \dots, P'_n\}$ for G' such that conditions (A) and (B) from Lemma 2.11 are satisfied.

In order to define the dynamical modular curves $Y_1^{\mathrm{dyn}}(G)$ for arbitrary admissible graphs G, we begin by defining a curve Y_G that will contain $Y_1^{\mathrm{dyn}}(G)$ as a Zariski open subset.

Let G be an admissible graph, minimally generated by the vertices $P_1, \dots, P_n$. For each $i \in \{1, \dots, n\}$, let $(N_i)_{M_i}$ be the preperiodic type of P_i, and let G_i, ℓ_i, j_i, and $\overline{\ell}_i$ be defined as above Lemma 2.11.

Let $R := K[X_1, \dots, X_n, C]$. To each generator P_i we associate a polynomial $\Psi_i \in R$:

$$\Psi_i := \begin{cases} \Phi_{N_i,M_i}(X_i, C), & \text{if } \mathcal{O}(P_i) \cap G_{i-1} = \emptyset; \\ f_C^{\ell_i}(X_i) + f_C^{\overline{\ell}_i}(X_{j_i}), & \text{if } \mathcal{O}(P_i) \cap G_{i-1} \neq \emptyset. \end{cases}$$

Let I_G be the ideal generated by $\Psi_1, \ldots, \Psi_n$, and define Y_G to be the reduced subscheme of $\mathbb{A}^{n+1}_{\mathbb{Z}}$ associated to $\operatorname{Spec} R/I_G$. We now show that Y_G is a (possibly reducible) curve over K.

Lemma 2.12. *Each of the irreducible components of Y_G has dimension one over $\overline{K}$.*

Proof. First, since Y_G is a subscheme of $\mathbb{A}^{n+1}$ defined by n polynomial equations, we immediately have that each component of Y_G has dimension at least one. Now consider the ideal

$$I'_G := \langle \Phi_{N_i,M_i}(X_i, C) : i \in \{1, \ldots, n\} \rangle,$$

and let Y'_G denote the reduced subscheme of $\mathbb{A}^{n+1}$ associated to $\operatorname{Spec} R/I'_G$. We claim that $I'_G \subseteq I_G$, in which case it follows that $Y_G \subseteq Y'_G$. Once we have shown this, it will suffice to show that each component of Y'_G has dimension at most one.

We show that $\Phi_{N_i,M_i}(X_i, C) \in I_G$ for each index $i \in \{1, \ldots, n\}$ by induction on i. Let $i \in \{1, \ldots, n\}$. If $\mathcal{O}(P_i)$ does not intersect G_{i-1}, then $\Phi_{N_i,M_i}(X_i, C) = \Psi_i \in I_G$ by definition. In particular, this means that $\Psi_1 \in I_G$, so we may assume $i > 1$. It also follows that we need only consider the case that $\mathcal{O}(P_i)$ intersects G_{i-1}, in which case $\Psi_i = f_C^{\ell_i}(X_i) + f_C^{\overline{\ell}_i}(X_{j_i})$. Since P_i and P_{j_i} lie in the same component of G, we must have $N_i = N_{j_i}$. Moreover, since P_i and P_{j_i} are generators of G, neither P_i nor P_{j_i} is periodic. Otherwise, the periodic generator would be redundant, contradicting minimality of the generating set. Therefore each of the vertices P_i and P_{j_i} is strictly preperiodic, and we consider separately the two cases depending on whether the orbits of P_i and P_{j_i} enter the periodic cycle at the same point or at different points.

Case 1: First, suppose $F^{M_i}(P_i) = F^{M_{j_i}}(P_{j_i})$. There is a unique point of type $(N_i)_1$ whose image under F is $F^{M_i}(P_i)$, so we also have $F^{M_i-1}(P_i) = F^{M_{j_i}-1}(P_{j_i})$. It follows from the definition of ℓ_i and $\overline{\ell}_i$ that $\ell_i < M_i - 1$ and $\overline{\ell}_i < M_{j_i} - 1$. Since also $F^{\ell_i+1}(P_i) = F^{\overline{\ell}_i+1}(P_{j_i})$, the hypothesis that $F^{M_i}(P_i) = F^{M_{j_i}}(P_{j_i})$ implies that

$$M_i - \ell_i = M_{j_i} - \overline{\ell}_i > 1.$$

Therefore, modulo I_G, we have

$$f_C^{M_i-1}(X_i) = f_C^{M_{j_i}-\overline{\ell}_i-1}(f_C^{\ell_i}(X_i)) \equiv f_C^{M_{j_i}-\overline{\ell}_i-1}(-f_C^{\overline{\ell}_i}(X_{j_i})) = f_C^{M_{j_i}-1}(X_{j_i}),$$

from which it follows that

$$\Phi_{N_i,M_i}(X_i, C) = \frac{\Phi_{N_i}(f_C^{M_i}(X_i), C)}{\Phi_{N_i}(f_C^{M_i-1}(X_i), C)} \equiv \frac{\Phi_{N_i}(f_C^{M_{j_i}}(X_{j_i}), C)}{\Phi_{N_i}(f_C^{M_{j_i}-1}(X_{j_i}), C)} = \Phi_{N_{j_i},M_{j_i}}(X_{j_i}, C).$$

By induction, we have $\Phi_{N_{j_i},M_{j_i}}(X_{j_i}, C) \in I_G$, so also $\Phi_{N_i,M_i}(X_i, C) \in I_G$.

Case 2: Now suppose the orbits of P_i and P_{j_i} enter the N_i-cycle at different points, so that $F^{M_i}(P_i) = F^{M_{j_i}+k}(P_{j_i})$ for some $k \in \{1, \ldots, N_i - 1\}$. In this case, $F^{M_i}(P_i)$ is the first element of $\mathcal{O}(P_i)$ to lie in the orbit of P_{j_i}, so we have $\ell_i = M_i - 1$ and $\overline{\ell}_i = M_{j_i} + k - 1$. Since $\Psi_i = f_c^{\ell_i}(X_i) + f_c^{\overline{\ell}_i}(X_{j_i}) \in I_G$, modulo I_G we have

$$\begin{aligned}\Phi_{N_i,M_i}(X_i, C) = \frac{\Phi_{N_i}(f_C^{M_i}(X_i), C)}{\Phi_{N_i}(f_C^{M_i-1}(X_i), C)} &= \frac{\Phi_{N_i}(f_C^{\ell_i+1}(X_i), C)}{\Phi_{N_i}(f_C^{\ell_i}(X_i), C)} \\ &\equiv \frac{\Phi_{N_i}(f_C^{\overline{\ell}_i+1}(X_{j_i}), C)}{\Phi_{N_i}(-f_C^{\overline{\ell}_i}(X_{j_i}), C)} = \Phi_{N_i}(f_C^{\overline{\ell}_i}(X_{j_i}), C),\end{aligned} \tag{2.2}$$

where the final equality follows from Lemma 2.6. Since $k - 1 \geq 0$, repeated applications of Lemma 2.6 show that the polynomial $\Phi_{N_i}(f^{M_{j_i}}(X_{j_i}), C)$ divides the polynomial

$$\Phi_{N_i}(f_C^{M_{j_i}+k-1}(X_{j_i}), C) = \Phi_{N_i}(f_C^{\overline{\ell}_i}(X_{j_i}), C).$$

Now, by our induction hypothesis, the polynomial $\Phi_{N_i,M_{j_i}}(X_{j_i})$ lies in I_G. Since $\Phi_{N_i,M_{j_i}}(X_{j_i})$ divides $\Phi_{N_i}(f_C^{M_{j_i}}(X_{j_i}), C)$, and therefore also $\Phi_{N_i}(f_C^{\bar{\ell}_i}(X_{j_i}), C)$, we conclude from (2.2) that $\Phi_{N_i,M_i}(X_i, C) \in I_G$.

We have shown that $Y_G \subseteq Y_G'$. Therefore, to complete the proof of Lemma 2.12, it suffices to show that each of the irreducible components of Y_G' has dimension at most one over $\overline{K}$. Observe that, for any given value of $c \in \overline{K}$, there are only finitely many $x_i \in \overline{K}$ for which $\Phi_{N_i,M_i}(x_i, c) = 0$. In particular, this means that the morphism $Y_G' \to \mathbb{A}^1$ given by projection onto the C-coordinate is dominant with finite fibers. Therefore each component has dimension at most one, completing the proof. □

The curve Y_G is generally too large to serve as the dynamical modular curve for the admissible graph G. For example, let G be the graph generated by two points of period 3 with disjoint orbits. Then Y_G is defined by the equations

$$\Phi_3(X_1, C) = \Phi_3(X_2, C) = 0.$$

However, a point $(x_1, x_2, c) \in Y_G(K)$ might not correspond to a pair $\{x_1, x_2\}$ of preperiodic points for f_c that generate a subgraph of $G(f_c, K)$ isomorphic to G. Indeed, let $c \in K$, and suppose $x \in K$ is a point of period 3 for f_c. Then $\Phi_3(x, c) = 0$, and the point (x, x, c) lies on Y_G. However, the point (x, x, c) only carries enough information to describe a single 3-cycle component for $G(f_c, K)$ — namely, the component containing the point x — rather than the two distinct 3-cycle components that make up G. The problem is that there is a component of Y_G defined by the additional equation $X_1 = X_2$, as well as components given by $X_1 = f_C(X_2)$ and $X_1 = f_C^2(X_2)$, whose points (x_1, x_2, c) fail to determine two distinct 3-cycles under f_c.

We see from the above example that, for a general admissible graph G minimally generated by $\{P_1, \ldots, P_n\}$, there are certain points $(x_1, \ldots, x_n, c) \in Y_G(K)$ that should be removed. For example, if P_i and P_j have disjoint orbits in G, then x_i and x_j should have disjoint orbits under f_c, so that we may avoid situations like the one described in the previous paragraph. Also, if P_i is a point of preperiodic type $(N_i)_{M_i}$ in G, then we should require that, for a point $(x_1, \ldots, x_n, c) \in Y_1^{\mathrm{dyn}}(G)(K)$, the element $f_c^{M_i}(x_i) \in K$ has *exact* period N_i and $f_c^{M_i-1}(x_i)$ does not.

With these restrictions in mind, we now define the dynamical modular curve $Y_1^{\mathrm{dyn}}(G)$.

Definition 2.13. Let G be an admissible graph, minimally generated by $\{P_1, \ldots, P_n\}$. Let $\mathcal{I} \subseteq \{1, \ldots, n\}$ denote the set of indices for which $\mathcal{O}(P_i)$ is disjoint from G_{i-1}; that is, $i \in \mathcal{I}$ if and only if $\Psi_i = \Phi_{N_i,M_i}(X_i, C)$. We define $Y_1^{\mathrm{dyn}}(G)$ to be the Zariski open subset of Y_G determined by the following additional conditions:

$$\Phi_{n_i,M_i}(X_i, C) \neq 0 \text{ for all } n_i < N_i \text{ with } n_i \mid N_i, \text{ for all } i \in \mathcal{I}; \tag{2.3}$$

$$\Phi_{N_i,M_i-1}(X_i, C) \neq 0 \text{ for all } i \in \mathcal{I} \text{ with } M_i > 0; \tag{2.4}$$

$$f_C^{M_i}(X_i) \neq f_C^{M_j+k}(X_j) \text{ for all } 0 \leq k \leq N_i - 1, \text{ for all } i, j \in \mathcal{I} \text{ with } i > j \text{ and } N_i = N_j. \tag{2.5}$$

The effect of the conditions (2.3) and (2.4) is to remove finitely many points from Y_G, and the effect of (2.5) is to remove some (but not all) of the irreducible components of Y_G, as was illustrated in the above example. We suspect that what remains is irreducible over $\mathbb{C}$ (hence over K, by the Lefschetz principle), and in some cases we have proven that this is the case (see Theorems 2.22 and 2.32), but we have not yet been able to prove this in general — see Conjecture 2.34.

We denote by $X_1^{\mathrm{dyn}}(G)$ the normalization of the projective closure of $Y_1^{\mathrm{dyn}}(G)$. We will refer to the curves $Y_1^{\mathrm{dyn}}(G)$ and $X_1^{\mathrm{dyn}}(G)$ as **dynamical modular curves**. We now justify this terminology by showing that $Y_1^{\mathrm{dyn}}(G)$ is the appropriate curve for our dynamical moduli problem.

Proposition 2.14. *Let K be any field of characteristic zero, and let G be an admissible graph minimally generated by vertices $\{P_1, \ldots, P_n\}$. Then $(x_1, \ldots, x_n, c) \in Y_1^{\mathrm{dyn}}(G)(K)$ if and only if $\{x_1, \ldots, x_n\}$ generates a subgraph of $G(f_c, K)$ that is isomorphic either to G or to a critical or fixed-point degeneration of G via an identification $P_i \mapsto x_i$.*

Proof. Suppose first that $(x_1, \ldots, x_n, c) \in Y_1^{\mathrm{dyn}}(G)(K)$. We observe that the relations defining Y_G imply that each of $x_1, \ldots, x_n$ is preperiodic under f_c, so we let G' be the (necessarily finite) subgraph of $G(f_c, K)$ generated by $\{x_1, \ldots, x_n\}$. We will show that conditions (A) and (B) of Lemma 2.11 are satisfied. The proposition will then follow immediately, since G' must be either admissible, critically degenerate, or fixed-point degenerate.

Let $i \in \{1, \ldots, n\}$ be such that P_i is disjoint from the graph G_{i-1}. Then $\Psi_i = \Phi_{N_i,M_i}(X_i, C)$, so we have $\Phi_{N_i,M_i}(x_i, c) = 0$; combining this with (2.3) and (2.4) tells us that x_i has preperiodic type $(N_i)_{M_i}$, which is precisely the preperiodic type of P_i. Furthermore, the condition (2.5) implies that the orbit of x_i under f_c is disjoint from the orbits of $x_1, \ldots, x_{i-1}$ under f_c. Therefore condition (A) of Lemma 2.11 is satisfied.

Now suppose the orbit of P_i intersects G_{i-1}. Then $\Psi_i = f_C^{\ell_i}(X_i) + f_C^{\bar{\ell}_i}(X_{j_i})$, so that $f_c^{\ell_i}(x_i) = -f_c^{\bar{\ell}_i}(x_{j_i})$, and therefore condition (B) of Lemma 2.11 is satisfied. We conclude that G' is isomorphic either to G or to a critical or fixed-point degeneration of G via an identification $P_i \mapsto x_i$, as claimed.

To prove the converse, suppose that $\{x_1, \ldots, x_n\} \subseteq \mathrm{PrePer}(f_c, K)$ generate a subgraph $G' \subseteq G(f_c, K)$ isomorphic either to G or to a critical or fixed-point degeneration of G via an identification $P_i \mapsto x_i$.

If the orbit of P_i is disjoint from G_{i-1}, then also the orbit of x_i is disjoint from G'_{i-1}, and both P_i and x_i have the same preperiodic type $(N_i)_{M_i}$. Therefore $\Phi_{N_i,M_i}(x_i, c) = 0$, $\Phi_{n,M_i}(x_i, c) \neq 0$ for all $n < N$ with $n \mid N$, and $\Phi_{N_i,M_i-1}(x_i, c) \neq 0$.

On the other hand, if the orbit of P_i intersects G_{i-1}, then the orbit of x_i also intersects G'_{i-1}. Moreover, we have $F^{\ell_i}(P_i) = -F^{\bar{\ell}_i}(P_{j_i})$, so the identification $P_i \mapsto x_i$ implies that $f_c^{\ell_i}(x_i) = -f^{\bar{\ell}_i}(x_{j_i})$.

The previous two paragraphs imply that $(x_1, \ldots, x_n, c) \in Y_1^{\mathrm{dyn}}(G)(K)$, as desired. $\square$

We now provide an equivalent formulation of Question 1.14 in terms of dynamical modular curves:

Question 2.15. *Given an admissible graph G, does $Y_1^{\mathrm{dyn}}(G)$ admit quadratic points? If so, can we determine the set of all such points?*

We now consider an example. Let G be the admissible graph shown in Figure 2.1. The graph G is generated by a point of type 1_2 and a point of period 3. Here we are using the fact that if G contains two fixed points, then according to Definition 2.9 only one of the fixed points needs to lie in the orbit of a generator for G. We therefore have

$$Y_1^{\mathrm{dyn}}(G)(K) = \{(a, b, c) \in \mathbb{A}^3(K) : \ a \text{ is of type } 1_2 \text{ and } b \text{ has period 3 for } f_c\}.$$

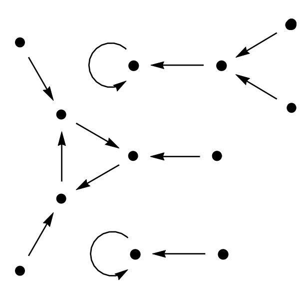


Figure 2.1: An admissible graph G

For a given admissible graph G, there is a natural morphism $X_1^{\mathrm{dyn}}(G) \to \mathbb{P}^1$, given by projection onto the C-coordinate, which is analogous to the map from the classical modular curve $X_1(N)$ to the j-line. We now show that if G and H are admissible graphs with $G \supsetneq H$, then there is a natural morphism $X_1^{\mathrm{dyn}}(G) \to X_1^{\mathrm{dyn}}(H)$ of degree at least two. We first prove this assertion in two special cases.

Lemma 2.16.

(A) (Addition of a cycle) Let H be an admissible graph, and suppose H contains ρ connected components terminating in cycles of length N, with $0 \leq \rho < r(N)$. Let G be the admissible graph obtained from H by adding a new point of period N (and therefore a full N-cycle together with its type N_1 preimages). Then there is a morphism of degree $(d(N) - N \cdot \rho) \geq \max\{2, N\}$ from $X_1^{\mathrm{dyn}}(G)$ to $X_1^{\mathrm{dyn}}(H)$.

(B) (Addition of preimages) Let H be an admissible graph, and let P_0 be an endpoint for H. Let G be the graph obtained by appending two preimages P and $-P$ to P_0, as in Figure 2.2. Then there is a degree two morphism from $X_1^{\mathrm{dyn}}(G)$ to $X_1^{\mathrm{dyn}}(H)$.

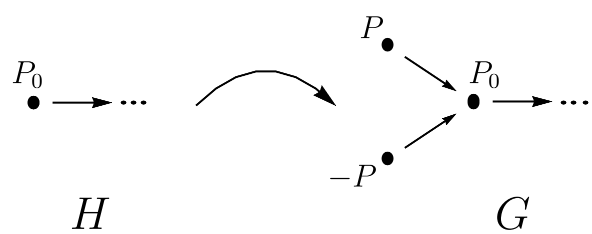


Figure 2.2: Appending two preimages to the vertex P_0 in H

Remark 2.17. We have not yet shown that the curves $X_1^{\mathrm{dyn}}(G)$ are irreducible over K for general admissible graphs G (see Theorem 2.32, however). Therefore, by the degree of a morphism $X_1^{\mathrm{dyn}}(G) \to X_1^{\mathrm{dyn}}(H)$, we simply mean the number of preimages, up to multiplicity, of a point on $X_1^{\mathrm{dyn}}(H)$.

Proof. We will show that there exist morphisms of the appropriate degrees between the Y_1^{dyn} curves, which one may then lift to the corresponding X_1^{dyn} curves.

First, let G and H be as in part (A). Say H is minimally generated by $\{P_1, \ldots, P_n\}$, in which case G is minimally generated by $\{P, P_1, \ldots, P_n\}$, where P is the additional point of period N. Points on $Y_1^{\mathrm{dyn}}(H)$ are of the form $(x_1, \ldots, x_n, c)$, satisfying $\Psi_i(x_1, \ldots, x_n, c) = 0$ for each $i \in \{1, \ldots, n\}$ together with the conditions given in (2.3)–(2.5). Similarly, points on $Y_1^{\mathrm{dyn}}(G)$ are of the form $(x, x_1, \ldots, x_n, c)$, satisfying the same relations as for H, with the additional conditions that $\Phi_N(x, c) = 0$ and that x does not lie in the orbit of x_i for any $i \in \{1, \ldots, n\}$. There is therefore a natural morphism

$$\varphi : Y_1^{\mathrm{dyn}}(G) \to Y_1^{\mathrm{dyn}}(H)$$
$$(x, x_1, \ldots, x_n, c) \mapsto (x_1, \ldots, x_n, c).$$

The preimages of a given point $(x_1, \ldots, x_n, c)$ under φ are those tuples $(x, x_1, \ldots, x_n, c)$ for which x has period N for f_c and x does not lie in the orbit of x_i for any $i \in \{1, \ldots, n\}$. Since H contains ρ components with N-cycles, and therefore contains $N \cdot \rho$ points of period N, x must avoid a set of size $N \cdot \rho$. Since a map f_c generically has $d(N)$ points of period N, that means the preimage of a point on $Y_1^{\mathrm{dyn}}(H)$ generically contains $(d(N) - N \cdot \rho)$ preimages; hence the degree of the morphism φ is equal to $(d(N) - N \cdot \rho)$.

Now let G and H be as in part (B), so that G is obtained from H by appending two preimages $\pm P$ to an endpoint P_0 for H. By construction, a minimal generating set for G must include exactly one of $\pm P$; without loss of generality, we choose P to be the new generator for G. We now consider two cases, depending on whether $-P_0$ is in a minimal generating set for H.

Case 1: Suppose there is a minimal generating set for H containing $-P_0$; write the generating set as $\{-P_0, P_1, \ldots, P_n\}$. Then $\{P, P_1, \ldots, P_n\}$ is a minimal generating set for G, and the morphism $X_1^{\rm dyn}(G) \to X_1^{\rm dyn}(H)$ is given by

$$(x, x_1, \dots, x_n, c) \mapsto (-f_c(x), x_1, \dots, x_n, c).$$

This morphism clearly has degree $\deg f_c = 2$.

Case 2: Now suppose that $-P_0$ cannot be contained in any minimal generating set for H. Let $\{P_1, \dots, P_n\}$ be a minimal generating set for H. Then one of $\pm P_0$ lies in the orbit of P_i for some $i \in \{1, \dots, n\}$. Since P_0 is an endpoint of H, it must be the case that $-P_0 = F^k(P_i)$ for some $k \in \mathbb{N}$. We therefore have $F(P) = P_0 = -F^k(P_i)$. Since $\{P, P_1, \dots, P_n\}$ is a minimal generating set for G, the morphism $X_1^{\rm dyn}(G) \to X_1^{\rm dyn}(H)$ may be given by

$$(x, x_1, \dots, x_n, c) \mapsto (x_1, \dots, x_n, c),$$

and the preimages of a given point $(x_1, \dots, x_n, c)$ on $X_1^{\rm dyn}(H)$ are precisely those points $(x, x_1, \dots, x_n, c)$ for which $f_c(x) = -f_c^k(x_i)$. Since this equation has degree two in x, the degree of the morphism must be two. $\square$

We will refer to cycle additions (as in part (A) of Lemma 2.16) and preimage additions (as in part (B) of Lemma 2.16) as **minimal additions**. Observe that an arbitrary admissible graph may be built up inductively from a single cycle using a finite number of minimal additions. Repeated applications of Lemma 2.16 therefore yield the following important fact.

Proposition 2.18. *Let G be an admissible graph, and let H be a proper admissible subgraph of G. Then there is a finite morphism $X_1^{\rm dyn}(G) \to X_1^{\rm dyn}(H)$ of degree at least two. Therefore, if $X_1^{\rm dyn}(G)$ is irreducible and $X_1^{\rm dyn}(H)$ has genus $g \geq 2$, then the genus of $X_1^{\rm dyn}(G)$ is strictly greater than g.*

Proof. Since G may be built from H via a finite number of minimal additions

$$G = H_n \supset H_{n-1} \supset \cdots \supset H_1 \supset H_0 = H,$$

we have a tower of curves

$$X_1^{\text{dyn}}(G) = X_1^{\text{dyn}}(H_n) \to X_1^{\text{dyn}}(H_{n-1}) \to \cdots \to X_1^{\text{dyn}}(H_1) \to X_1^{\text{dyn}}(H_0) = X_1^{\text{dyn}}(H).$$

Each of the morphisms in the tower has degree at least two by Lemma 2.16, so the composition $X_1^{\text{dyn}}(G) \to X_1^{\text{dyn}}(H)$ must have degree at least two, proving the first claim. The second claim follows from the Riemann-Hurwitz formula. $\square$

Finally, we describe a special family of dynamical modular curves. First, we observe that if $N \in \mathbb{N}$, then $Y_1^{\text{dyn}}(N)$ is equal to $Y_1^{\text{dyn}}(G)$ for the admissible graph G generated by a single point of period N. For each positive integer $n \le r(N)$, we define $Y_1^{\text{dyn}}(N^{(n)})$ to be the curve $Y_1^{\text{dyn}}(G)$ for the graph G minimally generated by n points of period N. In other words, points on $Y_1^{\text{dyn}}(N^{(n)})$ parametrize quadratic polynomials f_c together with n marked points of period N with disjoint orbits. If $n \ge 2$, then by Lemma 2.16(A) there is a natural morphism $X_1^{\text{dyn}}(N^{(n)}) \to X_1^{\text{dyn}}(N^{(n-1)})$ that essentially forgets one of the marked points of period N.

More generally, let $\{N_1, \ldots, N_m\}$ be a set of positive integers. For a sequence $(n_1, \ldots, n_m)$ of positive integers with $n_i \le r(N_i)$ for all i, we define $Y_1^{\text{dyn}}(N_1^{(n_1)}, \ldots, N_m^{(n_m)})$ to be the curve $Y_1^{\text{dyn}}(G)$ for the graph G minimally generated by a set $\{P_{i,j} : 1 \le i \le m\ ,\ 1 \le j \le n_i\}$ of periodic points, where $P_{i,j}$ has period N_i for each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n_i\}$. In other words, we have

$$
\begin{aligned}
&Y_1^{\rm dyn}(N_1^{(n_1)}, \dots, N_m^{(n_m)})(K) = \\
&\qquad \{(x_{1,1}, \dots, x_{1,n_1}, \dots, x_{m,1}, \dots, x_{m,n_m}, c) \in \mathbb{A}^{n_1+\cdots+n_m+1}(K) : \\
&\qquad\quad x_{i,j} \text{ has period } N_i, \text{ and } x_{i,j} \text{ and } x_{i,j'} \text{ have disjoint orbits when } j \neq j'\}.
\end{aligned} \tag{2.6}
$$

The points on $Y_1^{\rm dyn}(N_1^{(n_1)}, \dots, N_m^{(n_m)})$ parametrize quadratic polynomials f_c together with n_i marked points of period N_i for each $i \in \{1, \dots, m\}$, with the additional restriction that no two such points share a common orbit. As usual, we define $X_1^{\rm dyn}(N_1^{(n_1)}, \dots, N_m^{(n_m)})$ to be the normalization of the projective closure of $Y_1^{\rm dyn}(N_1^{(n_1)}, \dots, N_m^{(n_m)})$.

Lemma 2.19. *Let $m \geq 2$, and let $N_1, \dots, N_m$ be distinct positive integers. Let $(n_1, \dots, n_m)$ be a tuple of positive integers with $n_i \leq r(N_i)$ for all $i \in \{1, \dots, m\}$. Then the curve $X_1^{\rm dyn}(N_1^{(n_1)}, \dots, N_m^{(n_m)})$ is birational to the fiber product*

$$X_1^{\rm dyn}(N_1^{(n_1)}) \times_{\mathbb{P}^1} \cdots \times_{\mathbb{P}^1} X_1^{\rm dyn}(N_m^{(n_m)}),$$

where the fiber product is taken relative to the projection of $X_1^{\rm dyn}(N_i^{(n_i)})$ onto its C-coordinate.

Proof. It suffices to show that

$$Y_1^{\rm dyn}(N_1^{(n_1)}, \dots, N_m^{(n_m)}) \cong Y_1^{\rm dyn}(N_1^{(n_1)}) \times_{\mathbb{P}^1} \cdots \times_{\mathbb{P}^1} Y_1^{\rm dyn}(N_m^{(n_m)}).$$

A point on $Y_1^{\rm dyn}(N_i^{(n_i)})(\overline{K})$ is of the form $(x_{i,1}, \dots, x_{i,n_i}, c_i)$, where each $x_{i,j}$ has period N_i for f_{c_i}, and the $x_{i,j}$ lie in distinct orbits under f_{c_i}. Therefore, since the fiber product is taken relative to projection onto the C-coordinate, a point of the fiber product must have $c := c_1 = \cdots = c_m$, and therefore the point has the form

$$\Big((x_{1,1}, \dots, x_{1,n_1}, c), \dots, (x_{m,1}, \dots, x_{m,n_m}, c)\Big).$$

Using the expression for points on $Y_1^{\mathrm{dyn}}(N_1^{(n_1)}, \ldots, N_m^{(n_m)})$ given in (2.6), one can see that the desired isomorphism is given by

$$(x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{m,1}, \ldots, x_{m,n_m}, c) \mapsto \Big((x_{1,1}, \ldots, x_{1,n_1}, c), \ldots, (x_{m,1}, \ldots, x_{m,n_m}, c) \Big).$$

$\square$

There are natural automorphisms on the curves $Y_1^{\mathrm{dyn}}(N_1^{(n_1)}, \ldots, N_m^{(n_m)})$, induced by the action of f_C on each coordinate. More precisely, for each $i \in \{1, \ldots, m\}$, there are n_i automorphisms $\sigma_{i,1}, \ldots, \sigma_{i,n_i}$ of order N_i on $Y_1^{\mathrm{dyn}}(N_1^{(n_1)}, \ldots, N_m^{(n_m)})$, with $\sigma_{i,j}$ defined by taking the $X_{i,j}$-coordinate to $f_C(X_{i,j})$ and leaving all other coordinates fixed. Indeed, if $x_{i,j}$ is a point of period N_i for f_c whose orbit is disjoint from that of $x_{i,j'}$, then the same must be true for $f_c(x_{i,j})$. Moreover, since $f_c^{N_i}(x_{i,j}) = x_{i,j}$, we can see that $\sigma_{i,j}^{-1}$ is induced by the action of $f_C^{N_i-1}$ on the $X_{i,j}$-coordinate and the identity on all other coordinates.

If we let $\Gamma \cong \oplus_{i=1}^m (\mathbb{Z}/N_i\mathbb{Z})^{n_i}$ denote the automorphism group generated by $\sigma_{i,j}$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n_i\}$, then we define $Y_0^{\mathrm{dyn}}(N_1^{(n_1)}, \ldots, N_m^{(n_m)})$ to be the quotient of $Y_1^{\mathrm{dyn}}(N_1^{(n_1)}, \ldots, N_m^{(n_m)})$ by Γ. A point on $Y_0^{\mathrm{dyn}}(N_1^{(n_1)}, \ldots, N_m^{(n_m)})(K)$ corresponds to a quadratic map f_c defined over K together with n_i distinct marked K-rational *cycles* of lengths N_i for f_c, for all $i \in \{1, \ldots, m\}$. Each automorphism $\sigma_{i,j}$ extends to an automorphism of $X_1^{\mathrm{dyn}}(N_1^{(n_1)}, \ldots, N_m^{(n_m)})$, and we define $X_0^{\mathrm{dyn}}(N_1^{(n_1)}, \ldots, N_m^{(n_m)})$ similarly.

2.2 Irreducibility of dynamical modular curves

In this section, we study the geometry of dynamical modular curves over $\mathbb{C}$. We begin with the following result of Bousch, which is a restatement of Theorem 2.2.

Theorem 2.20 (Bousch [2]). *Let $N \in \mathbb{N}$. Then the curve $X_1^{\mathrm{dyn}}(N)$ is irreducible over $\mathbb{C}$.*

We now build on Bousch's result to show irreducibility (over $\mathbb{C}$) for a more general class of dynamical modular curves.

Recall that $X_1^{\mathrm{dyn}}(N)$ has an affine model given by the equation $\Phi_N(x,c) = 0$. Let $\pi_N : X_1^{\mathrm{dyn}}(N) \to \mathbb{P}^1$ be the degree $d := d(N)$ projection $(x,c) \mapsto c$. Because $X_1^{\mathrm{dyn}}(N)$ is irreducible over $\mathbb{C}$, this corresponds to a degree d field extension $F/\mathbb{C}(c)$, where F is the function field $\mathbb{C}(x,c)$ with $\Phi_N(x,c) = 0$.

Observe that the Galois group $\mathcal{G} := \mathrm{Gal}(F/\mathbb{C}(c))$ — by which we mean the automorphism group of the Galois closure of $F/\mathbb{C}(c)$ — cannot be the full symmetric group S_d. Indeed, let L be the Galois closure of $F/\mathbb{C}(c)$ in some algebraic closure of $\mathbb{C}(c)$. If $\sigma \in \mathcal{G}$ maps $x \mapsto x'$ in L, then σ must also map $f_c^k(x) \mapsto f_c^k(x')$ for all $k \in \{0, \ldots, N-1\}$. In other words, $\mathcal{G}$ must respect the cycle structure of the roots of $\Phi_N(x,c)$. The following result, also due to Bousch in his thesis, states that this is the only restriction on the elements of $\mathcal{G}$.

Theorem 2.21 (Bousch [2], 1992)**.** *Let N be any positive integer, let F be the function field $\mathbb{C}(x,c)$ with $\Phi_N(x,c) = 0$, and let $\mathcal{G}$ be the Galois group of $F/\mathbb{C}(c)$. Then $\mathcal{G}$ consists of all permutations of the roots of $\Phi_N(x,c) \in \mathbb{C}(c)[x]$ that commute with $x \mapsto f_c(x)$.*

In [23, p. 320], Morton remarks that $\mathcal{G}$ is isomorphic to the *wreath product* of the cyclic group $\mathbb{Z}/N\mathbb{Z}$ with the symmetric group S_r, which is a certain semi-direct product $(\mathbb{Z}/N\mathbb{Z})^r \rtimes S_r$, where $r := r(N)$. See [34, §3.9] for a discussion (and definition) of the wreath product in this setting. The group $\mathcal{G}$ is allowed to permute the r different N-cycles *as sets* by any permutation in S_r, and $\mathcal{G}$ is allowed to act by a cyclic permutation within each N-cycle independently.

We will use Theorems 2.20 and 2.21 to prove the following theorem, which extends Bousch's irreducibility result to curves of the form $X_1^{\mathrm{dyn}}(N^{(n)})$.

Theorem 2.22. *Let N be any positive integer, and let $1 \le n \le r(N)$. Then the curve $X_1^{\mathrm{dyn}}(N^{(n)})$ is irreducible over $\mathbb{C}$.*

To prove the theorem, we first describe the tower of field extensions of $\mathbb{C}(c)$ obtained by repeatedly adjoining roots of $\Phi_N(x,c)$. As above, let F denote the extension of $\mathbb{C}(c)$ determined by the polynomial $\Phi_N(x,c) \in \mathbb{C}(c)[x]$, and let L be the Galois closure of $F/\mathbb{C}(c)$.

Lemma 2.23. *Let $r := r(N)$. Define a tower of fields $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_r$ as follows: Set $F_0 := \mathbb{C}(c)$, and set $g_0 := \Phi_N(x,c) \in F_0[x]$. For each $n \in \{0,\ldots,r-1\}$, we recursively define*

$$F_{n+1} := F_n(\alpha_n)\ , \textit{ where } \alpha_n \in L \textit{ is some root of } g_n\textit{; and}$$
$$g_{n+1} := \frac{\Phi_N(x,c)}{\textit{(linear factors of } \Phi(x,c) \textit{ over } F_{n+1}\textit{)}} = \prod_{\substack{\Phi_N(\alpha,c)=0\\ \alpha\notin F_{n+1}}} (x-\alpha) \in F_{n+1}[x].$$

Then, for each $n \in \{0,\ldots,r-1\}$:

(A) The polynomial g_n is irreducible over $F_n[x]$.

(B) The degree of the field extension F_{n+1}/F_n is equal to $(r-n)N$.

(C) The polynomial g_n has precisely N roots in F_{n+1}.

Proof. The proof is by induction; we begin with the case $n=0$. In this case, statements (A) and (B) follow immediately from Theorem 2.20, since the degree in x of $\Phi_N(x,c)$ is $d = rN$.

To prove (C), we first observe that $F_1 = F$ contains *at least* N roots of g_0. Indeed, if α is any root of $\Phi_N(x,c)$, then so too are $f_c^k(\alpha)$ for all $k \in \{0,\ldots,N-1\}$. If $E/\mathbb{C}(c)$ is *any* field extension, and if $\alpha \in E$, then we must also have $f_c^k(\alpha) \in E$ for all k, since f_c^k is a polynomial with coefficients in $\mathbb{C}(c)$. Since F_1 contains a root of $\Phi_N(x,c)$ by definition, it must therefore contain at least N roots.

Now suppose F_1 contains more than N roots of g_0. By the argument in the previous paragraph, F_1 must contain at least $2N$ roots, so there are at most $(r-2)N$ roots of $\Phi_N(x,c)$ in $L \setminus F_1$. This means that $[F_2 : F_1] \le (r-2)N$ and, since roots of $\Phi_N(x,c)$ necessarily come in sets of N (in the sense of the previous paragraph), the largest $[L : F_1]$ could possibly be is $(r-2)N \cdot (r-3)N \cdots 2N \cdot N = (r-2)!N^{r-2}$. Therefore,

$$[L : F_0] = [L : F_1] \cdot [F_1 : F_0] \leq (r-2)!N^{r-2} \cdot rN = r \cdot (r-2)! \cdot N^{r-1} < r!N^r,$$

contradicting the fact that L is the Galois closure of $F/\mathbb{C}(c)$ and that $\mathcal{G} = \mathrm{Gal}(F/\mathbb{C}(c))$ has cardinality $r!N^r$ by Theorem 2.21 (and the paragraph that follows).

Now let $n > 0$. We first note that

$$\begin{aligned}
[F_n : F_0] &= [F_n : F_{n-1}] \cdot [F_{n-1} : F_{n-2}] \cdots [F_1 : F_0] \\
&= (r-n+1)N \cdot (r-n+2)N \cdots rN \\
&= \frac{r!}{(r-n)!} \cdot N^n.
\end{aligned}$$

by the induction hypothesis on (B). Since $n < r$, we have $[F_n : F_0] < r!N^r$, so F_n is a proper subfield of L.

To prove (A), it suffices to show that $\mathrm{Gal}(L/F_n)$ acts transitively on the roots of g_n, which are precisely the roots of $\Phi_N(x, c)$ that do not lie in F_n. Let β_1 and β_2 be any two distinct roots of g_n; we must show that there exists $\sigma \in \mathrm{Gal}(L/f_n)$ that maps $\beta_1 \mapsto \beta_2$. Note that, since $\beta_1, \beta_2 \notin F_n$, we also have $f_c^k(\beta_i) \notin F_n$ for all $k \in \{0, \ldots, N-1\}$ and $i \in \{1, 2\}$.

By Theorem 2.21, there is an automorphism $\sigma \in \mathrm{Gal}(L/\mathbb{C}(c))$ that fixes all roots of $\Phi_N(x, c)$ except for those in the N-cycle(s) containing β_1 and β_2, and maps the cycle containing β_1 to the cycle containing β_2 (and vice versa). Replacing σ with the composition of σ with a cyclic permutation if necessary, the automorphism σ maps $\beta_1 \mapsto \beta_2$. Since this automorphism only affects the cycle(s) containing β_1 and β_2, it follows from the previous paragraph that σ fixes all of the roots of $\Phi_N(x, c)$ that lie in F_n; hence, $\sigma \in \mathrm{Gal}(L/F_n)$. We therefore conclude that $\mathrm{Gal}(L/F_n)$ acts transitively on the roots of g_n.

Since we now know that (A) holds for F_n and g_n, (B) is equivalent to saying that the degree of g_n is equal to $(r-n)N$. Since the roots of g_n are precisely those roots of $\Phi_N(x,c)$ that do not lie in F_n, it suffices to show that there are nN roots of $\Phi_N(x,c)$ that *do* lie in F_n. This follows immediately from the induction hypothesis on statement (C) of the theorem, since for each $0 \leq k < n$, there are precisely N roots of $\Phi_N(x,c)$ in $F_{k+1} \setminus F_k$.

Finally, we prove (C) by a counting argument very similar to the one used in the $n = 0$ case. We have already mentioned that the induction hypothesis on (B) gives us

$$[F_n : F_0] = \frac{r!}{(r-n)!} \cdot N^n,$$

and we have already said that $\deg g_n = (r-n)N$. Therefore, since g_n is irreducible over F_n,

$$[F_{n+1} : F_0] = (r-n)N \cdot [F_n : F_0] = \frac{r!}{(r-n-1)!} \cdot N^{n+1}.$$

Recall that since g_n has a root in F_{n+1}, it must have at least N roots in F_{n+1}. Now suppose for the sake of contradiction that g_n has more than N roots in F_{n+1}. Then, as we argued in the $n = 0$ case, there are at least $2N$ roots of g_n in F_{n+1}. Thus there are at most $\deg g_n - 2N = (r-n-2)N$ roots of g_n in $L \setminus F_{n+1}$, hence at most $(r-n-2)N$ roots of $\Phi_N(x,c)$ in $L \setminus F_{n+1}$. Since roots of $\Phi_N(x,c)$ come in sets of N, this implies that

$$[L : F_{n+1}] \leq (r-n-2)N \cdot (r-n-3)N \cdots 2N \cdot N = (r-n-2)!N^{r-n-2}.$$

Therefore the degree of L over $F_0 = \mathbb{C}(c)$ satisfies

$$\begin{aligned}[L : F_0] &= [L : F_{n+1}] \cdot [F_{n+1} : F_0] \\ &\leq (r-n-2)!N^{r-n-2} \cdot \frac{r!}{(r-n-1)!} N^{n+1} = \frac{r!}{r-n-1} N^{r-1} < r!N^r.\end{aligned}$$

However, this contradicts the fact mentioned previously that $[L : F_0] = r!N^r$. Therefore F_{n+1} contains only N roots of g_n, and this completes the proof. $\square$

Corollary 2.24. *The field F_r from Lemma 2.23 is equal to the Galois closure L of $F/\mathbb{C}(c)$.*

Proof. By definition, $F_r \subseteq L$. Since $F = F_0$, we apply Lemma 2.23 to get

$$[F_r : F_0] = \prod_{n=0}^{r-1} [F_{n+1} : F_n] = \prod_{n=0}^{r-1} (r-n)N = r!N^r = [L : F_0],$$

so F_r and L must coincide. $\square$

To complete the proof of Theorem 2.22, we prove the following proposition:

Proposition 2.25. *Let $r := r(N)$, and let $n \in \{1, \ldots, r-1\}$. Then the curve*

$$X_1^{\mathrm{dyn}}(N^{(n)}) \times_{\mathbb{P}^1} X_1^{\mathrm{dyn}}(N)$$

has precisely $nN+1$ irreducible components $C_1, \ldots, C_{nN}, C'$ over $\mathbb{C}$, where $C_i \cong X_1^{\mathrm{dyn}}(N^{(n)})$ for each $i \in \{1, \ldots, nN\}$ and $C' \cong X_1^{\mathrm{dyn}}(N^{(n+1)})$ has a function field isomorphic to F_{n+1}. Here the fiber product is taken relative to the map $X_1^{\mathrm{dyn}}(\cdot) \to \mathbb{P}^1$ given by projection onto the c-coordinate. In particular, $X_1^{\mathrm{dyn}}(N^{(n+1)})$ is irreducible over $\mathbb{C}$.

Proof. Let F_n and g_n be defined as in Lemma 2.23. We again proceed by induction, so consider the case $n = 1$. In this case, we want to show that

$$Z := X_1^{\mathrm{dyn}}(N) \times_{\mathbb{P}^1} X_1^{\mathrm{dyn}}(N)$$

has N irreducible components isomorphic to $X_1^{\mathrm{dyn}}(N)$ and a single irreducible component isomorphic to $X_1^{\mathrm{dyn}}(N^{(2)})$. The curve $X_1^{\mathrm{dyn}}(N)$ is irreducible by Theorem 2.20, and F_1 is the function field of $X_1^{\mathrm{dyn}}(N)$. Lemma 2.23 says that $g_0 = \Phi_N(x, c)$ has exactly N roots

$\alpha_1, \ldots, \alpha_N$ in F_1, and

$$g_0 = (x - \alpha_1) \cdots (x - \alpha_N) \cdot g_1,$$

with g_1 irreducible over F_1. We therefore compute the total quotient ring of Z to be

$$F_1 \otimes_{\mathbb{C}(c)} F_1 \cong F_1[x]/g_0 \cong \left(\bigoplus_{i=1}^{N} F_1[x]/(x - \alpha_i) \right) \oplus F_1[x]/g_1(x) \cong F_1^N \oplus F_2,$$

where the isomorphism $F_1[x]/g_1(x) \cong F_2$ comes from Lemma 2.23. Therefore Z has N irreducible components with function field F_1, which are therefore isomorphic to $X_1^{\text{dyn}}(N)$, and one irreducible component C' with function field F_2. We must now show that $C' \cong X_1^{\text{dyn}}(N^{(2)})$.

Let $Y \subseteq \mathbb{A}^3$ be the affine curve defined by

$$\Phi_N(x_1, c) = \Phi_N(x_2, c) = 0, \tag{2.7}$$

and let $Y' \subseteq Y$ denote the open subset defined by the N conditions

$$x_2 \neq f_c^k(x_1) \text{ for all } 0 \leq k \leq N - 1. \tag{2.8}$$

Now observe that (2.7) defines a curve birational to the fiber product Z, and each of the N conditions in (2.8) removes a component of Y birational to $X_1^{\text{dyn}}(N)$. Therefore Y' is birational to the remaining component C' of Z. Since $Y_1^{\text{dyn}}(N^{(2)})$ is by definition a (dense) open subset of Y', this completes the proof in the case $n = 1$.

Now suppose $n > 1$. In this case, we set

$$Z := X_1^{\text{dyn}}(N^{(n)}) \times_{\mathbb{P}^1} X_1^{\text{dyn}}(N).$$

By the induction hypothesis, $X_1^{\mathrm{dyn}}(N^{(n)})$ and $X_1^{\mathrm{dyn}}(N)$ are both irreducible, with function fields F_n and F_1, respectively. Arguing as before, we find that the total quotient ring of Z is

$$F_n \otimes_{\mathbb{C}(c)} F_1 \cong F_n[x]/g_0 \cong F_n^{nN} \oplus F_n[x]/g_n,$$

where the second isomorphism comes from Lemma 2.23, recalling from the proof that there are exactly nN roots of $\Phi_N(x,c)$ lying inside F_n. Therefore Z has nN irreducible components with function field F_n, which are therefore isomorphic to $X_1^{\mathrm{dyn}}(N^{(n)})$. Also, by Lemma 2.23, the summand $F_n[x]/g_n$ is isomorphic to F_{n+1}, so Z contains one more irreducible component C' with function field F_{n+1}. It remains to show that $C' \cong X_1^{\mathrm{dyn}}(N^{(n+1)})$. We argue in a similar manner as in the proof for the $n = 1$ case.

Let $Y \subseteq \mathbb{A}^{n+2}$ be the curve defined by

$$\Phi_N(x_1, c) = \Phi_N(x_2, c) = \cdots = \Phi_N(x_{n+1}, c) = 0, \tag{2.9}$$

let $Y' \subseteq Y$ be the open subset defined by the conditions

$$x_j \neq f_c^k(x_i) \text{ for all } 1 \leq i < j \leq n\ ,\ 0 \leq k \leq N-1, \tag{2.10}$$

and let $Y'' \subseteq Y'$ be the open subset defined by the nN conditions

$$x_{n+1} \neq f_c^k(x_i) \text{ for all } 1 \leq i \leq n\ ,\ 0 \leq k \leq N-1. \tag{2.11}$$

We begin by observing that the curve Y given by (2.9) is birational to the $(n+1)$-fold fiber product of $X_1^{\mathrm{dyn}}(N)$ with itself, and the curve Y' given by (2.9) and (2.10) is birational to $Z = X_1^{\mathrm{dyn}}(N^{(n)}) \times_{\mathbb{P}^1} X_1^{\mathrm{dyn}}(N)$. Each of the nN conditions in (2.11) removes an irreducible component of Y' that is birational to $X_1^{\mathrm{dyn}}(N^{(n)})$; hence Y'' is birational to the remaining component C'. By definition, $Y_1^{\mathrm{dyn}}(N^{(n+1)})$ is an open subset of Y'', so $Y_1^{\mathrm{dyn}}(N^{(n+1)})$ is birational to C', completing the proof. $\square$

Remark 2.26. It follows from the proof of Theorem 2.22 that the cover $X_1^{\mathrm{dyn}}(N^{(r)}) \to \mathbb{P}^1$ is the Galois closure of the cover $X_1^{\mathrm{dyn}}(N) \to \mathbb{P}^1$, since the function field of $X_1^{\mathrm{dyn}}(N^{(r)})$ is the Galois closure of the extension $F/\mathbb{C}(c)$.

We now turn our attention to dynamical modular curves of the form $X_1^{\mathrm{dyn}}(N_1^{(n_1)}, \ldots, N_m^{(n_m)})$ with $n_i \le r_i := r(N_i)$. In this more general setting, we still maintain irreducibility over $\mathbb{C}$. In order to prove this, it will suffice for us to show that the curve $X_1^{\mathrm{dyn}}(N_1^{r_1}, \ldots, N_m^{r_m})$ is irreducible over $\mathbb{C}$, since this curve admits a finite morphism to each curve of the form $X_1^{\mathrm{dyn}}(N_1^{(n_1)}, \ldots, N_m^{(n_m)})$ by Proposition 2.18.

We must first recall some standard facts from the classical theory of rational dynamics over $\mathbb{C}$. Let φ be a rational function of degree $d \ge 2$ defined over $\mathbb{C}$. Let P be a point of formal period N for φ, and let $\omega := \{P, \varphi(P), \ldots, \varphi^{N-1}(P)\}$ be the corresponding formal N-cycle. The N-**multiplier**[1] of ω is the quantity

$$\lambda_N = \lambda_N(\omega) := \left(\varphi^N\right)'(P).$$

By the chain rule, $\lambda_N(\omega)$ does not depend on the choice of representative $P \in \omega$. If P has *exact* period equal to N, then we drop the dependence on N in the notation and simply refer to the **multiplier** of ω, which we denote by λ. If P has exact period n strictly dividing N, then one can see that

$$\lambda_N = \left(\left(\varphi^n\right)'(P)\right)^{N/n} = \lambda^{N/n}.$$

[1]This is not standard terminology; usually the prefix "N-" is omitted. However, we reserve the term "multiplier" for cycles of *exact* period N.

We say that the cycle ω is

$$\begin{aligned} \textit{superattracting}, &\quad \text{if } \lambda = 0; \\ \textit{attracting}, &\quad \text{if } 0 < |\lambda| < 1; \\ \textit{indifferent}, &\quad \text{if } |\lambda| = 1; \text{ and} \\ \textit{repelling}, &\quad \text{if } |\lambda| > 1. \end{aligned}$$

If ω is an indifferent cycle, then we say that ω is *rationally indifferent* if λ is a root of unity, and we say that ω is *irrationally indifferent* otherwise. One fact that we require is the following:

Lemma 2.27 (Douady)**.** *Let $\varphi \in \mathbb{C}[z]$ be a polynomial of degree $d \geq 2$. Then φ has at most $d-1$ nonrepelling cycles in the complex plane.*

Proof. See [3, Thm. VI.1.2]. □

Corollary 2.28. *Let $c \in \mathbb{C}$. Then the map f_c has at most one rationally indifferent cycle.*

We now use Corollary 2.28 to show that, for a given element $c_0 \in \mathbb{C}$, the polynomial $\Phi_N(x, c_0) \in \mathbb{C}[x]$ has multiple roots for at most one $N \in \mathbb{N}$.

Lemma 2.29. *Let $c_0 \in \mathbb{C}$. Then there exists at most one $N \in \mathbb{N}$ for which $\Phi_N(x, c_0)$ has a multiple root x_0. Equivalently, there exists at most one $N \in \mathbb{N}$ for which the map*

$$\begin{aligned} X_1^{\mathrm{dyn}}(N) &\to \mathbb{P}^1 \\ (x, c) &\mapsto c \end{aligned}$$

is ramified over c_0.

Proof. Suppose that $\Phi_N(x, c_0) \in \mathbb{C}[x]$ has a multiple root x_0. Then x_0 is also a multiple root for $f_{c_0}^N(x) - x$, which has $\Phi_N(x, c_0)$ as a factor by (2.1). Hence, if ω is the cycle of exact period $n \mid N$ containing x_0, then

$$\lambda(\omega)^{N/n} = \left(f_{c_0}^N\right)'(x_0) = 1,$$

so ω is a rationally indifferent cycle. By Corollary 2.28, there is only one rationally indifferent cycle for f_{c_0}. Therefore, if N' is any positive integer for which $\Phi_{N'}(x, c_0)$ has a multiple root, then x_0 is one such multiple root.

Let r be the positive integer for which $\lambda(\omega)$ is a primitive rth root of unity. First suppose that $r = 1$, so that $\lambda = 1$. Then, by [34, Thm. 4.5(b)], the only N for which $\Phi_N(x_0, c_0) = 0$ is $N = n$, in which case we are done.

Now suppose $r \geq 2$. We again apply [34, Thm. 4.5(b)], which in this case says that the only N for which $\Phi_N(x_0, c_0) = 0$ are $N \in \{n, rn\}$. However, since $\lambda(\omega) = \lambda_n(\omega) \neq 1$, it follows from [26, p. 572] that $\operatorname{disc}_x \Phi_n(x, c_0) \neq 0$, so x_0 must be a simple root of $\Phi_n(x, c_0)$. Therefore, the only N for which $\Phi_N(x, c_0)$ may have a multiple root is $N = rn$.

In both cases, we have shown that there is at most one $N \in \mathbb{N}$ for which $\Phi_N(x, c_0)$ has a multiple root, completing the proof. $\square$

By general Galois theory, since the cover $X_1^{\text{dyn}}(N^{(r)}) \to \mathbb{P}^1$ is the Galois closure of the cover $X_1^{\text{dyn}}(N) \to \mathbb{P}^1$, an element $c_0 \in \mathbb{C}$ is a branch point for $X_1^{\text{dyn}}(N^{(r)}) \to \mathbb{P}^1$ if and only if it is a branch point for $X_1^{\text{dyn}}(N) \to \mathbb{P}^1$. We use this fact to prove the following consequence of Lemma 2.29.

Corollary 2.30. *Let $N_1 \neq N_2$ be two positive integers. Then the only branch point common to both maps $\pi_1 : X_1^{\text{dyn}}(N_1^{(r_1)}) \to \mathbb{P}^1$ and $\pi_2 : X_1^{\text{dyn}}(N_2^{(r_2)}) \to \mathbb{P}^1$ is ∞.*

Proof. By Lemma 2.29, if c_0 is a finite branch point for π_1, then c_0 cannot be the branch point of $X_1^{\text{dyn}}(N^{(r)}) \to \mathbb{P}^1$ for any $N \neq N_1$. On the other hand, ∞ is a branch point for *all* maps $X_1^{\text{dyn}}(N^{(r)}) \to \mathbb{P}^1$, since the map $X_1^{\text{dyn}}(N) \to \mathbb{P}^1$ is ramified over ∞ for all $N \in \mathbb{N}$ by [23, Prop. 10]. $\square$

We require one final lemma before proving our main result on irreducibility.

Lemma 2.31. *Let X_1 and X_2 be two nonsingular, reduced, irreducible curves over $\mathbb{C}$, and for each $i \in \{1,2\}$ let $\pi_i : X_i \to \mathbb{P}^1$ be a finite morphism of degree $d_i \geq 2$. Let $\mathcal{B}_i \subset \mathbb{P}^1(\mathbb{C})$ be the branch locus for π_i. Suppose that $\mathcal{B}_1 \cap \mathcal{B}_2$ contains at most one point. Then the curve*

$$X_1 \times_{\mathbb{P}^1} X_2$$

is irreducible over $\mathbb{C}$. Here the fiber product is taken relative to the maps π_i.

Proof. For each $i \in \{1,2\}$, let K_i denote the function field of X_i, considered as subfields of a common algebraic closure of $\mathbb{C}(t)$. To prove the lemma, it suffices to show that K_1 and K_2 are linearly disjoint over $\mathbb{C}(t)$, since this means that $K_1 \otimes_{\mathbb{C}(t)} K_2 = K_1K_2$ is the function field of the fiber product. In fact, if we let L_i denote the Galois closure of $K_i/\mathbb{C}(t)$, then it suffices to show that $L_1 \cap L_2 = \mathbb{C}(t)$, since this implies that L_1 and L_2 (and hence K_1 and K_2) are linearly disjoint over $\mathbb{C}(t)$.

We begin by showing that if X is an irreducible curve with a map $\pi : X \to \mathbb{P}^1$ of degree $d \geq 2$, then π has at least two branch points in $\mathbb{P}^1(\mathbb{C})$. Let g denote the genus of X. By the Riemann-Hurwitz formula, we have

$$\sum_{P \in X(\mathbb{C})} (e_P - 1) = 2d + 2(g-1), \tag{2.12}$$

where e_P denotes the ramification index of the map π at P. Since $d \geq 2$, the right hand side of (2.12) is always positive, so π must be ramified over at least one point $P_0 \in \mathbb{P}^1(\mathbb{C})$.

Now suppose P_0 is the only branch point for π. Then (2.12) becomes

$$\sum_{P \in \pi^{-1}(P_0)} (e_P - 1) = 2d + 2(g-1).$$

Since $\sum_{P\in\pi^{-1}(P_0)} e_P = d$, we may rewrite this equation as

$$n = 2 - d - 2g,$$

where n is the number of distinct preimages of P_0. We now have a contradiction, since a triple of integers (g, n, d) with $g \geq 0$, $n \geq 1$, and $d \geq 2$ cannot satisfy this equation. Therefore π has at least two branch points in $\mathbb{P}^1(\mathbb{C})$.

Now suppose that $L_1 \cap L_2 = L \supsetneq \mathbb{C}(t)$. By the previous paragraph, there are at least two places $\mathfrak{p}$, $\mathfrak{p}'$ of $\mathbb{C}(t)$ which ramify in L, and therefore $\mathfrak{p}$ and $\mathfrak{p}'$ both ramify in each of the extensions L_1 and L_2. It follows that $\mathfrak{p}$ and $\mathfrak{p}'$ are both ramified in each of the extensions K_1 and K_2, since a place unramified in K_i would remain unramified in the Galois closure L_i. However, this contradicts our assumption that the maps π_1 and π_2 shared at most a single branch point. Therefore the field extensions K_1 and K_2 must be linearly disjoint over $\mathbb{C}(t)$, completing the proof. $\square$

We are now ready to prove the following theorem.

Theorem 2.32. *Let $\{N_1, \ldots, N_m\}$ be any nonempty set of positive integers. Then*

(A) The branch locus for the map $X_1^{\mathrm{dyn}}(N_1^{(r_1)}, \ldots, N_m^{(r_m)}) \to \mathbb{P}^1$ is the union

$$\bigcup_{i=1}^{m} \mathcal{B}_i,$$

where $\mathcal{B}_i$ is the branch locus for the map $X_1^{\mathrm{dyn}}(N_i^{(r_i)}) \to \mathbb{P}^1$.

(B) The curve $X_1^{\mathrm{dyn}}(N_1^{(r_1)}, \ldots, N_m^{(r_m)})$ is irreducible over $\mathbb{C}$.

Proof. We proceed by induction on m. For the $m = 1$ case, (A) is trivial, and (B) follows from Theorem 2.22. Now suppose $m \geq 2$. By Lemma 2.19, we can write

$$X_1^{\mathrm{dyn}}(N_1^{(r_1)}, \ldots, N_m^{(r_m)}) = X_1^{\mathrm{dyn}}(N_1^{(r_1)}, \ldots, N_{m-1}^{(r_{m-1})}) \times_{\mathbb{P}^1} X_1^{\mathrm{dyn}}(N_m^{(r_m)}).$$

The induction hypothesis tells us that the curves $X_1^{\mathrm{dyn}}(N_1^{(r_1)}, \ldots, N_{m-1}^{(r_{m-1})})$ and $X_1^{\mathrm{dyn}}(N_m^{(r_m)})$ are irreducible over $\mathbb{C}$. Moreover, the branch locus of $X_1^{\mathrm{dyn}}(N_1^{(r_1)}, \ldots, N_{m-1}^{(r_{m-1})}) \to \mathbb{P}^1$ is the union of the branch loci $\mathcal{B}_i$ for the maps $X_1^{\mathrm{dyn}}(N_i^{(r_i)}) \to \mathbb{P}^1$, $i \in \{1, \ldots, m-1\}$. By Corollary 2.30, $\mathcal{B}_m \cap \mathcal{B}_i = \{\infty\}$ for all $i \in \{1, \ldots, m-1\}$, so the only common branch point for the maps $X_1^{\mathrm{dyn}}(N_1^{(r_1)}, \ldots, N_{m-1}^{(r_{m-1})}) \to \mathbb{P}^1$ and $X_1^{\mathrm{dyn}}(N_m^{(r_m)}) \to \mathbb{P}^1$ is ∞. It now follows from Lemma 2.31 that $X_1^{\mathrm{dyn}}(N_1^{(r_1)}, \ldots, N_m^{(r_m)})$ is irreducible. Finally, note that the branch locus of the map $X_1^{\mathrm{dyn}}(N_1^{(r_1)}, \ldots, N_m^{(r_m)}) \to \mathbb{P}^1$ certainly contains $\bigcup_{i=1}^m \mathcal{B}_m$. Equality comes from the fact that if $P \in \mathbb{P}^1(\mathbb{C})$ is unramified for both maps $X_1^{\mathrm{dyn}}(N_1^{(r_1)}, \ldots, N_{m-1}^{(r_{m-1})}) \to \mathbb{P}^1$ and $X_1^{\mathrm{dyn}}(N_m^{(r_m)}) \to \mathbb{P}^1$, then P remains unramified for the map from the fiber product. $\square$

Corollary 2.33. *Let $\{N_1, \ldots, N_m\}$ be a set of positive integers, and let $(n_1, \ldots, n_m)$ be a sequence of positive integers satisfying $n_i \le r(N_i)$ for all $i \in \{1, \ldots, m\}$. Then the curve*

$$X_1^{\mathrm{dyn}}(N_1^{(n_1)}, \ldots, N_m^{(n_m)})$$

is irreducible over $\mathbb{C}$.

Proof. Since there is a finite morphism

$$X_1^{\mathrm{dyn}}(N_1^{(r_1)}, \ldots, N_m^{(r_m)}) \to X_1^{\mathrm{dyn}}(N_1^{(n_1)}, \ldots, N_m^{(n_m)}),$$

the result follows from Theorem 2.32. $\square$

We would like to know whether these irreducibility results hold in general, that is, for all curves $X_1^{\mathrm{dyn}}(G)$ with G an admissible graph. However, we are currently unable to prove that $X_1^{\mathrm{dyn}}(G)$ is irreducible in general when G contains points of type N_M for $M \ge 2$. However, it is true that all of the dynamical modular curves studied in this thesis, as well as all of the curves studied in [8], are irreducible over $\mathbb{C}$. We are therefore motivated to make the following conjecture.

Conjecture 2.34. *Let G be an admissible graph. Then $X_1^{\text{dyn}}(G)$ is irreducible over $\mathbb{C}$.*

Chapter 3

Quadratic points on algebraic curves

In order to answer Question 2.15, we require the ability to understand the sets of quadratic points on various algebraic curves. In this chapter, we set up the necessary notation and record a number of results that will be useful throughout the remainder of this thesis. Throughout this chapter, k will denote a number field.

Let X be an algebraic curve defined over k. We say that $P \in X(\overline{k})$ is **quadratic over** k if the field of definition of P — denoted $k(P)$ — is a quadratic extension K/k. We will mostly be working in the situation that $k = \mathbb{Q}$, in which case we will simply say that P is **quadratic**. Note that our definition does not include k-rational points on X. We will denote by $X(k,2)$ the set of points on X that are rational or quadratic over k; i.e.,

$$X(k,2) := \{P \in X(\overline{k}) : [k(P) : k] \leq 2\}.$$

Many of the curves we consider will be affine curves. If X is an affine curve, then the **genus** of X will be understood to be the geometric genus; i.e., the genus of the nonsingular projective curve birational to X.

A special class of curves that we will often need to consider is the class of hyperelliptic curves. A **hyperelliptic** curve defined over k is a nonsingular projective curve X of genus $g \geq 2$ for which there exists a degree two morphism $X \to \mathbb{P}^1$. Such a curve has an affine model of the form $y^2 = f(x)$ for some polynomial $f(x) \in k[x]$ of degree $2g+1$ or $2g+2$ with no repeated roots. (In fact, one can always take f to have degree $2g+2$.) Conversely, any curve given by such an equation (with $\deg f \geq 5$) is hyperelliptic, with the degree two map to $\mathbb{P}^1$ being given by $(x, y) \mapsto x$. If $\deg f$ is odd, then the curve X has a single k-rational point at infinity. If $\deg f$ is even, then the curve X has two points at infinity, and they are k-rational if and only if the leading coefficient of f is a square in k. (Otherwise, the two points at infinity are defined over the quadratic extension of k generated by the square root of the leading coefficient of f.) A **Weierstrass point** on X is a ramification point for the double cover of $\mathbb{P}^1$ given by $(x, y) \mapsto x$; i.e., P is a Weierstrass point on X if $P = (x, y)$ with $y = 0$, or if $\deg f$ is odd and P is the point at infinity. We will refer to the set of Weierstass points as the **Weierstrass locus** of X, which we will denote by W_X.

There is a simple way to generate infinitely many quadratic (over k) points on a hyperelliptic curve X: if $x \in k$ is such that $f(x)$ is not a k-rational square, then the point $\left(x, \sqrt{f(x)}\right)$ is quadratic over k. The fact that there are infinitely many such $x \in k$ is a consequence of Hilbert's irreducibility theorem, which says in this case that there are infinitely many $x \in k$ for which the polynomial $Y^2 - f(x)$ is irreducible over k. Such quadratic points on X will be called **obvious** quadratic points. Any quadratic point with $x \notin \mathbb{Q}$ will be called a **non-obvious** quadratic point. We will use the same terminology for genus one curves of the form $y^2 = f(x)$ with $\deg f \in \{3, 4\}$.

3.1 Elliptic curves

Let k be a number field, and let E be an elliptic curve defined over k. There is a model for E of the form $y^2 = f(x)$ with $f(x) \in k[x]$ of degree three. It will be useful for us to be able to describe the non-obvious quadratic points on E, so we provide here a lemma to this end.

Suppose $P = (x, y) \in E(\overline{k})$ is a non-obvious quadratic point over k, and let $\sigma(P) = (\sigma(x), \sigma(y))$ be the Galois conjugate of P, where σ is the nontrivial automorphism of $k(P)/k$. Consider the line passing through P and $\sigma(P)$. The slope v of this line is k-rational, so the third point $Q = (x_0, y_0) \in E(\overline{k})$ lying on this line is necessarily a k-rational point (different from ∞, since $x \neq \sigma(x)$). The following result, which appears in [15, Lem. 4.5.3] and [8, Lem. 2.2], makes this geometric discussion more explicit.

Lemma 3.1. *Let k be a number field, let E be an elliptic curve defined over k, and fix a model for E of the form*

$$y^2 = ax^3 + bx^2 + cx + d$$

with $a, b, c, d \in k$. Let (x, y) be a quadratic point on E with $x \notin k$. Then there exist a point $(x_0, y_0) \in E(k)$ and an element $v \in k$ such that $y = y_0 + v(x - x_0)$ and

$$x^2 + \frac{ax_0 - v^2 + b}{a}x + \frac{ax_0^2 + v^2x_0 + bx_0 - 2y_0v + c}{a} = 0.$$

3.2 Curves of genus two

Now let X be a curve of genus two defined over a number field k. A nice treatment of the arithmetic of genus two curves may be found in [4], and much of what we say in this section may be found there. Every genus two curve is hyperelliptic, so X has an affine model given by an equation of the form $y^2 = f(x)$ with $f(x) \in k[x]$ of degree $d \in \{5, 6\}$ having no repeated roots. We denote by ι the hyperelliptic involution of X:

$$\iota(x, y) = (x, -y).$$

The Jacobian J of X is a two-dimension abelian variety defined over k. If we assume that X has a k-rational point (this is guaranteed if $d = 5$), then we may identify the Mordell-Weil group $J(k)$ with the group of k-rational degree zero divisors of X modulo linear equivalence ([20, p. 168], [4, p. 39]). For $n \geq 2$, the n-torsion subgroup of $J(k)$ is the subgroup

$$J(k)[n] = \{\mathcal{D} \in J(k) : n\mathcal{D} = 0\}.$$

We denote by $J(k)_{\text{tors}}$ the full torsion subgroup of $J(k)$; i.e.,

$$J(k)_{\text{tors}} = \{\mathcal{D} \in J(k) : n\mathcal{D} = 0 \text{ for some } n \in \mathbb{N}\} = \bigcup_{n \in \mathbb{N}} J(k)[n].$$

If $d = 6$, then X has two points at infinity, which we call ∞^+ and ∞^-. We distinguish the points in the following way: The curve X can be covered by two affine patches — the first is given by the equation $y^2 = f(x)$, and the second is given by the equation $v^2 = u^6 f(1/u)$, together with the identification $x = 1/u$, $y = v/u^3$. Letting a denote the leading coefficient of $f(x)$, we take $\infty^\pm$ to be the point on X corresponding to the point $(u, v) = (0, \pm\sqrt{a})$. Of course, we are making a choice of $+\sqrt{a}$ versus $-\sqrt{a}$; in the case that $\sqrt{a} \in \mathbb{R}$, we make the standard choice. This explicit distinction is typically unnecessary, but we require it for an application in §7. If $d = 5$, then we take $\infty^+ = \infty^- = \infty$ to be the unique (rational!) point at infinity, which corresponds to the point $(u, v) = (0, 0)$. In both the $d = 5$ and $d = 6$ cases, we have $\iota(\infty^\pm) = \infty^\mp$.

With this notation, the divisor $\infty^+ + \infty^-$ is a *rational* divisor on X, regardless of whether $d = 5$ or $d = 6$. The divisor class $\mathcal{K}$ containing $\infty^+ + \infty^-$ is the canonical divisor class. It is a consequence of the Riemann-Roch theorem that every degree two divisor class $\mathcal{D}$ contains an effective divisor, and this effective divisor is unique if and only if $\mathcal{D} \neq \mathcal{K}$. The effective divisors in the canonical class $\mathcal{K}$ are precisely those divisors of the form $P + \iota P$.

A useful consequence of the above discussion is that every nontrivial element of $J(k)$ may be represented *uniquely* by a divisor of the form $P + Q - \infty^+ - \infty^-$, up to reordering of P and Q, where $P + Q$ is a k-rational divisor on X; that is, either P and Q are both k-rational points on X or P and Q are quadratic points on X that are Galois conjugates. With this in mind, we will represent points of $J(k)$ as unordered pairs $\{P, Q\}$, with the identification

$$\{P, Q\} = \left[P + Q - \infty^+ - \infty^-\right].$$

Here $[D]$ denotes the divisor class of the divisor D. Note that $\{P, Q\} = 0$ if and only if $[P + Q] = \mathcal{K}$; that is, if and only if $P = \iota(Q)$. If Σ is any subset of $X(\overline{k})$, we say that the element $\mathcal{D} \in J(\overline{k})$ is **supported on** Σ if $\mathcal{D} = \{P, Q\}$ with $P, Q \in \Sigma$.

Observe that $P \in X(\overline{k})$ is a Weierstrass point if and only if $\iota P = P$. The two-torsion subgroup of $J(\overline{k})$ is precisely the set of elements of $J(\overline{k})$ supported on the Weierstrass locus:

$$J(\overline{k})[2] = \{\mathcal{D} \in J(\overline{k}) : \ \mathcal{D} \text{ is supported on } W_X \ \}.$$

More precisely, the fifteen points of order two in $J(\overline{k})$ are those points $\mathcal{D} = \{P, Q\}$ where P and Q are *distinct* Weierstrass points on X, since if $P = Q$, then $\{P, P\} = \{P, \iota(P)\} = 0$. This characterization of 2-torsion points reduces the problem of determining $J(k)[2]$ to the problem of finding the linear and quadratic factors of $f(x)$ in $k[x]$.

The identification of points on $J(k)$ as pairs $\{P, Q\}$ is also useful for describing the non-obvious quadratic points on the genus two curve X. Let P be any quadratic point on X, and let σ be the nontrivial automorphism of $k(P)/k$.

First suppose that P is an obvious quadratic point. We can write $P = (x, \sqrt{z})$ for some $x, z \in k$ with z not a square in k. Then $\sigma(P) = (\sigma(x), \sigma(\sqrt{z})) = (x, -\sqrt{z}) = \iota(P)$, so that the point $\mathcal{D} = \{P, \sigma(P)\} \in J(k)$ is equal to $\{P, \iota(P)\} = 0$.

On the other hand, if $P = (x, y)$ is a non-obvious quadratic point, then $x \notin k$, hence $\sigma(x) \neq x$. It follows that $\sigma(P)$ is equal to neither P nor $\iota(P)$, so $\{P, \sigma(P)\}$ is a nontrivial element of $J(k)$.

This distinction between obvious and non-obvious quadratic points $P \in X(\overline{k})$ in terms of the corresponding points $\{P, \sigma(P)\} \in J(k)$ is not new. In fact, it is noted in [1, p. 11] that this is well known. However, we record here a useful consequence.

Lemma 3.2. *Let X be a genus two curve defined over a number field k. Then $J(k)$ is finite if and only if X admits only finitely many non-obvious quadratic points over k. Moreover, if $J(k)$ is finite and all elements of $J(k)$ are known, then the finite set of non-obvious quadratic points on X may be explicitly determined.*

Proof. First, we note that $X(k)$ is finite by Faltings' theorem, so we can have only finitely many points $\{P, Q\} \in J(k)$ for which $P, Q \in X(k)$.

Suppose $J(k)$ is infinite. Since $X(k)$ is finite, there must be infinitely many points $0 \neq \mathcal{D} \in J(k)$ for which $\mathcal{D} = \{P, \sigma(P)\}$ with P a quadratic point on X. By the discussion preceding the lemma, P must in fact be a non-obvious quadratic point, so there are infinitely many non-obvious quadratic points on X.

Now suppose that there are infinitely many non-obvious quadratic points on X. Choose an infinite sequence $P_1, P_2, \ldots$ of such points, where P_i is not in the Galois orbit of P_j for any $i \neq j$. Then the points $\mathcal{D}_i := \{P_i, \sigma_i(P_i)\} \in J(k)$ are distinct, where σ_i is the nontrivial element of the Galois group of $k(P_i)/k$. Therefore $J(k)$ is infinite.

Finally, if $J(k)$ is finite and we know all elements $\mathcal{O}, \{P_1, Q_1\}, \cdots, \{P_n, Q_n\}$ of $J(k)$, then the non-obvious quadratic points will be among the P_i and Q_i. We can therefore explicitly list all non-obvious quadratic points by determining which of the P_i and Q_i are quadratic. $\square$

3.3 Curves of genus at least two

We now describe a number of results about curves of arbitrary genus. Many are known, though we introduce some new results as well.

The first collection of results may be classified as *Chabauty-type* results, as they stem from the important work of Chabauty [5] in 1941 and a subsequent improvement due to Coleman [6] in 1985. An excellent introduction to the method of Chabauty and Coleman is the survey [18]. We first record the main consequence of Chabauty's original work. As we will do throughout this thesis, for an abelian variety A defined over a number field k, we denote by $\operatorname{rk} A(k)$ the *rank* of the Mordell-Weil group $A(k)$.

Theorem 3.3 (Chabauty [5], 1941)**.** *Let X be a curve of genus $g \geq 1$ defined over a number field k, and let J be the Jacobian of X. If $\operatorname{rk} J(k) < g$, then $X(k)$ is finite.*

As noted in [18], though Chabauty's theorem is weaker than Faltings' theorem because of the condition on the rank of $J(k)$, it has the advantage that the method involved can lead to bounds on the number of k-rational points on X. Like Theorem 3.3, the next few results hold over arbitrary number fields (with minor modifications), but we will state them only over $\mathbb{Q}$ since that is the only case we will require in later chapters.

Coleman was the first to use Chabauty's method to give explicit bounds on the number of rational points on a curve of genus g satisfying the conditions of Theorem 3.3.

Theorem 3.4 (Coleman [6], 1985)**.** *Let X be a curve of genus $g \geq 1$ defined over $\mathbb{Q}$, let J be the Jacobian of X, and let $p > 2g$ be a prime of good reduction for X. Suppose that $\operatorname{rk} J(\mathbb{Q}) < g$. Then*

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2(g-1).$$

Various improvements have since been made to Coleman's result. First, Lorenzini and Tucker [16] have extended Coleman's result to the case of primes of bad reduction.

Theorem 3.5 (Lorenzini-Tucker [16], 2002)**.** *Let X be a curve of genus $g \geq 1$ defined over $\mathbb{Q}$, let J be the Jacobian of X, and let $p > 2g$ be a prime. Let $\mathscr{X}/\mathbb{Z}_p$ be any regular model for $X/\mathbb{Q}_p$. Suppose that* $\operatorname{rk} J(\mathbb{Q}) < g$*. Then*

$$\#X(\mathbb{Q}) \leq \#\overline{\mathscr{X}}_{\mathrm{ns}}(\mathbb{F}_p) + 2(g-1),$$

where $\overline{\mathscr{X}}_{\mathrm{ns}}$ denotes the set of nonsingular points on the special fiber of $\mathscr{X}$.

Lorenzini and Tucker [16, p. 59] ask whether the term $(g-1)$ in the bound may be improved to depend on the rank r of $J(\mathbb{Q})$ rather than the genus of the curve; more precisely, they suggest that perhaps $(g-1)$ may be replaced with r.[1] Stoll [36] has shown that we may indeed improve the bound as Lorenzini and Tucker suggest, in the case that p is a prime of good reduction.

Theorem 3.6 (Stoll [36], 2006)**.** *Let X be a curve of genus $g \geq 1$ defined over $\mathbb{Q}$, let J be the Jacobian of X, and let p be a prime of good reduction for X. Suppose that* $r := \operatorname{rk} J(\mathbb{Q}) < g$*. Then*

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2r + \left\lfloor \frac{2r}{p-2} \right\rfloor .$$

In particular, if $p > 2(r+1)$, then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2r.$$

Now, if A is an abelian variety over a field k, and if K/k is any field extension, then there is an obvious inclusion $A(k) \hookrightarrow A(K)$. The goal of the remainder of this section is to give a

[1] Lorenzini and Tucker actually suggest that $(g-1)$ may be replaced with the "Chabauty rank" of X, which is a quantity that is always at most the rank r of $J(\mathbb{Q})$. It follows from work of Stoll [36] that this is indeed the case if p is a prime of good reduction for X. However, we present here a weaker version of Stoll's result that is sufficient for our purposes.

sufficient condition for the Mordell-Weil group to remain unchanged upon base change from k to K. We will then give a consequence for quadratic points on curves of genus two.

If G is a finitely generated abelian group and $H \subseteq G$ is a subgroup, then the **saturation** of H in G is the largest subgroup $H' \subseteq G$ containing H such that $[H' : H] < \infty$. We will say that H is **saturated** in G if the saturation of H in G is H itself.

Proposition 3.7. *Let k be a field, and let A be an abelian variety defined over k such that $A(k)$ is finitely generated. Let K be a finite Galois extension of k of degree $n := [K : k]$. Suppose that $A(K)_{\mathrm{tors}} = A(k)_{\mathrm{tors}}$ and $A(k)[n] = 0$. Then $A(k)$ is saturated in $A(K)$.*

Remark 3.8. If k is finitely generated over its prime subfield, for example, then Néron's generalization [29] of the Mordell-Weil theorem states that $A(k)$ is necessarily finitely generated.

Proof. Let $P \in A(K)$ lie in the saturation of $A(k)$; we wish to show that $P \in A(k)$. Let $\sigma \in G := \mathrm{Gal}(K/k)$ be arbitrary. Since P lies in the saturation of $A(k)$, there exists some $\ell \in \mathbb{Z}$ for which $\ell P \in A(k)$, and therefore $\sigma(\ell P) = \ell P$. Writing this as $\ell(\sigma P - P) = 0$ shows that $\sigma P - P$ must be a torsion element of $A(K)$. Since $A(K)_{\mathrm{tors}} = A(k)_{\mathrm{tors}}$, the element $\sigma P - P$ must in fact be k-rational.

Set

$$P' := \sum_{\tau \in G} \tau(\sigma P - P).$$

On one hand, since $\sigma P - P \in A(k)$, we have $\tau(\sigma P - P) = \sigma P - P$ for all $\tau \in G$. Since $\#G = n$, this means that $P' = n(\sigma P - P)$.

On the other hand, we can rewrite the sum defining P' as

$$P' = \sum_{\tau \in G} \tau\sigma P - \sum_{\tau \in G} \tau P = \sum_{\tau' \in G} \tau' P - \sum_{\tau \in G} \tau P = 0.$$

Therefore $n(\sigma P - P) = 0$. Since we assumed that $A(k)[n] = 0$, it follows that $\sigma P - P = 0$; i.e., $\sigma P = P$. Since this holds for all $\sigma \in G$, it follows that $P \in A(k)$. $\square$

Corollary 3.9. *Let k be a field, and let A be an abelian variety defined over k for which $A(k)$ is finitely generated. Let K be a finite Galois extension of k of degree $[K : k] = n$. Suppose that $\operatorname{rk} A(K) = \operatorname{rk} A(k)$, $A(K)_{\text{tors}} = A(k)_{\text{tors}}$, and $A(k)[n] = 0$. Then $A(K) = A(k)$.*

Proof. Since $A(k)$ is finitely generated and $\operatorname{rk} A(K) = \operatorname{rk} A(k)$, the index $[A(K) : A(k)]$ is finite. Therefore $A(K)$ is the saturation of $A(k)$ in $A(K)$. By Proposition 3.7, it follows that $A(K) = A(k)$. $\square$

Remark 3.10. It is clear that the conditions that $\operatorname{rk} A(K) = \operatorname{rk} A(k)$ and $A(K)_{\text{tors}} = A(k)_{\text{tors}}$ are necessary for the conclusion of Corollary 3.9. We now give an example to show that we cannot, in general, omit the restriction on the n-torsion. Let $k = \mathbb{Q}$, let A be the elliptic curve defined by $y^2 + xy = x^3 - x$ (curve 65A1 in [7]), and let K be the field $\mathbb{Q}(\sqrt{5})$. By [7] we know that $\operatorname{rk} A(\mathbb{Q}) = 1$ and $A(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z}$; in particular, the n-torsion condition fails in this case. A computation in Magma shows that $\operatorname{rk} A(K) = \operatorname{rk} A(\mathbb{Q}) = 1$ and $A(K)_{\text{tors}} = A(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z}$. However, one can quickly verify that $\left(\frac{1+\sqrt{5}}{2}, 1\right)$ is a point on A, hence we have a K-rational point which is not rational.

We now provide a useful consequence of Corollary 3.9.

Proposition 3.11. *Let X be a genus two curve defined over a number field k, and let J be the Jacobian of X. Let K be a degree n Galois extension of k. Suppose that $\operatorname{rk} J(K) = \operatorname{rk} J(k)$, $J(K)_{\text{tors}} = J(k)_{\text{tors}}$, and $J(k)[n] = 0$. Then*

$$X(K) = X(k) \cup W_X(K).$$

In other words, the only additional points gained by X upon base change from k to K are the K-rational Weierstrass points on X.

Proof. The conditions on the Jacobian allow us to apply Corollary 3.9 in order to conclude that $J(K) = J(k)$. Now suppose $X(K)$ is strictly larger than $X(k)$, and let $P \in X(K)\backslash X(k)$. Consider the point $\mathcal{D} := \{P, P\} \in J(K)$. Since $J(K) = J(k)$, we must have $\mathcal{D} \in J(k)$. However, because $P \notin X(k)$, there exists $\sigma \in \mathrm{Gal}(K/k)$ for which $\sigma P \neq P$, so the divisor $P+P$ (and hence $P+P-\infty^+-\infty^-$) is not fixed by the Galois action. By the discussion in §3.2, this implies that the divisor class $[P+P-\infty^+-\infty^-]$ is the trivial class; i.e., $\{P, P\} = 0$. It follows that $P = \iota P$, so P is a Weierstrass point for X. $\square$

Chapter 4

Periodic points for quadratic polynomials

The purpose of this and the following chapter is to prove results that provide evidence for Conjecture 1.13. As was mentioned in §1.3, we will consider a number of graphs G that were not found in the search in [8] (and hence do not appear in Appendix A), and for many such graphs we will show that G cannot be realized as a subgraph of $G(f_c, K)$ for any quadratic pair (K, c).

As we mentioned in Chapter 1, it has been conjectured [12] that, for any quadratic field K and any $c \in K$, f_c cannot admit a K-rational point of period $N > 6$. For each $N \in \{1, 2, 3, 4\}$, it is known that there are infinitely many quadratic pairs (K, c) that admit points of period N. On the other hand, there are no known quadratic pairs admitting points of period 5, and there is only one known pair $(K, c) = \big(\mathbb{Q}(\sqrt{33}), -71/48\big)$ that admits points of period 6. We will henceforth restrict our attention to points of period at most 4.

The results of this chapter fall into two categories:

First, for each $N \in \{1, 2, 3, 4\}$, we bound the number of points of period N a quadratic pair (K, c) may admit. For example, over $\overline{\mathbb{Q}}$, the map f_c generically admits six points of period 3; however, we show that if (K, c) is a quadratic pair, then f_c may admit at most *three* K-rational points of period 3.

Second, for period lengths up to 4, we determine which combinations of period lengths f_c may admit. For example, we will show that, although f_c may simultaneously admit K-rational points of periods 1 and 2, periods 1 and 3, or periods 2 and 3, f_c may *not* simultaneously admit K-rational points of periods 1, 2, and 3.

The chapter is arranged as follows: For each of the different combinations of period lengths that we consider, we will first provide (and prove the correctness of) a model for the corresponding dynamical modular curve $Y_1^{\mathrm{dyn}}(\cdot)$. The genus of each curve was computed in Magma. We then describe the sets of quadratic points on each of these models. Since the model we give for each curve is birational to $Y_1^{\mathrm{dyn}}(\cdot)$ *over* $\mathbb{Q}$, this is equivalent to describing the set of quadratic points on $Y_1^{\mathrm{dyn}}(\cdot)$. Finally, we use this information to describe the set of quadratic pairs (K, c) that simultaneously admit points with periods in the given combination.

To see how one might derive a particular model for $Y_1^{\mathrm{dyn}}(\cdot)$, we recommend the articles [21], [22], and [32]. For the purposes of this thesis, however, we omit the construction of the models and simply prove that they are isomorphic to the appropriate curves $Y_1^{\mathrm{dyn}}(\cdot)$.

In this and the following chapter, K will always be a number field.

4.1 Bounds on the number of points of period N

Let K be a number field. We begin with a result that describes those maps f_c, with $c \in K$, for which f_c admits K-rational points of period 1 or period 2. The following result may be found in [37].

Proposition 4.1. *Let K be a number field, and let $c \in K$.*

(A) The map f_c admits a K-rational point P of period 1 (i.e., a K-rational fixed point) if and only if there exists $r \in K$ such that

$$c = 1/4 - r^2 \ , \quad P = 1/2 + r.$$

In this case, there are exactly two fixed points,

$$P = 1/2 + r \ , \quad P' = 1/2 - r,$$

unless $r = 0$, in which case there is only one. Hence $X_1^{\mathrm{dyn}}(1) \cong X_0^{\mathrm{dyn}}(1) \cong \mathbb{P}^1$.

(B) The map f_c admits a K-rational point P of period 2 if and only if there exists $s \in K$, $s \neq 0$, such that

$$c = -3/4 - s^2 \ , \quad P = -1/2 + s.$$

In this case, there are exactly two points of period 2, and they form a 2-cycle:

$$P = -1/2 + s \ , \quad f_c(P) = -1/2 - s.$$

Hence $X_1^{\mathrm{dyn}}(2) \cong X_0^{\mathrm{dyn}}(2) \cong \mathbb{P}^1$.

Proof. The map f_c admits a K-rational fixed point P if and only if P is a K-rational solution to the equation

$$\Phi_1(x, c) := f_c(x) - x = x^2 - x + c = 0.$$

Thus

$$P = \frac{1}{2} \pm \sqrt{\frac{1}{4} - c},$$

with $1/4 - c = r^2$ for some $r \in K$. The expressions for c, P, and P' now follow immediately.

Now suppose f_c admits a K-rational point P of period 2. Then $f_c^2(P) = P$ but $f_c(P) \neq P$, so P is a root of

$$\Phi_2(x, c) := \frac{f_c^2(x) - x}{f_c(x) - x} = x^2 + x + c + 1 = 0.$$

Thus

$$P = -\frac{1}{2} \pm \sqrt{-\frac{3}{4} - c},$$

with $-3/4 - c = s^2$ for some $s \in K$. The expressions for c, P, and $f_c(P)$ again follow immediately.

Certainly any P which is a root of $\Phi_2(x, c) = 0$ satisfies $f_c^2(P) = P$; however, it may be the case that $f_c(P) = P$, in which case P is actually a fixed point. If we have both $\Phi_2(P, c) = 0$ and $f_c(P) = P$, then

$$\begin{cases} P^2 + c &= -(P + 1); \\ P^2 + c &= P. \end{cases}$$

One can easily verify that this implies[1] $P = -1/2$ and $c = -3/4$, which is precisely the case $s = 0$ excluded from the theorem. □

These two cases are relatively simple since, for a fixed c, each of the polynomials $\Phi_1(x, c)$ and $\Phi_2(x, c)$ has degree two in x. In fact, the existence of one K-rational point of period 1 or period 2 for f_c guarantees (generically) a second such point, and even over the algebraic closure $\overline{K}$ there are only two such points.

The situation becomes somewhat more complicated for larger periods N. The third and fourth dynatomic polynomials are

[1] One could also phrase this in terms of resultants: the resultant of the polynomials $\Phi_1(x, c)$ and $\Phi_2(x, c)$, considered as polynomials over $K[c]$, is $4c + 3$, which is zero precisely when $c = -3/4$. See also [34, §4.2.4], where resultants of this form are called *bifurcation polynomials*.

$$
\begin{aligned}
\Phi_3(x,c) &:= \frac{f_c^3(x)-x}{f_c(x)-x} \\
&= x^6 + x^5 + (3c+1)x^4 + (2c+1)x^3 + (3c^2+3c+1)x^2 \\
&\qquad + (c^2+2c+1)x + c^3 + 2c^2 + c + 1; \\
\Phi_4(x,c) &:= \frac{f_c^4(x)-x}{f_c^2(x)-x} \\
&= x^{12} + 6cx^{10} + x^9 + (15c^2+3c)x^8 + 4cx^7 + (20c^3+12c^2+1)x^6 + (6c^2+2c)x^5 \\
&\qquad + (15c^4+18c^3+3c^2+4c)x^4 + (4c^3+4c^2+1)x^3 \\
&\qquad + (6c^5+12c^4+6c^3+5c^2+c)x^2 + (c^4+2c^3+c^2+2c)x \\
&\qquad + (c^6+3c^5+3c^4+3c^3+2c^2+1).
\end{aligned}
$$

Therefore, for a generic $c \in K$, f_c has six points of period 3 and twelve points of period 4 over $\overline{K}$. However, if K is a number field and $c \in K$, not all of the points of period 3 or period 4 necessarily lie in K. In fact, we will now show that if K is a quadratic field and $c \in K$, then f_c may have at most one 3-cycle (Theorem 4.4) and at most one 4-cycle (Theorem 4.11) defined pointwise over K.

4.1.1 Period 3

We begin by stating a result from [37] that describes, for an arbitrary number field K, those $c \in K$ for which f_c admits a K-rational point of period 3.

Proposition 4.2 ([37, Thm. 3])**.** *Let K be a number field, and let $c \in K$. The map f_c admits a K-rational point a of period 3 if and only if there exists $t \in K$, with $t(t+1)(t^2+t+1) \neq 0$, such that*

$$a = \frac{t^3+2t^2+t+1}{2t(t+1)}, \quad c = -\frac{t^6+2t^5+4t^4+8t^3+9t^2+4t+1}{4t^2(t+1)^2}.$$

In this case, the 3-cycle containing a consists of

$$a = \frac{t^3 + 2t^2 + t + 1}{2t(t+1)}\ , \quad f_c(a) = \frac{t^3 - t - 1}{2t(t+1)}\ , \quad f_c^2(a) = -\frac{t^3 + 2t^2 + 3t + 1}{2t(t+1)}.$$

Hence $X_1^{\mathrm{dyn}}(3) \cong X_0^{\mathrm{dyn}}(3) \cong \mathbb{P}^1$.

Remark 4.3. A quick calculation shows that t may be recovered from a and c by the identity $t = a + f_c(a)$. From this, we can see that the order 3 automorphism on $X_1^{\mathrm{dyn}}(3)$ given by $(x, c) \mapsto (f_c(x), c)$ corresponds to the order 3 automorphism on $\mathbb{P}^1$ that takes $t \mapsto -\frac{t+1}{t} \mapsto -\frac{1}{t+1} \mapsto t$.

The main result of this section is the following theorem, which strengthens Proposition 4.2 in the case that K is a quadratic field.

Theorem 4.4. *Let K be a quadratic field, and let $c \in K$. The map f_c admits a K-rational point a of period 3 if and only if there exists $t \in K$, with $t(t+1)(t^2+t+1) \neq 0$, such that*

$$a = \frac{t^3 + 2t^2 + t + 1}{2t(t+1)}\ , \quad c = -\frac{t^6 + 2t^5 + 4t^4 + 8t^3 + 9t^2 + 4t + 1}{4t^2(t+1)^2}.$$

In this case, there are precisely *three points of period 3, and they form a 3-cycle*

$$a = \frac{t^3 + 2t^2 + t + 1}{2t(t+1)}\ , \quad f_c(a) = \frac{t^3 - t - 1}{2t(t+1)}\ , \quad f_c^2(a) = -\frac{t^3 + 2t^2 + 3t + 1}{2t(t+1)}.$$

To prove Theorem 4.4 we will require the following result of Morton which, in particular, implies Theorem 4.4 with the quadratic field K replaced with $\mathbb{Q}$.

Lemma 4.5 ([21, Thm. 3])**.** *Let $c \in \mathbb{Q}$. The degree six polynomial $\Phi_3(x, c) \in \mathbb{Q}[x]$ cannot have irreducible quadratic factors, nor can it split completely into linear factors, over $\mathbb{Q}$.*

Corollary 4.6. *Let $c \in \mathbb{Q}$. Then f_c admits at most three rational points of period 3 and admits no quadratic points of period 3.*

In order to prove Theorem 4.4, we must show that the curve

$$Y_1^{\mathrm{dyn}}(3^{(2)}) = \{(a,b,c) \in \mathbb{A}^3 : a \text{ and } b \text{ have period 3 and lie in distinct orbits under } f_c\}.$$

has no quadratic points. We first provide a model for $Y_1^{\mathrm{dyn}}(3^{(2)})$.

Proposition 4.7. *Let X be the affine curve of genus 4 defined by the equation*

$$h(t,u) := t(t+1)u^3 + (t^3+2t^2-t-1)u^2 + (t^3-t^2-4t-1)u - t(t+1) = 0, \tag{4.1}$$

and let Y be the open subset of X defined by

$$t(t+1)(t^2+t+1)(u^2+u+1)(u^3+u^2-2u-1) \neq 0. \tag{4.2}$$

Consider the morphism $\Phi : Y \to \mathbb{A}^3$, $(t,u) \mapsto (a,b,c)$, given by

$$a = \frac{t^3+2t^2+t+1}{2t(t+1)} \ , \ b = \frac{u^3+2u^2+u+1}{2u(u+1)} \ , \ c = -\frac{t^6+2t^5+4t^4+8t^3+9t^2+4t+1}{4t^2(t+1)^2}. \tag{4.3}$$

Then Φ maps Y isomorphically onto $Y_1^{\mathrm{dyn}}(3^{(2)})$, with the inverse map given by

$$t = a + f_c(a) \ , \ u = b + f_c(b). \tag{4.4}$$

Remark 4.8. Though the expression for $h(t,u)$ is different, the function field of the curve X is isomorphic to the function field N discussed in [21]. We have introduced a change of variables in order to keep our parameter t consistent with the parameter τ in [37, 32]. Though his result was not phrased in this way, the fact that Φ maps onto $Y_1^{\mathrm{dyn}}(3^{(2)})$ is due to Morton [21, pp. 354–5].

Proof. A simple computation shows that (4.4) provides a left inverse for Φ, which in particular shows that Φ is injective.

Let $(a, b, c) = \Phi(t, u)$ for some $(t, u) \in Y$. We must show that $(a, b, c) \in Y_1^{\text{dyn}}(3^{(2)})$. One can easily verify that $f_c^3(a) = a$, $f_c^3(b) = b$, and

$$a - f_c(a) = \frac{t^2 + t + 1}{t(t+1)} \ , \ b - f_c(b) = \frac{u^2 + u + 1}{u(u+1)}.$$

Since $t^2 + t + 1$ and $u^2 + u + 1$ are nonzero for points (t, u) on Y, a and b cannot be fixed points. Hence a and b have period exactly 3.

We also need for a and b to lie in distinct orbits under f_c. It suffices to show that $b \notin \{a, f_c(a), f_c^2(a)\}$. A simple calculation shows that

$$a - b = \frac{(t + u + 1)\,(tu^2 + (t-2)u - 1)}{u(u+1)} \ , \ f_c(a) - b = \frac{(t^2 - t - 1)\,u - t}{tu} \ ,$$

$$f_c^2(a) - b = -\frac{u^2 + (t+2)u + t}{u+1}.$$

- If $t + u + 1 = 0$, then $u = -(t + 1)$, and thus $0 = h(t, u) = -t^2(t + 1)^2$, contradicting (4.2).

- If $tu^2 + (t - 2)u - 1 = 0$, then $t = \frac{2u+1}{u(u+1)}$, and thus $0 = h(t, u) = \frac{u^3+u^2-2u-1}{u+1}$, again contradicting (4.2).

- If $(t^2 - t - 1)u - t = 0$, then $u = \frac{t}{t^2-t-1}$, and thus $0 = h(t, u) = \frac{t^2(t+1)(t^3+t^2-2t-1)}{(t^2-t-1)^3}$. We cannot have $t(t + 1) = 0$, so we must have $t^3 + t^2 - 2t - 1 = 0$; however, one can verify that if $t^3 + t^2 - 2t - 1 = 0$ and $u = \frac{t}{t^2-t-1}$, then u satisfies $u^3 + u^2 - 2u - 1 = 0$, which contradicts (4.2).

- If $u^2 + (t + 2)u + t = 0$, then $t = -\frac{u(u+2)}{u+1}$, and thus $0 = h(t, u) = -u(u^3 + u^2 - 2u - 1)$, again giving us a contradiction.

Therefore b cannot lie in the orbit of a, from which it now follows that the image of Φ is contained in $Y_1^{\mathrm{dyn}}(3^{(2)})$. It now remains only to show that Φ surjects onto $Y_1^{\mathrm{dyn}}(3^{(2)})$.

Suppose a and b are points of period 3 for f_c such that a and b lie in distinct orbits. Since a is a point of period 3 for f_c, Proposition 4.2 says that there exists t satisfying $t(t+1)(t^2+t+1) \neq 0$ such that

$$a = \frac{t^3+2t^2+t+1}{2t(t+1)}\ ,\ c = -\frac{t^6+2t^5+4t^4+8t^3+9t^2+4t+1}{4t^2(t+1)^2}.$$

Similarly, since b is a point of period 3 for f_c, we have

$$b = \frac{u^3+2u^2+u+1}{2u(u+1)}\ ,\ c = -\frac{u^6+2u^5+4u^4+8u^3+9u^2+4u+1}{4u^2(u+1)^2}$$

for some u with $u(u+1)(u^2+u+1) \neq 0$. Subtracting these two expressions for c and clearing denominators, we find that

$$(t-u)(tu+t+1)(tu+u+1)\cdot h(t,u) = 0.$$

We must therefore show that $(t-u)(tu+t+1)(tu+u+1) \neq 0$. It is clear that $t-u \neq 0$, since $t = u$ implies that $a = b$.

Now suppose $tu+t+1 = 0$. Then we can write $u = -(t+1)/t$. However, substituting $u = -(t+1)/t$ into the above expression for b yields

$$b = \frac{t^3-t-1}{2t(t+1)} = f_c(a),$$

where the second equality follows from Proposition 4.2.

Finally, suppose $tu+u+1 = 0$. Substituting $u = -1/(t+1)$ into the above expression for b yields

$$b = -\frac{t^3 + 2t^2 + 3t + 1}{2t(t+1)} = f_c^2(a),$$

where the second equality again comes from Proposition 4.2. We have therefore shown that if $(t-u)(tu+t+1)(tu+u+1) = 0$, then b lies in the orbit of a under f_c, a contradiction. Thus $h(t,u) = 0$, so (t,u) lies on the curve X. That (4.2) must be satisfied follows from the discussion at the beginning of the proof, so in fact $(t,u) \in Y$ and $(a,b,c) = \Phi(t,u)$. $\square$

We are now left with the task of finding all quadratic points on $Y_1^{\mathrm{dyn}}(3^{(2)})$. The proof of the following theorem will occupy the remainder of this section.

Theorem 4.9. *Let X be the genus 4 affine curve defined by* (4.1)*. Then*

$$X(\mathbb{Q}, 2) = X(\mathbb{Q}) = \{(0,-1), (0,0), (-1,0), (-1,-1)\}.$$

The computation of rational points was done by Morton in [21, pp. 362–364], though with a different model for X (as explained above). We therefore need only show that X has no quadratic points.

Consider the automorphism σ on X defined by

$$\sigma(t) = -\frac{t+1}{t}\ , \quad \sigma(u) = -\frac{1}{u+1}.$$

By the relations in (4.3) and the remark following Proposition 4.2, the automorphism σ corresponds to the automorphism on $X_1^{\mathrm{dyn}}(3^{(2)})$ that takes $(a,b,c) \mapsto (f_c(a), f_c^2(b), c)$. Set

$$\begin{aligned} w &:= t + \sigma(t) + \sigma^2(t); \\ x &:= tu + \sigma(tu) + \sigma^2(tu); \\ y &:= (w+1)(x-2)(x-3). \end{aligned} \tag{4.5}$$

Recalling from Theorem 4.7 that $c = -\dfrac{t^6 + 2t^5 + 4t^4 + 8t^3 + 9t^2 + 4t + 1}{4t^2(t+1)^2}$, a Magma computation verifies that

$$c = \frac{x^3 - 8x^2 + 19x - 13}{4(x-2)(x-3)}. \tag{4.6}$$

Furthermore, one can show that w and x satisfy the equation

$$(w+1)^2 = -\frac{x^3 - x^2 - 16x + 29}{(x-2)(x-3)}.$$

Since $y = (w+1)(x-2)(x-3)$, we let X' denote the genus two curve given by the equation

$$y^2 = -(x-2)(x-3)(x^3 - x^2 - 16x + 29).$$

By construction, we have a map $\psi : X \to X'$ given by $(t,u) \mapsto (x,y)$, and any quadratic point on X must map under ψ to a quadratic or rational point on X'.

To complete our argument, we need to know all of the non-obvious quadratic points on X'.

Lemma 4.10. *Let X' be the genus two curve defined by*

$$y^2 = -(x-2)(x-3)(x^3 - x^2 - 16x + 29).$$

The only non-obvious quadratic points on X' are the points of the form

$$(x_1, \pm(17x_1 - 36))\ ,\quad (x_2, \pm(11x_2 - 34))\ ,\quad (x_3, \pm(3x_3 - 8))\ ,\quad \left(x_4, \pm\frac{1}{27}(x_4 + 2)\right)\ ,$$

where

$$x_1^2 + 5x_1 - 15 = x_2^2 + 3x_2 - 19 = x_3^2 - 5x_3 + 7 = 9x_4^2 - 47x_4 + 59 = 0.$$

Proof. Let J denote the Jacobian of X'. A Magma computation shows that $\operatorname{rk} J(\mathbb{Q}) = 0$, so we may apply Lemma 3.2 to find all non-obvious quadratic points on X'. We find that $J(\mathbb{Q})$ consists of the following twelve points:

$$\begin{array}{cc}
\mathcal{O} & \{(x_1, \pm(17x_1 - 36)), (x_1', \pm(17x_1' - 36))\} \\
\{\infty, (2,0)\} & \{(x_2, \pm(11x_2 - 34)), (x_2', \pm(11x_2' - 34))\} \\
\{\infty, (3,0)\} & \{(x_3, \pm(3x_3 - 8)), (x_3', \pm(3x_3' - 8))\} \\
\{(2,0), (3,0)\} & \left\{\left(x_4, \pm\tfrac{1}{27}(x_4 + 2)\right), \left(x_4', \pm\tfrac{1}{27}(x_4' + 2)\right)\right\}
\end{array}$$

where

$$x_1^2 + 5x_1 - 15 = x_2^2 + 3x_2 - 19 = x_3^2 - 5x_3 + 7 = 9x_4^2 - 47x_4 + 59 = 0,$$

and x_i' denotes the Galois conjugate of x_i over $\mathbb{Q}$. Applying Lemma 3.2 yields the result. $\square$

We may now complete the proof of Theorem 4.9. First, a computation in Magma verifies that the preimages under ψ of the non-obvious quadratic points listed in Lemma 4.10 have degree strictly greater than two. Therefore a quadratic point on X must map to an obvious quadratic point or a rational point on X'. In either case, we have $x \in \mathbb{Q}$, which implies $c \in \mathbb{Q}$ by (4.6).

Let K be the quadratic extension of $\mathbb{Q}$ generated by t and u. The points a and b of period 3 for f_c — defined as in the statement of Proposition 4.7 — must also lie in K. By Corollary 4.6, f_c cannot admit quadratic points of period 3, so $a, b \in \mathbb{Q}$. Finally, since $a, b, c \in \mathbb{Q}$, we have

$$t = a + f_c(a) \in \mathbb{Q}\,, \quad u = b + f_c(b) \in \mathbb{Q},$$

a contradiction. Therefore X has no quadratic points, completing the proof of Theorem 4.9 and, consequently, Theorem 4.4.

4.1.2 Period 4

In this section, we prove the following analogue of Theorem 4.4 for points of period four:

Theorem 4.11. *Let K be a quadratic field, and let $c \in K$. The map f_c admits a K-rational point a of period 4 if and only if there exist $u, v \in K$, with $v^2 = -u(u^2+1)(u^2-2u-1)$ and $v(u+1)(u-1) \neq 0$, such that*

$$a = \frac{u-1}{2(u+1)} + \frac{v}{2u(u-1)}\ , \quad c = \frac{(u^2-4u-1)(u^4+u^3+2u^2-u+1)}{4u(u+1)^2(u-1)^2}.$$

In this case, there are precisely *four points of period 4, and they form a 4-cycle*

$$\begin{aligned} a &= \frac{u-1}{2(u+1)} + \frac{v}{2u(u-1)} \\ f_c(a) &= -\frac{u+1}{2(u-1)} + \frac{v}{2u(u+1)} \\ f_c^2(a) &= \frac{u-1}{2(u+1)} - \frac{v}{2u(u-1)} \\ f_c^3(a) &= -\frac{u+1}{2(u-1)} - \frac{v}{2u(u+1)}. \end{aligned}$$

Moreover, we must have $c \in \mathbb{Q}$, and each K-rational point P of period 4 is the $\mathrm{Gal}(K/\mathbb{Q})$*-conjugate of $f_c^2(P)$.*

Before we prove Theorem 4.11, we first establish a model for

$$Y_1^{\mathrm{dyn}}(4) = \{(a,c) \in \mathbb{A}^2 : a \text{ is a point of period 4 for } f_c\}.$$

Though we include slightly more details, the following proposition is due to Morton [22].

Proposition 4.12. *Let X be the affine curve of genus 2 defined by the equation*

$$v^2 = -u(u^2+1)(u^2-2u-1), \tag{4.7}$$

and let Y be the open subset of X defined by

$$v(u-1)(u+1) \neq 0. \tag{4.8}$$

Consider the morphism $\Phi : Y \to \mathbb{A}^2$, $(u,v) \mapsto (a,c)$, given by

$$a = \frac{u-1}{2(u+1)} + \frac{v}{2u(u-1)}\ , \quad c = \frac{(u^2-4u-1)(u^4+u^3+2u^2-u+1)}{4u(u+1)^2(u-1)^2}.$$

Then Φ maps Y isomorphically onto $Y_1^{\mathrm{dyn}}(4)$, with the inverse map given by

$$u = -\frac{f_c^2(a)+a+1}{f_c^2(a)+a-1}\ , \quad v = \frac{u(u-1)(2au+2a-u+1)}{u+1}. \tag{4.9}$$

Proof. First, observe that the condition $v \neq 0$ implies $u \neq 0$, so Φ is well-defined. Also, it is easy to verify that (4.9) provides a left inverse for Φ, so Φ is injective.

Suppose that $(u,v) \in Y$ and $(a,b,c) = \Phi(u,v)$. A simple calculation verifies that $f_c^4(a) = a$ and

$$a - f_c^2(a) = \frac{v}{u(u-1)},$$

which is nonzero by (4.8), so a must have exact period 4. Thus Φ maps Y into $Y_1^{\mathrm{dyn}}(4)$.

It remains to show that Φ maps Y onto $Y_1^{\mathrm{dyn}}(4)$. Let a be a point of period 4 for f_c, and define u and v as in (4.9). It suffices to show that $(f_c^2(a)+a-1)(u+1) \neq 0$ (so that u and v are well-defined) and that $v(u-1)(u+1) \neq 0$ (so that (u,v) lies on Y), since one can verify explicitly that if u and v are defined as in (4.9), then $v^2 = -u(u^2+1)(u^2-2u-1)$ and $(a,c) = \Phi(u,v)$.

We first show that $f_c^2(a)+a-1$ cannot be zero. Suppose to the contrary that $f_c^2(a)+a-1 = 0$, so $f_c^2(a) = -(a-1)$. Since a is a point of period 4, we also have $a = f_c^4(a) = f_c^2(-(a-1))$.

Hence

$$f_c^2(a) + a - 1 = 0 = f_c^2(-(a-1)) - a. \tag{4.10}$$

Subtracting the right side of (4.10) from the left gives us the new equation

$$2(2a-1)(c + a^2 - a + 1) = 0.$$

Hence either $a = 1/2$ or $c = -(a^2 - a + 1)$. If $a = 1/2$, then (4.10) implies that $c \in \{1/4, -7/4\}$. However, 1/2 is a fixed point for $f_{1/4}$ and is a point of period 2 for $f_{-7/4}$, so we cannot have $a = 1/2$. Similarly, if $c = -(a^2 - a + 1)$, then one may easily verify that a is a point of type 2_1 for f_c, not a point of period 4.

It is somewhat more apparent that $u+1 \neq 0$: if $u = -1$, then $f_c^2(a)+a+1 = f_c^2(a)+a-1$ by (4.9), which is impossible.

Now suppose $v = 0$. By (4.9), this means that $u(u-1)(2au + 2a - u + 1) = 0$.

- If $u = 0$, then $f_c^2(a) = -(a+1)$. An argument similar to the one showing that $f_c^2(a) + a - 1 \neq 0$ shows that this cannot happen.

- If $u = 1$, then $-(f_c^2(a) + a + 1) = f_c^2(a) + a - 1$, so that $f_c^2(a) = -a$. However, since a and $-a$ map to the same point under f_c, $-a$ cannot lie in the orbit of a.

- Finally, if $2au + 2a - u + 1 = 0$, then $u = -\frac{2a+1}{2a-1}$, which we rewrite using (4.9) as

$$\frac{f_c^2(a) + a + 1}{f_c^2(a) + a - 1} = \frac{2a+1}{2a-1}.$$

  Subtracting one side from the other and clearing denominators yields the equation

$$(a^2 - a + c)(a^2 + a + c + 1) = 0.$$

Observe that the left hand side of this equation is precisely equal to $\Phi_1(a,c) \cdot \Phi_2(a,c)$, which means that a is a point of period 1 or 2, not 4 as we had assumed.

In each case, we reached a contradiction to the fact that a is a point of period 4 for f_c, so v must be nonzero. We have already ruled out the possibility that $u = \pm 1$, so we have shown that Φ is surjective, therefore completing the proof. $\square$

Remark 4.13. As noted in [22, Prop. 3], the dynamical modular curve $X_1^{\mathrm{dyn}}(4)$ is birational to the classical modular curve $X_1(16)$, since (4.7) is the model for $X_1(16)$ appearing in [38].

We now give a proof of the fact that $X_0^{\mathrm{dyn}}(4)$ is isomorphic to $\mathbb{P}^1$. This is already well known (see [34, p. 164]), but certain details of the proof will be useful later. The proof we give is taken from [22, pp. 91–93].

Proposition 4.14. *The curve $X_0^{\mathrm{dyn}}(4)$ is isomorphic to $\mathbb{P}^1$, and $Y_0^{\mathrm{dyn}}(4)$ is isomorphic to $\mathbb{P}^1$ with five points — three defined over $\mathbb{Q}$ and two defined over $\mathbb{Q}(\sqrt{-1})$ — removed.*

Proof. Let σ be the automorphism of $X_1^{\mathrm{dyn}}(4)$ given by

$$\sigma(a,c) = (f_c(a), c).$$

Morton [22] shows directly that the quotient of $X_1^{\mathrm{dyn}}(4)$ by σ is given by

$$\begin{aligned} \pi : X_1^{\mathrm{dyn}}(4) &\to \mathbb{P}^1 \\ (a,c) &\mapsto z := a + f_c(a) + f_c^2(a) + f_c^3(a) \end{aligned}$$

so $X_0^{\mathrm{dyn}}(4) \cong \mathbb{P}^1$.

Now suppose $z \in \mathbb{P}^1 \setminus \{\infty, 0, -2, \pm 2\sqrt{-1}\}$. Morton shows that one may recover the map f_c and the 4-cycle $\{a, f_c(a), f_c^2(a), f_c^3(a)\}$ via

$$c = -\frac{z^3+3z+4}{4z} \tag{4.11}$$
$$\{a, f_c(a), f_c^2(a), f_c^3(a)\} = \{X : p(X,z) = 0\},$$

where p is the polynomial

$$p(X,z) := X^4 - zX^3 - \frac{z^2+3z+4}{2z}X^2 + \frac{z^3+2z^2+5z+8}{4}X \\ - \frac{z^6+2z^5+4z^4+6z^3-5z^2-8z-16}{16z^2}.$$

The condition $z \notin \{\infty, 0\}$ is to ensure that c is finite, and the condition $z \notin \{-2, \pm 2\sqrt{-1}\}$ is to make sure the discriminant

$$\operatorname{disc}_X p(X,z) = \frac{(z+2)^2(z^2+4)^3}{z^4} \tag{4.12}$$

is nonzero, so that the roots of $p(X,z)$ are distinct (thus making up a full 4-cycle rather than a 1- or 2-cycle). $\square$

Since $X_1^{\mathrm{dyn}}(4)$ is a hyperelliptic curve, it has infinitely many quadratic points. The only rational points on the curve X defined by (4.7) are the point at infinity and the five points $(0,0)$, $(\pm 1, \pm 2)$ (see [38, p.774], for example). Therefore, for any $u \in \mathbb{Q} \setminus \{0, \pm 1\}$, the number $v = \sqrt{-u(u^2+1)(u^2-2u-1)}$ generates a quadratic extension $K/\mathbb{Q}$. For such u and v, (u,v) is a quadratic point on X; since $u \notin \{0, \pm 1\}$, defining c as in Proposition 4.12 yields a quadratic pair (K,c) that admits a point of period 4. For a fixed c, there are at most six values of u that could yield c in this way, so this method yields infinitely many quadratic pairs (K,c) that admit points of period 4.

In the previous paragraph, u was chosen to be rational. By Proposition 4.12, this means that actually $c \in \mathbb{Q}$. One may then ask whether we can find a quadratic field K and an element $c \in K \backslash \mathbb{Q}$ for which f_c admits K-rational points of period 4. We answer this question in the negative.

Proposition 4.15. *Let K be a quadratic field, and let $c \in K$ be such that f_c admits a K-rational point a of period four. Then $c \in \mathbb{Q}$. Moreover, a and $f_c^2(a)$ are Galois conjugates.*

Proof. Consider the Jacobian J of the curve X defined in (4.7). Since $X_1^{\mathrm{dyn}}(4) \cong X_1(16)$, it is known that $\operatorname{rk} J(\mathbb{Q}) = 0$ and $\#J(\mathbb{Q}) = \#J(\mathbb{Q})_{\mathrm{tors}} = 20$. (A proof is given in [1, Lem. 14].) Therefore the following twenty points are all of the points on $J(\mathbb{Q})$:

$$\begin{array}{cccc}
\mathcal{O} & \{\infty, (-1,-2)\} & \{(1,2),(1,2)\} & \{(1,-2),(-1,-2)\} \\
\{\infty, (0,0)\} & \{(0,0),(1,2)\} & \{(1,2),(-1,2)\} & \{(-1,2),(-1,2)\} \\
\{\infty, (1,2)\} & \{(0,0),(1,-2)\} & \{(1,2),(-1,-2)\} & \{(-1,-2),(-1,-2)\} \\
\{\infty, (1,-2)\} & \{(0,0),(-1,2)\} & \{(1,-2),(1,-2)\} & \{(\sqrt{-1},0),(-\sqrt{-1},0)\} \\
\{\infty, (-1,2)\} & \{(0,0),(-1,-2)\} & \{(1,-2),(-1,2)\} & \{(1+\sqrt{2},0),(1-\sqrt{2},0)\}
\end{array}$$

Applying Lemma 3.2, we find that if (u, v) is a non-obvious quadratic point on X, then $(u^2 + 1)(u^2 - 2u - 1) = 0$. However, as this implies $v = 0$, such points are not allowed by Proposition 4.12. For every other quadratic point on X, we have $u \in \mathbb{Q}$. Writing

$$a = \frac{u-1}{2(u+1)} + \frac{v}{2u(u-1)}\ , \quad c = \frac{(u^2-4u-1)(u^4+u^3+2u^2-u+1)}{4u(u+1)^2(u-1)^2}$$

shows that $c \in \mathbb{Q}$, proving the first claim in the proposition.

To prove the second claim, we observe that a simple computation shows that

$$f_c^2(a) = \frac{u-1}{2(u+1)} - \frac{v}{2u(u-1)}.$$

Since $u \in \mathbb{Q}$, $v \notin \mathbb{Q}$, and $v^2 \in \mathbb{Q}$, it follows that $f_c^2(a)$ is the Galois conjugate of a. $\square$

We are now ready to prove the following result which, together with Propositions 4.12 and 4.15, completes the proof of Theorem 4.11.

Proposition 4.16. *Let K be a quadratic field, and let $c \in K$. The map f_c admits* at most *four K-rational points of period 4.*

Proof. In his dissertation [31], Panraksa proves that for a given $c \in \mathbb{Q}$, the degree 12 polynomial $\Phi_4(x, c) \in \mathbb{Q}[x]$ cannot have four distinct quadratic factors over $\mathbb{Q}$. Since each point of period 4 for f_c is a root of $\Phi_4(x, c)$, and since having more than four points of period 4 implies at least eight such points, it follows that f_c cannot admit more than four quadratic points of period 4 whenever $c \in \mathbb{Q}$. Combining this with Proposition 4.15 yields the result. □

We can actually prove a somewhat stronger statement than Proposition 4.16. We know from Proposition 4.15 that if (K, c) is a quadratic pair that admits a K-rational point a of period 4, then $c \in \mathbb{Q}$ and a and $f_c^2(a)$ (resp. $f_c(a)$ and $f_c^3(a)$) are Galois conjugates. This means that the 4-cycle containing a is a *rational* cycle. We could therefore prove Proposition 4.16 by showing that, if $c \in \mathbb{Q}$, then f_c may admit at most one rational 4-cycle. In order to do so, we will need to find all rational points on $X_0^{\mathrm{dyn}}(4^{(2)})$.

By the proof of Proposition 4.14, the data of a map f_c together with a marked 4-cycle $\{a, f_c(a), f_c^2(a), f_c^3(a)\}$ is equivalent to the data of the trace $z := a + f_c(a) + f_c^2(a) + f_c^3(a)$. Thus we may write

$$Y_0^{\mathrm{dyn}}(4^{(2)}) = \{(z_1, z_2) \in \mathbb{A}^2 : \ z_1 \text{ and } z_2 \text{ are distinct traces of 4-cycles under the map } f_c\ \}.$$

In fact, since f_c has only three 4-cycles over $\overline{\mathbb{Q}}$, the third trace z_3 is determined by (z_1, z_2), so $X_0^{\mathrm{dyn}}(4^{(2)}) \cong X_0^{\mathrm{dyn}}(4^{(3)})$. We now give a model for $X_0^{\mathrm{dyn}}(4^{(2)})$.

Proposition 4.17. *Let X be the affine curve of genus 1 defined by the equation*

$$y^2 = x^3 + 4, \tag{4.13}$$

and let Y be the open subset of X defined by

$$x(x+2)(x^2+4)(y-6)\Big(y+2(x-1)\Big)\Big(4y-(x^3+4x^2+8)\Big) \neq 0 \tag{4.14}$$

Consider the morphism $\Phi : Y \to \mathbb{A}^2$, $(x, y) \mapsto (z_1, z_2)$, given by

$$z_1 = \frac{y-2}{x}\ , \quad z_2 = \frac{4}{x}. \tag{4.15}$$

Then Φ maps Y isomorphically onto $Y_0^{\mathrm{dyn}}(4^{(2)})$, with the inverse map given by

$$x = \frac{4}{z_2}\ , \quad y = \frac{4z_1 + 2z_2}{z_2}. \tag{4.16}$$

Remark 4.18. The equation (4.13) is an affine model for elliptic curve 108A1 in [7].

Proof. The identifications (4.16) give a left inverse for Φ, so Φ is injective.

To show that Φ maps Y into $Y_0^{\mathrm{dyn}}(4^{(2)})$, let $(x, y) \in Y$ and let $(z_1, z_2) = \Phi(x, y)$. The condition $x \neq 0$ implies that each z_i is finite, and the condition $y \neq 6$ implies that $z_1 \neq z_2$. The values of z_i which are not allowed by the proof of Proposition 4.14 are those satisfying $z_i(z_i + 2)(z_i^2 + 4)$; one can verify that $z_1 z_2$ can never be zero and that

$$(z_1+2)\cdot(z_1^2+4)\cdot(z_2+2)\cdot(z_2^2+4) = \frac{y+2(x-1)}{x} \cdot \frac{-(4y-(x^3+4x^2+8))}{x^2} \cdot \frac{2(x+2)}{x} \cdot \frac{4(x^2+4)}{x^2},$$

which is nonzero by (4.14). Finally, we must verify that the expression for c given in (4.11) is the same for $z = z_1$ and $z = z_2$. Indeed,

$$\frac{z_1^3 + 3z_1 + 4}{4z_1} - \frac{z_2^3 + 3z_2 + 4}{4z_2} = \frac{(y-6)\big(y^2 - (x^3+4)\big)}{4x^2(y-2)} = 0,$$

which shows that $(z_1, z_2) \in Y_0^{\mathrm{dyn}}(4^{(2)})$.

Finally, we must show that Φ is surjective onto $Y_0^{\mathrm{dyn}}(4^{(2)})$, so suppose that z_1 and z_2 are distinct traces of 4-cycles for a map f_c. Let z_3 be the remaining 4-cycle trace for f_c. From (4.11), we see that each z_i is a root of the equation

$$Z^3 + (4c+3)Z + 4 = 0.$$

Hence

$$z_1 + z_2 + z_3 = 0\ , \quad z_1 z_2 z_3 = -4,$$

and therefore

$$-(z_1 + z_2) = z_3 = -\frac{4}{z_1 z_2}.$$

This shows that z_3 is determined by z_1 and z_2, and it also shows that

$$z_1 z_2 (z_1 + z_2) = 4.$$

This equation defines a curve C birational to X via the identifications (4.15) and (4.16). $\square$

Corollary 4.19. *If $c \in \mathbb{Q}$, then f_c may admit at most one rational 4-cycle.*

Proof. According to [7], the curve 108A1 has only three rational points, so the three points ∞, $(0, \pm 2)$ on the curve X defined by (4.13) are the only rational points on X. However, each of these points falls outside the open subset $Y \subset X$ defined in Proposition 4.17, and therefore $Y_1^{\mathrm{dyn}}(4^{(2)})$ has no rational points. $\square$

4.2 Combinations of cycles of different lengths

In this section, we consider whether a quadratic pair (K, c) may admit cycles of different lengths. We begin by studying the curves

$$Y_1^{\text{dyn}}(N_1, N_2) := \{(a, b, c) \in \mathbb{A}^3 : \ a \text{ has period } N_1 \text{ and } b \text{ has period } N_2 \text{ for } f_c\}$$

for the pairs $(N_1, N_2) \in \{(1, 2), (1, 3), (2, 3)\}$.

Poonen [32, Thm. 2] gives a rational parametrization for those $c \in \mathbb{Q}$ for which f_c admits rational points of periods 1 and 2, and he also shows that f_c cannot simultaneously admit rational points of periods 1 and 3 or periods 2 and 3 whenever $c \in \mathbb{Q}$. The dynamical modular curve $X_1^{\text{dyn}}(1, 2)$ is isomorphic to $\mathbb{P}^1$, and the curves $X_1^{\text{dyn}}(1, 3)$ and $X_1^{\text{dyn}}(2, 3)$ have genus two and are therefore hyperelliptic. It follows, then, that all three of these curves will have points over infinitely many quadratic fields. The following result is due to Poonen ([32, Thm. 2]); see also [8].

Proposition 4.20.

(A) Let K be a number field, and let $c \in K$. The map f_c has K-rational points of period 1 and period 2 if and only if

$$c = -\frac{3q^4 + 10q^2 + 3}{4(q^2 - 1)^2}$$

for some $q \in K$, $q \notin \{0, \pm 1\}$. In this case, the parameters r and s of Proposition 4.1 are

$$r = -\frac{q^2 + 1}{q^2 - 1}\ , \ \ s = \frac{2q}{q^2 - 1}.$$

Hence $X_1^{\text{dyn}}(1, 2) \cong X_0^{\text{dyn}}(1, 2) \cong \mathbb{P}^1$.

(B) Let X be the affine curve of genus 2 defined by the equation

$$y^2 = t^6 + 2t^5 + 5t^4 + 10t^3 + 10t^2 + 4t + 1, \tag{4.17}$$

and let Y be the open subset of X defined by

$$t(t+1)(t^2+t+1) \neq 0. \tag{4.18}$$

Consider the morphism $\Phi : Y \to \mathbb{A}^3$, $(t,y) \mapsto (a,b,c)$, given by

$$a = \frac{t^2+t+y}{2t(t+1)}\ , \quad b = \frac{t^3+2t^2+t+1}{2t(t+1)}\ , \quad c = -\frac{t^6+2t^5+4t^4+8t^3+9t^2+4t+1}{4t^2(t+1)^2}.$$

Then Φ maps Y isomorphically onto $Y_1^{\mathrm{dyn}}(1,3)$, with the inverse map given by

$$t = b + f_c(b)\ , \quad y = (2a-1)t(t+1).$$

(C) Let X' be the affine curve of genus 2 defined by the equation

$$z^2 = t^6+2t^5+t^4+2t^3+6t^2+4t+1, \tag{4.19}$$

and let Y' be the open subset of X' defined by

$$t(t+1)(t^2+t+1)z \neq 0. \tag{4.20}$$

Consider the morphism $\Phi' : Y' \to \mathbb{A}^3$, $(t,z) \mapsto (a,b,c)$, given by

$$a = -\frac{t^2+t-z}{2t(t+1)}\ , \quad b = \frac{t^3+2t^2+t+1}{2t(t+1)}\ , \quad c = -\frac{t^6+2t^5+4t^4+8t^3+9t^2+4t+1}{4t^2(t+1)^2}.$$

Then Φ' maps Y' isomorphically onto $Y_1^{\mathrm{dyn}}(2,3)$, with the inverse map given by

$$t = b + f_c(b)\ , \quad z = (2a+1)t(t+1).$$

In particular, for each pair $(N_1, N_2) \in \{(1,2),(1,3),(2,3)\}$, there exist infinitely many quadratic pairs (K, c) that admit points of periods N_1 and N_2.

Proof. Statement (A) is part (1) of [32, Thm. 2]. That Φ and Φ' map onto $Y_1^{\mathrm{dyn}}(1,3)$ and $Y_1^{\mathrm{dyn}}(2,3)$, respectively, follows from the proof of part (2) of the same theorem. A simple computation shows that the proposed inverse maps for parts (B) and (C) are indeed *left* inverse maps; in particular, this means that Φ and Φ' are injective. All that remains to be shown is that the images of Φ and Φ' lie in $Y_1^{\mathrm{dyn}}(1,3)$ and $Y_1^{\mathrm{dyn}}(2,3)$, respectively.

Let $(a, b, c) = \Phi(t, y)$. One can easily show that $f_c(a) = a$, so that a is a fixed point for f_c. Also, we have $f_c^3(b) = b$ and

$$b - f_c(b) = \frac{t^2+t+1}{t(t+1)};$$

since $t^2 + t + 1 \neq 0$, b is a point of exact period 3 for f_c. Therefore $(a, b, c) \in Y_1^{\mathrm{dyn}}(1,3)$.

Now let $(a, b, c) = \Phi'(t, z)$. One can check that $f_c^2(a) = a$, $f_c^3(b) = b$, and

$$a - f_c(a) = \frac{z}{t(t+1)}\ , \quad b - f_c(b) = \frac{t^2+t+1}{t(t+1)},$$

which are nonzero by assumption. Therefore a and b have exact periods 2 and 3 for f_c, respectively, which means that $(a, b, c) \in Y_1^{\mathrm{dyn}}(2,3)$.

The final assertion of the proposition follows from the fact that, for each pair $(N_1, N_2) \in \{(1,2),(1,3),(2,3)\}$, the curve $X_1^{\mathrm{dyn}}(N_1, N_2)$ has genus at most two and therefore has infinitely many quadratic points. $\square$

Remark 4.21. As explained in [32], the curves $X_1^{\mathrm{dyn}}(1,3)$ and $X_1^{\mathrm{dyn}}(2,3)$ are birational to $X_1(18)$ and $X_1(13)$, respectively.

We now describe the quadratic points on the curves appearing in parts (B) and (C) of Theorem 4.20.

Lemma 4.22 ([15, 8]). *Let X and X' be the curves defined by* (4.17) *and* (4.19)*, respectively.*

(A) The set of rational points on X is

$$\{(-1, \pm 1), (0, \pm 1)\},$$

and the set of quadratic points on X consists of the obvious quadratic points

$$\left\{\left(t, \pm\sqrt{t^6 + 2t^5 + 5t^4 + 10t^3 + 10t^2 + 4t + 1}\right) : t \in \mathbb{Q}, t(t+1) \neq 0\right\}$$

and the non-obvious quadratic points

$$\left\{(t, \pm(t-1)) : t^2 + t + 1 = 0\right\}.$$

(B) The set of rational points on X' is

$$\{(-1, \pm 1), (0, \pm 1)\},$$

and the set of quadratic points on X' consists only of the obvious quadratic points

$$\left\{\left(t, \pm\sqrt{t^6 + 2t^5 + t^4 + 2t^3 + 6t^2 + 4t + 1}\right) : t \in \mathbb{Q}, t(t+1) \neq 0\right\}.$$

Proof. First, it is easy to verify that all of the points listed are points on X and X' of the appropriate types.

Let J be the Jacobian of X. A computation in Magma shows that $\operatorname{rk} J(\mathbb{Q}) = 0$, and $J(\mathbb{Q})$ consists of the identity $\mathcal{O}$ and the following 20 nontrivial torsion points:

$$
\begin{array}{cccc}
\{\infty^+,\infty^+\} & \{\infty^-,(0,1)\} & \{(0,1),(0,1)\} & \{(0,-1),(0,-1)\} \\
\{\infty^-,\infty^-\} & \{\infty^-,(0,-1)\} & \{(0,1),(-1,-1)\} & \{(0,1),(-1,1)\} \\
\{\infty^+,(-1,1)\} & \{\infty^+,(0,1)\} & \{(0,-1),(-1,1)\} & \{(-1,-1),(-1,-1)\} \\
\{\infty^+,(-1,-1)\} & \{\infty^+,(0,-1)\} & \{(0,-1),(-1,-1)\} & \{(\zeta,\zeta-1),(\zeta^2,\zeta^2-1)\} \\
\{\infty^-,(-1,1)\} & \{\infty^-,(-1,-1)\} & \{(-1,1),(-1,1)\} & \{(\zeta,-\zeta+1),(\zeta^2,-\zeta^2+1)\}
\end{array}
$$

Here ζ is a primitive cube root of unity. Applying Lemma 3.2 shows that the only rational points and non-obvious quadratic points are those listed in part (A).

Similarly, if we let J' denote the Jacobian of X', then a Magma computation shows that $\operatorname{rk} J'(\mathbb{Q}) = 0$, and $J(\mathbb{Q})$ consists of the identity $\mathcal{O}$ and the following 18 nontrivial torsion points:

$$
\begin{array}{cccc}
\{\infty^+,\infty^+\} & \{\infty^-,(0,1)\} & \{\infty^-,(-1,1)\} & \{(0,1),(0,1)\} \\
\{\infty^-,\infty^-\} & \{\infty^-,(0,-1)\} & \{\infty^-,(-1,-1)\} & \{(0,-1),(0,-1)\} \\
\{\infty^+,(0,1)\} & \{\infty^+,(-1,1)\} & \{(0,1),(-1,1)\} & \{(0,1),(-1,-1)\} \\
\{\infty^+,(0,-1)\} & \{\infty^+,(-1,-1)\} & \{(0,-1),(-1,-1)\} & \{(0,-1),(-1,1)\} \\
 & \{(-1,1),(-1,1)\} & \{(-1,-1),(-1,-1)\} &
\end{array}
$$

Again, we apply Lemma 3.2 to determine all rational points and non-obvious quadratic points on X'. In this case, we get precisely the rational points listed in part (B), and we find that there are no non-obvious quadratic points. □

Proposition 4.23. *Let K be a quadratic field, and let $c \in K$. If f_c has K-rational points of periods 1 and 3 (resp. 2 and 3), then $c \in \mathbb{Q}$. In this case, the points of period 3 are rational, and the points of period 1 (resp. 2) are quadratic.*

Proof. We first suppose that f_c admits K-rational points of period 1 and 3. By Theorem 4.20(B), there exists a K-rational point on the curve X defined by (4.17), with $t(t+1)(t^2+t+1) \neq 0$, for which

$$c = -\frac{t^6 + 2t^5 + 4t^4 + 8t^3 + 9t^2 + 4t + 1}{4t^2(t+1)^2}.$$

By Lemma 4.22, the only quadratic points on X satisfying $t^2 + t + 1 \neq 0$ are those with $t \in \mathbb{Q}$. It follows from Proposition 4.20 that c and the points of period 3 are also in $\mathbb{Q}$ but, since $y \notin \mathbb{Q}$, the fixed points for f_c are not in $\mathbb{Q}$.

A similar argument applies to the case that f_c admits K-rational points of period 2 and 3, using the fact that the only quadratic points on the curve X' defined by (4.19) have $t \in \mathbb{Q}$ by Lemma 4.22. □

For each of the three different cycle combinations described in Theorem 4.20, we can find infinitely many quadratic pairs (K, c) for which $\mathrm{PrePer}(f_c, K)$ contains the given combination. For the remainder of this section we show that for every other possible combination of cycles of length at most 4, there are at most finitely many quadratic pairs (K, c) for which $\mathrm{PrePer}(f_c, K)$ contains the given combination.

4.2.1 Periods 1 and 4

We are now interested in determining whether a quadratic pair (K, c) may simultaneously admit K-rational points of periods 1 and 4. Unfortunately, we have thus far been unable to fully answer this question. The issue is that the general Chabauty-type bound we obtain for the number of rational points on a certain hyperelliptic curve does not appear to be sharp. We are able, however, to make the following statement.

Theorem 4.24. *There is at most one quadratic pair (K, c) for which f_c simultaneously admits K-rational points of period 1 and period 4. Moreover, if such a quadratic pair exists, then $c \in \mathbb{Q}$ and the points of period 1 and 4, respectively, lie in $K \setminus \mathbb{Q}$.*

As usual, we first give a more computationally useful model for the dynamical modular curve $Y_1^{\mathrm{dyn}}(1,4)$, defined by

$$Y_1^{\mathrm{dyn}}(1,4) = \{(a,b,c) \in \mathbb{A}^3 : \ a \text{ is a fixed point and } b \text{ is a point of period 4 for } f_c\}.$$

Proposition 4.25. *Let X be the affine curve of genus 9 defined by the equation*

$$\begin{cases} v^2 &= -u(u^2+1)(u^2-2u-1) \\ w^2 &= -u(u^6-4u^5-3u^4-8u^3+3u^2-4u-1), \end{cases} \tag{4.21}$$

and let Y be the open subset of X defined by

$$(u-1)(u+1)v \neq 0. \tag{4.22}$$

Consider the morphism $\Phi : Y \to \mathbb{A}^3$, $(u,v,w) \mapsto (a,b,c)$, given by

$$a = \frac{1}{2} + \frac{w}{2u(u-1)(u+1)} \ , \quad b = \frac{u-1}{2(u+1)} + \frac{v}{2u(u-1)} \ ,$$

$$c = \frac{(u^2-4u-1)(u^4+u^3+2u^2-u+1)}{4u(u+1)^2(u-1)^2}.$$

Then Φ maps Y isomorphically onto $Y_1^{\mathrm{dyn}}(1,4)$, with the inverse map given by

$$u = -\frac{b+f_c^2(b)+1}{b+f_c^2(b)-1} \ , \quad v = \frac{u(u-1)(2bu+2b-u+1)}{u+1} \ , \quad w = u(u-1)(u+1)(2a-1). \tag{4.23}$$

Proof. Since $v \neq 0$ implies $u \neq 0$, the condition $(u-1)(u+1)v \neq 0$ implies that Φ is well-defined. It is easy to see that $f_c(a) = a$, so a is a fixed point for f_c, and that $f_c^4(b) = b$. Moreover,

$$b - f_c^2(b) = \frac{v}{u(u-1)},$$

which is nonzero by hypothesis, so b is a point of exact period 4 for f_c. Therefore Φ maps Y into $Y_1^{\mathrm{dyn}}(1,4)$. A quick computation shows that (4.23) serves as a left inverse for Φ, so Φ is injective.

It now remains to show that Φ is surjective. Suppose $(a,b,c) \in Y_1^{\mathrm{dyn}}(1,4)$. By Proposition 4.12, there exist $u, v \in K$ such that

$$v^2 = -u(u^2+1)(u^2-2u-1)\ , \quad b = \frac{u-1}{2(u+1)} + \frac{v}{2u(u-1)}\ ,$$

$$c = \frac{(u^2-4u-1)(u^4+u^3+2u^2-u+1)}{4u(u+1)^2(u-1)^2}.$$

Also, by Proposition 4.1, there exists $r \in K$ such that

$$a = 1/2 + r\ , \quad c = 1/4 - r^2.$$

Equating these two expressions for c yields the equation

$$r^2 = -\frac{u^6 - 4u^5 - 3u^4 - 8u^3 + 3u^2 - 4u - 1}{4u(u-1)^2(u+1)^2}.$$

Setting $w := 2u(u-1)(u+1)r$ yields the second equation in (4.21) and allows us to rewrite a as

$$a = \frac{1}{2} + \frac{w}{2u(u-1)(u+1)},$$

completing the proof. $\square$

We now offer a bound on the number of quadratic points on the affine curve X birational to $X_1^{\mathrm{dyn}}(1,4)$.

Theorem 4.26. *Let X be the genus 9 affine curve defined by* (4.21)*. Then*

$$X(\mathbb{Q},2) = X(\mathbb{Q}) \cup \{(t,0,\pm(2t+2)) : t^2+1=0\} \cup \mathcal{S}$$
$$= \{(0,0,0),(\pm 1,\pm 2,\pm 4)\} \cup \{(t,0,\pm(2t+2)) : t^2+1=0\} \cup \mathcal{S},$$

where $\#\mathcal{S} \in \{0,8\}$. Moreover, if $\mathcal{S}$ is nonempty and $(u,v,w) \in \mathcal{S}$, then $u \in \mathbb{Q}$, $v,w \notin \mathbb{Q}$, and

$$\mathcal{S} = \left\{(u,\pm v,\pm w), \left(-\frac{1}{u}, \pm\frac{v}{u^3}, \pm\frac{w}{u^4}\right)\right\}. \tag{4.24}$$

Assuming Theorem 4.26 for the moment, we now prove Theorem 4.24.

Proof of Theorem 4.24. Suppose (K,c) is a quadratic pair that simultaneously admits points a and b of periods 1 and 4, respectively, for f_c. By Proposition 4.25, there exists a K-rational point (u,v,w) on the curve X defined by (4.21) such that

$$a = \frac{1}{2} + \frac{w}{2u(u-1)(u+1)}\ , \quad b = \frac{u-1}{2(u+1)} + \frac{v}{2u(u-1)}\ ,$$

$$c = \frac{(u^2-4u-1)(u^4+u^3+2u^2-u+1)}{4u(u+1)^2(u-1)^2}.$$

The point (u,v,w) cannot be one of the known points listed in Theorem 4.26, since all such points satisfy $(u-1)(u+1)v = 0$ and therefore lie outside the open subset of X defined by (4.22). Hence (u,v,w) is a K-rational point in $\mathcal{S}$. This implies that $\#\mathcal{S} = 8$, and from (4.24) all eight elements of $\mathcal{S}$ lie in K. Since the expression for c given above is invariant under $u \mapsto -1/u$, all eight points yield the same value of c, and therefore the same quadratic pair (K,c).

Finally, since $u \in \mathbb{Q}$ and $v,w \notin \mathbb{Q}$, we necessarily have $c \in \mathbb{Q}$ and $a,b \notin \mathbb{Q}$. $\square$

The bound in Theorem 4.26 for the number of quadratic points on X will rely on the following lemma, which describes the set of *rational* points on two different quotients of X.

Lemma 4.27.

(A) Let C_3 be the genus 3 hyperelliptic curve defined by

$$w^2 = -u(u^6 - 4u^5 - 3u^4 - 8u^3 + 3u^2 - 4u - 1). \tag{4.25}$$

Then

$$C_3(\mathbb{Q}) = \{(0,0), (\pm 1, \pm 4), \infty\}.$$

(B) Let C_4 be the genus 4 hyperelliptic curve defined by

$$y^2 = (u^2+1)(u^2-2u-1)(u^6 - 4u^5 - 3u^4 - 8u^3 + 3u^2 - 4u - 1). \tag{4.26}$$

Then

$$C_4(\mathbb{Q}) = \{(0, \pm 1), (\pm 1, \pm 8), \infty^{\pm}\} \cup \mathcal{S}_4,$$

where $\#\mathcal{S}_4 \in \{0, 4\}$. Moreover, if $\mathcal{S}_4$ is nonempty and $(u, y) \in \mathcal{S}_4$, then

$$\mathcal{S}_4 = \left\{(u, \pm y), (-1/u, \pm y/u^5)\right\}.$$

Proof. We first consider the curve C_3. Let J_3 denote the Jacobian of C_3. A two-descent using Magma's `RankBound` function gives an upper bound of 1 for $\operatorname{rk} J_3(\mathbb{Q})$. The point $[(1,4) - \infty] \in J_3(\mathbb{Q})$ has infinite order, so $\operatorname{rk} J_3(\mathbb{Q}) = 1$. Because the rank is less than the genus, the method of Chabauty and Coleman may be used to bound the number of rational points on C_3. We apply the bound due to Stoll [36], stated in Theorem 3.6.

The prime $p = 7$ is a prime of good reduction for C_3, and a quick computation shows that $C_3(\mathbb{F}_7)$ consists of six points. Since $\operatorname{rk} J_3(\mathbb{Q}) = 1$, applying Theorem 3.6 yields the bound

$$\#C_3(\mathbb{Q}) \leq \#C_3(\mathbb{F}_7) + 2 \cdot \operatorname{rk} J_3(\mathbb{Q}) = 8.$$

We have already listed six rational points, so there can be at most two additional rational points on C_3.

Let ι be the hyperelliptic involution $(u, w) \mapsto (u, -w)$, and let σ be the automorphism given by

$$\sigma(u, w) = \left(-\frac{1}{u}, \frac{w}{u^4}\right).$$

It is easy to check that the automorphism σ also has order 2. We claim that if $(u, w) \notin \{(0, 0), \infty\}$ is a rational point on C_3, then the set

$$\{(u, w), \iota(u, w), \sigma(u, w), \sigma\iota(u, w)\}$$

contains exactly four elements. This will complete the proof of (A), since we have already said that there can be at most *two* rational points in addition to the six we have already found.

First, we note that the only points fixed by ι are the Weierstrass points, and the only rational Weierstrass points on C_3 are $(0, 0)$ and ∞, since the sextic polynomial appearing in (4.25) is irreducible over $\mathbb{Q}$.

Now suppose (u, w) is fixed by one of σ and $\sigma\iota$. In this case, we necessarily have $u = -1/u$, which implies that $u = \pm\sqrt{-1}$ and thus (u, w) is not rational.

Since just one additional rational point would actually yield at least four such points, we conclude that $C_3(\mathbb{Q})$ consists only of the six points listed in the lemma.

We prove (B) in a similar way. Let J_4 denote the Jacobian of C_4. Magma's `RankBound` function gives a bound of 2 for $\operatorname{rk} J_4(\mathbb{Q})$, so we may again apply Theorem 3.6 to bound the number of rational points on C_4. The prime $p = 5$ is a prime of good reduction for C_4, and we find that $\#C_4(\mathbb{F}_5) = 10$, so

$$\#C_4(\mathbb{Q}) \le \#C_4(\mathbb{F}_5) + 2 \cdot \operatorname{rk} J_4(\mathbb{Q}) + \left\lfloor \frac{2 \cdot \operatorname{rk} J_4(\mathbb{Q})}{3} \right\rfloor \le 15.$$

We have listed eight points, so there can be at most seven additional points.

Suppose now that $C_4(\mathbb{Q})$ contains an additional rational point (u, y). Consider the automorphism σ given by

$$\sigma(u, y) = \left(-\frac{1}{u}, \frac{y}{u^5}\right).$$

The automorphism σ has order four, since

$$(u, y) \stackrel{\sigma}{\mapsto} \left(-\frac{1}{u}, \frac{y}{u^5}\right) \stackrel{\sigma}{\mapsto} (u, -y) \stackrel{\sigma}{\mapsto} \left(-\frac{1}{u}, -\frac{y}{u^5}\right) \stackrel{\sigma}{\mapsto} (u, y).$$

Hence, the points $\sigma^k(u, y)$ are distinct for $k \in \{0, 1, 2, 3\}$, unless (u, y) is fixed by σ^k for some $k \in \{1, 2, 3\}$, in which case (u, y) must be fixed by the hyperelliptic involution $\iota = \sigma^2$. However, since the quadratic and sextic polynomials appearing in (4.26) are irreducible over $\mathbb{Q}$, C_4 has no finite rational Weierstrass points. Therefore $\{\sigma^k(u, y) : k \in \{0, 1, 2, 3\}\}$ consists of four distinct points.

In fact, this argument shows that $\#C_4(\mathbb{Q})$ is divisible by four. Since we have already found eight points and have obtained an upper bound of 15 for $\#C_4(\mathbb{Q})$, we have that $\#C_4(\mathbb{Q}) \in \{8, 12\}$ and therefore $\#\mathcal{S}_4 \in \{0, 4\}$, as claimed. If $\mathcal{S}_4$ is nonempty, then the four rational points in $\mathcal{S}_4$ are the orbit of (u, y) under σ. $\square$

We are now ready to prove Theorem 4.26.

Proof of Theorem 4.26. Since the only rational points on the affine curve defined by

$$v^2 = -u(u^2 + 1)(u^2 - 2u - 1)$$

have $u \in \{0, \pm 1\}$ (see [38, p. 774]), one can easily check that $X(\mathbb{Q})$ consists only of the nine rational points listed in the statement of the theorem; that is,

$$X(\mathbb{Q}) = \{(0, 0, 0), (\pm 1, \pm 2, \pm 4)\}.$$

Now suppose (u, v, w) is a quadratic point on X different from those listed in the theorem (i.e., an element of $\mathcal{S}$). We first show that $vw \neq 0$. Indeed, suppose $v = 0$. Then either $u = 0$, in which case $w = 0$ and (u, v, w) is a rational point; $u^2 + 1 = 0$, in which case we have one of the known quadratic points on X; or $u^2 - 2u - 1 = 0$, in which case one can check that $[\mathbb{Q}(w) : \mathbb{Q}] > 2$. In any case, we get a contradiction to the assumption that $v = 0$. A similar argument shows that $w \neq 0$.

We now observe that, since $vw \neq 0$, the following eight points must be distinct quadratic points on X, all defined over the same quadratic extension of $\mathbb{Q}$:

$$\left\{(u, \pm v, \pm w), \left(-\frac{1}{u}, \pm\frac{v}{u^3}, \pm\frac{w}{u^4}\right)\right\}.$$

Therefore, the addition of a single quadratic point implies the existence of at least eight additional quadratic points. We now show that there can be *at most* eight additional quadratic points.

Now, if $(u, v, w) \in \mathcal{S}$, it follows from the proof of Proposition 4.15 that either $u \in \mathbb{Q}$ or $(u^2 + 1)(u^2 - 2u - 1) = 0$. However, we have already ruled out the possibility that $(u^2 + 1)(u^2 - 2u - 1) = 0$, so we must have $u \in \mathbb{Q}$.

Since $u \in \mathbb{Q}$, our previous remarks imply that $v \in \mathbb{Q}$ if and only if $u \in \{0, \pm 1\}$. Similarly, by Lemma 4.27(A) we know that the only rational solutions to the equation

$$w^2 = -u(u^6 - 4u^5 - 3u^4 - 8u^3 + 3u^2 - 4u - 1)$$

must also satisfy $u \in \{0, \pm 1\}$. Hence $u \in \mathbb{Q} \setminus \{0, \pm 1\}$, $v, w \notin \mathbb{Q}$, and $v^2, w^2 \in \mathbb{Q}$. We may therefore write $v = v'\sqrt{d}$, $w = w'\sqrt{d}$ for some $v', w' \in \mathbb{Q}$ and $d \neq 1$ a squarefree integer, which means $vw \in \mathbb{Q}$. Since

$$(vw)^2 = u^2(u^2 + 1)(u^2 - 2u - 1)(u^6 - 4u^5 - 3u^4 - 8u^3 + 3u^2 - 4u - 1),$$

setting $y = vw/u$ gives a (finite) rational point on the curve C_4 from Lemma 4.27(B). Since we have $u \notin \{0, \pm 1\}$, the point (u, y) must be one of the four possible points in the set $\mathcal{S}_4$ appearing in Lemma 4.27(B). Since the map

$$\begin{aligned} X &\to C_4 \\ (u, v, w) &\mapsto \left(u, \frac{vw}{u}\right) \end{aligned}$$

has degree two, this allows at most eight possibilities for the point (u, v, w) on X, thus completing the proof. □

4.2.2 Periods 2 and 4

For the question of whether a quadratic pair (K, c) may admit K-rational points of periods 2 and 4, we refer to the following result from [8]:

Theorem 4.28. *In addition to the known pair $(\mathbb{Q}(\sqrt{-15}), -31/48)$, there is at most one quadratic pair (K, c) for which f_c simultaneously admits K-rational points of periods 2 and 4. Moreover, if an additional example exists, it must have $c \in \mathbb{Q}$.*

Proof. See [8, §3.17]. □

4.2.3 Periods 3 and 4

We now begin to prove the following theorem:

Theorem 4.29. *Let K be a quadratic field, and let $c \in K$. The map f_c cannot simultaneously admit K-rational points of periods 3 and 4.*

The appropriate dynamical modular curve to study in this case is

$$Y_1^{\text{dyn}}(3, 4) = \{(a, b, c) \in \mathbb{A}^3 : \ a \text{ has period 3 and } b \text{ has period 4 for } f_c\}.$$

Proposition 4.30. *Let X be the affine curve of genus 49 defined by the equation*

$$\begin{cases} -(t^6+2t^5+4t^4+8t^3+9t^2+4t+1)u(u+1)^2(u-1)^2 \\ \qquad\qquad = (u^2-4u-1)(u^4+u^3+2u^2-u+1)t^2(t+1)^2 \\ v^2 = -u(u^2+1)(u^2-2u-1), \end{cases} \tag{4.27}$$

and let Y be the open subset of X defined by

$$t(t+1)(t^2+t+1)(u-1)(u+1)v \neq 0. \tag{4.28}$$

Consider the morphism $\Phi : Y \to \mathbb{A}^3$, $(t,u,v) \mapsto (a,b,c)$, given by

$$\begin{gathered} a = \frac{t^3+2t^2+t+1}{2t(t+1)} \ , \quad b = \frac{u-1}{2(u+1)} + \frac{v}{2u(u-1)} \ , \\ c = -\frac{t^6+2t^5+4t^4+8t^3+9t^2+4t+1}{4t^2(t+1)^2}. \end{gathered} \tag{4.29}$$

Then Φ maps Y isomorphically onto $Y_1^{\rm dyn}(3,4)$, with the inverse map given by

$$t = a + f_c(a) \ , \quad u = -\frac{f_c^2(b)+b+1}{f_c^2(b)+b-1} \ , \quad v = \frac{u(u-1)(2bu+2b-u+1)}{u+1}. \tag{4.30}$$

Proof. This follows from Propositions 4.2 and 4.12, noting that the first equation in (4.27) comes from equating the two expressions for c given in those two propositions, and the second equation is precisely (4.7) from Proposition 4.12. □

In order to find all quadratic points on $X_1^{\rm dyn}(3,4)$, it will be beneficial for us to first understand the quotient $X_0^{\rm dyn}(3,4)$. Recall from §2.1 that there are automorphisms σ_3 and σ_4 on $Y_1^{\rm dyn}(3,4)$ — and hence $X_1^{\rm dyn}(3,4)$ — given by

$$\sigma_3(a,b,c) = (f_c(a),b,c) \ \text{ and } \ \sigma_4(a,b,c) = (a, f_c(b), c),$$

for which $X_0^{\mathrm{dyn}}(3,4)$ is the quotient of $X_1^{\mathrm{dyn}}(3,4)$ by the group of automorphisms generated by σ_3 and σ_4. Abusing notation, we will also denote by σ_3 and σ_4 the corresponding automorphisms on the curve X defined in (4.27), which is birational to $X_1^{\mathrm{dyn}}(3,4)$. By using the relations given in (4.29) and (4.30), we verify using Magma that σ_3 and σ_4 act on X as follows:

$$\begin{aligned}\sigma_3(t,u,v) &= \left(-\frac{t+1}{t}, u, v\right)\\ \sigma_4(t,u,v) &= \left(t, -\frac{1}{u}, \frac{v}{u^3}\right).\end{aligned}$$

Since σ_3 has order 3 and σ_4 has order 4, the group $\langle \sigma_3, \sigma_4\rangle$ is cyclic of order 12, generated by the automorphism $\sigma := \sigma_3 \circ \sigma_4$ defined by

$$\sigma(t,u,v) = \left(-\frac{t+1}{t}, -\frac{1}{u}, \frac{v}{u^3}\right).$$

The curve X' is therefore the quotient of X by σ. It will be simpler, however, to first consider the quotient X'' of X by the automorphism σ^6, and then take the quotient of X'' by the automorphism $\tau \in \mathrm{Aut}(X'')$ that descends from $\sigma \in \mathrm{Aut}(X)$. Observe that

$$\sigma^6(t,u,v) = (t,u,-v),$$

so the curve X'' is simply given by the first equation from (4.27), which we rewrite for convenience as

$$-\frac{t^6+2t^5+4t^4+8t^3+9t^2+4t+1}{4t^2(t+1)^2} = \frac{(u^2-4u-1)(u^4+u^3+2u^2-u+1)}{4u(u+1)^2(u-1)^2}. \tag{4.31}$$

Note that this is precisely the equation obtained by equating the expressions for c found in Theorems 4.4 and 4.11.

We now seek the quotient of X'' by the automorphism τ, which is given by

$$\tau(t,u) := \left(-\frac{t+1}{t}, -\frac{1}{u}\right).$$

Consider the following invariants of τ:

$$T := t - \frac{t+1}{t} - \frac{1}{t+1} = \frac{t^3-3t-1}{t(t+1)}\ ,\ U := u - \frac{1}{u} = \frac{(u-1)(u+1)}{u}. \tag{4.32}$$

To find the quotient of X'' by τ, we rewrite (4.31) in terms of T and U. Doing so yields the equation

$$-(T^2+2T+8) = \frac{(U^2+U+4)(U-4)}{U^2}.$$

Clearing denominators, we see that X' is the curve defined by

$$-(T^2+2T+8)U^2 = (U^2+U+4)(U-4). \tag{4.33}$$

This curve has genus one and is birational to $X_1(11)$, which is labeled 11A3 in [7] and given by

$$y^2+y = x^3-x.$$

The inverse birational maps between X' and $X_1(11)$ are

$$\begin{aligned}(x,y) &\mapsto \left(-\frac{x-2y-1}{x}, -4x\right)\\ (T,U) &\mapsto \left(-\frac{U}{4}, -\frac{TU+U+4}{8}\right).\end{aligned} \tag{4.34}$$

We have just proven the following:

Proposition 4.31. *The curve* $X_0^{\mathrm{dyn}}(3,4)$ *is isomorphic to the elliptic curve* $X_1(11)$ *(labeled 11A3 in [7]), given by the equation*

$$y^2 + y = x^3 - x.$$

We will now show that the curve X defined by (4.27) has no quadratic points.

Theorem 4.32. *Let* X *be the genus 49 affine curve defined by* (4.27). *Then*

$$X(\mathbb{Q}, 2) = X(\mathbb{Q}) = \{(0,0,0), (0, \pm 1, \pm 2), (-1,0,0), (-1, \pm 1, \pm 2)\}.$$

Assuming this theorem for the moment, we may now complete the proof of Theorem 4.29. Theorem 4.32 implies that all points of degree at most 2 on X fail to satisfy (4.28). It follows that if K is a quadratic field, then $Y_1^{\mathrm{dyn}}(3,4)(K)$ is empty. Therefore, there can be no element $c \in K$ such that f_c admits K-rational points of periods 3 and 4. $\square$

Proof of Theorem 4.32. As mentioned in the proof of Theorem 4.26, the only rational solutions to

$$v^2 = -u(u^2+1)(u^2-2u-1)$$

have $u \in \{0, \pm 1\}$, so the rational points listed above must be all rational points on X.

The rational points all fail to satisfy condition (4.28). One can verify that all *other* points failing condition (4.28) have degree strictly greater than two over $\mathbb{Q}$. Thus any quadratic point (t,u,v) on X lies on the curve Y defined in Proposition 4.30. Therefore, if we let K be the field of definition of (t,u,v) and set

$$c = -\frac{t^6 + 2t^5 + 4t^4 + 8t^3 + 9t^2 + 4t + 1}{4t^2(t+1)^2} = \frac{(u^2-4u-1)(u^4+u^3+2u^2-u+1)}{4u(u+1)^2(u-1)^2} \in K,$$

then the map f_c admits K-rational points of periods 3 and 4.

By the proof of Proposition 4.15, any rational or quadratic solution to the equation $v^2 = -u(u^2+1)(u^2-2u-1)$ for which $(u^2+1)(u^2-2u-1) \neq 0$ must have $u \in \mathbb{Q}$. It follows that $c \in \mathbb{Q}$, and therefore the point a of period 3 is rational by Corollary 4.6. Finally, by the remark following Proposition 4.2, we have $t = a + f_c(a) \in \mathbb{Q}$. This implies that $t, u \in \mathbb{Q}$, so that the image of (t, u, v) under the quotient map $X \to X'$ is a rational point.

Let $\widetilde{X'}$ denote the projective closure of the curve defined by (4.33); that is, $\widetilde{X'}$ is the curve in $\mathbb{P}^2$ given by the homogeneous equation

$$-(T^2 + 2TZ + 8Z^2)U^2 = (U^2 + UZ + 4Z^2)(U - 4Z)Z. \tag{4.35}$$

A quick search reveals the four rational points

$$[T : U : Z] \in \{[0 : -4 : 1], [-2 : -4 : 1], [1 : 0 : 0], [0 : 1 : 0]\}.$$

We claim that this is the complete list of rational points on $\widetilde{X'}$.

Let $\varphi : 11A3 \to X'$ be the rational map given by (4.34). In projective coordinates, we have

$$\varphi([x : y : z]) = [xz - 2yz - z^2 : 4x^2 : -xz].$$

The curve 11A3 has only five rational points, namely

$$\{[0 : 0 : 1], [0 : -1 : 1], [1 : -1 : 1], [1 : 0 : 1], [0 : 1 : 0]\},$$

and they map to $\widetilde{X'}$ as follows:

$$\begin{aligned}
[0:0:1],[0:-1:1] &\mapsto [1:0:0]\\
[1:-1:1] &\mapsto [-2:-4:1]\\
[1:0:1] &\mapsto [0:-4:1]\\
[0:1:0] &\mapsto [0:1:0].
\end{aligned}$$

Note that $[1:0:0]$ is the unique singular point on $\widetilde{X'}$. If $\widetilde{X'}$ had another (necessarily nonsingular) rational point P, then P would be the image under φ of a rational point on 11A3. However, we have accounted for the images of all five rational points on 11A3, so $\widetilde{X'}$ cannot have any additional rational points.

Now, since we are assuming that the point (t,u,v) satisfies the open condition (4.28), then in particular $t(t+1)u \neq 0$, so T and U are finite. The only finite rational points on $\widetilde{X'}$ are $[T:U:Z] = [0:-4:1]$ and $[T:U:Z] = [-2:-4:1]$. In other words, the only rational points on X' are $(T,U) = (0,4)$ and $(T,U) = (-2,-4)$. In both cases, we have $U = -4$, and therefore $(u-1)(u+1)/u = -4$. Rewriting this equation yields $u^2+4u-1 = 0$, hence $u \notin \mathbb{Q}$, contradicting our previous assertion regarding u. We therefore conclude that X has no quadratic points. $\square$

4.2.4 Periods 1, 2, and 3

In this section, we prove the following result:

Theorem 4.33. *Let K be a quadratic field, and let $c \in K$. The map f_c cannot simultaneously admit K-rational points of periods 1, 2, and 3.*

To prove Theorem 4.33, we must find all rational and quadratic points on the curve

$$Y_1^{\mathrm{dyn}}(1,2,3) = \{(p_1,p_2,p_3,c) \in \mathbb{A}^4 : \; p_i \text{ has period } i \text{ for } f_c \text{ for each } i \in \{1,2,3\}\}.$$

Proposition 4.34. *Let X be the affine curve of genus 9 defined by the equation*

$$\begin{cases} y^2 &= t^6 + 2t^5 + 5t^4 + 10t^3 + 10t^2 + 4t + 1 \\ z^2 &= t^6 + 2t^5 + t^4 + 2t^3 + 6t^2 + 4t + 1, \end{cases} \tag{4.36}$$

and let Y be the open subset of X defined by

$$t(t+1)(t^2+t+1)z \neq 0. \tag{4.37}$$

Consider the morphism $\Phi : Y \to \mathbb{A}^4$, $(t,y,z) \mapsto (p_1,p_2,p_3,c)$, given by

$$p_1 = \frac{t^2+t+y}{2t(t+1)} \; , \;\; p_2 = -\frac{t^2+t-z}{2t(t+1)} \; , \;\; p_3 = \frac{t^3+2t^2+t+1}{2t(t+1)} \; ,$$
$$c = -\frac{t^6+2t^5+4t^4+8t^3+9t^2+4t+1}{4t^2(t+1)^2}.$$

Then Φ maps Y isomorphically onto $Y_1^{\mathrm{dyn}}(1,2,3)$, with the inverse map given by

$$t = p_3 + f_c(p_3) \; , \;\; y = (2p_1 - 1)t(t+1) \; , \;\; z = (2p_2+1)t(t+1). \tag{4.38}$$

Proof. This follows from parts (B) and (C) of Proposition 4.20. □

Theorem 4.35. *Let X be the genus 9 affine curve defined by (4.36). Then*

$$X(\mathbb{Q},2) = X(\mathbb{Q}) = \{(0,\pm1,\pm1),(-1,\pm1,\pm1)\}.$$

Theorem 4.33 now follows immediately, since Theorem 4.35 says that the only rational or quadratic points on the curve X defined in (4.36) lie outside the open set given by (4.37).

Proof of Theorem 4.35. Let $f(t) := t^6 + 2t^5 + 5t^4 + 10t^3 + 10t^2 + 4t + 1$ and $g(t) := t^6 + 2t^5 + t^4 + 2t^3 + 6t^2 + 4t + 1$. By Lemma 4.22, the only rational points on $y^2 = f(t)$ and $z^2 = g(t)$ are those points with $t \in \{-1, 0\}$, so the eight points listed in the theorem are the only rational points. It now remains to show that the curve X admits no quadratic points.

Suppose to the contrary that (t, y, z) is a quadratic point on X. From Lemma 4.22, we know that $t \in \mathbb{Q}$; furthermore, if $t \notin \{0, -1\}$, then neither y nor z is in $\mathbb{Q}$. Thus y and z generate the same quadratic extension $K = \mathbb{Q}(\sqrt{d})$, with $d \neq 1$ a squarefree integer. Since $y^2, z^2 \in \mathbb{Q}$, we may write $y = u\sqrt{d}$, $z = v\sqrt{d}$ for some $u, v \in \mathbb{Q}$. This allows us to rewrite (4.36) as

$$\begin{cases} du^2 &= f(t) = t^6 + 2t^5 + 5t^4 + 10t^3 + 10t^2 + 4t + 1 \\ dv^2 &= g(t) = t^6 + 2t^5 + t^4 + 2t^3 + 6t^2 + 4t + 1, \end{cases} \tag{4.39}$$

where $t, u, v \in \mathbb{Q}$.

Let $p \in \mathbb{Z}$ be a prime dividing d; we first show that t, u, and v are integral at p. Suppose instead that $\operatorname{ord}_p(t) < 0$. Then $\operatorname{ord}_p(f(t)) = 6 \operatorname{ord}_p(t)$ is even, but $\operatorname{ord}_p(du^2) = 2 \operatorname{ord}_p(u) + 1$ is odd, a contradiction. Therefore $\operatorname{ord}_p(t) \geq 0$, and it can easily be seen now that $\operatorname{ord}_p(u)$ and $\operatorname{ord}_p(v)$ must also be nonnegative. We can therefore reduce (4.39) mod p to get

$$f(t) \equiv g(t) \equiv 0 \mod p.$$

From this we see that p divides the resultant of f and g. We compute $\operatorname{Res}(f, g) = 2^{12}$, so the only prime that can divide d is 2, and therefore $d \in \{-1, \pm 2\}$. However, $f(t)$ is strictly positive for all $t \in \mathbb{R}$, so we must have $d = 2$.

Let $C_f^{(2)}$ and $C_g^{(2)}$ be the curves defined by $2u^2 = f(t)$ and $2v^2 = g(t)$, respectively, and let $J_f^{(2)}$ and $J_g^{(2)}$ be their Jacobians. A Magma computation shows that each of $J_f^{(2)}$ and $J_g^{(2)}$

has a trivial Mordell-Weil group over $\mathbb{Q}$, so in particular neither curve has rational points. Therefore X has no quadratic points. $\square$

Chapter 5

Strictly preperiodic points for quadratic polynomials

We now concern ourselves with preperiodic graphs that contain *strictly* preperiodic points for the map $f_c(z) := z^2 + c$; that is, points that are preperiodic but not periodic. Recall that P is a point of type m_n for f_c if $f_c^n(P)$ is a point of period m, but $f^{n-1}(P)$ is not. In other words, P is a point of type m_n if P enters into an m-cycle after exactly n iterations.

We first observe that a point P is of type m_1 for f_c if and only if P is the negative of a nonzero point of period m.[1] Indeed, if P is a point of type m_1, then $Q := f_c(P)$ is a point of period m. Therefore also $f_c^{m-1}(Q)$ is a point of period m, and we have

$$f_c(f_c^{m-1}(Q)) = f_c^m(Q) = Q = f_c(P).$$

Since $f_c^{m-1}(Q)$ and P are distinct preimages of Q, we must have $P = -f_c^{m-1}(Q) \neq 0$, showing that P is the negative of a nonzero point of period m. Conversely, let P be a nonzero point of period m. Then $f_c(-P) = f_c(P)$ is also a point of period m. Since P is nonzero, $P \neq -P$, so $-P$ cannot also be periodic. Therefore $-P$ is a point of type m_1.

[1]One could also see this by applying Corollary 2.7, which says that $\Phi_{N,1}(X, C) = \Phi_N(-X, C)$.

From this discussion, it follows that if f_c admits a nonzero K-rational point of period m, then f_c automatically admits a K-rational point of type m_1. Therefore our discussion of strictly preperiodic points will focus on points of type m_n for $n \geq 2$.

5.1 Period 1 and type 2_3

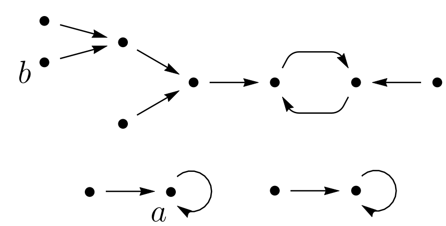


Figure 5.1: The graph generated by a fixed point a and a point b of type 2_3

Let G be the graph shown in Figure 5.1. Then G is a subgraph of the graph 14(2,1,1) appearing in Appendix A. In [8], we showed that the graph 14(2,1,1) occurs as a subgraph of $G(f_c, K)$ for a single quadratic pair $(K, c) = (\mathbb{Q}(\sqrt{17}), -21/16)$, in which case we actually have $G(f_c, K) \cong 14(2,1,1)$. Since G is a subgraph of 14(2,1,1), this gives one quadratic pair (K, c) for which G is a subgraph of $G(f_c, K)$. In this section, we show that there are at most three additional quadratic pairs with this property.

Since G is generated by a fixed point and a point of type 2_3, we may write $Y_1^{\text{dyn}}(G)$ as

$$Y_1^{\text{dyn}}(G) = \{(a, b, c) \in \mathbb{A}^3 : \ a \text{ is a fixed point and } b \text{ has type } 2_3 \text{ for } f_c.\}$$

Proposition 5.1. *Let X be the affine curve of genus 5 defined by the equation*

$$\begin{cases} y^2 &= 2(x^3 + x^2 - x + 1) \\ z^2 &= 5x^4 + 8x^3 + 6x^2 - 8x + 5, \end{cases} \tag{5.1}$$

and let Y be the open subset of X defined by

$$x(x-1)(x+1)(x^2+4x-1) \neq 0. \tag{5.2}$$

Consider the morphism $\Phi : Y \to \mathbb{A}^3$, $(x,y,z) \mapsto (a,b,c)$, given by

$$a = \frac{x^2-1+z}{2(x^2-1)} \ , \ b = \frac{y}{x^2-1} \ , \ c = -\frac{x^4+2x^3+2x^2-2x+1}{(x^2-1)^2}.$$

Let G be the graph shown in Figure 5.1. Then Φ maps Y isomorphically onto $Y_1^{\mathrm{dyn}}(G)$, with the inverse map given by

$$x = \frac{f_c(b)-1}{f_c^2(b)} \ , \ y = b(x^2-1) \ , \ z = \frac{2a-1}{x^2-1}. \tag{5.3}$$

Proof. The condition that $(x-1)(x+1) \neq 0$ ensures that Φ is a well-defined morphism to $\mathbb{A}^3$, and it is not difficult to show that (5.3) provides a left inverse for Φ, so that Φ is injective.

Now let $(x,y,z) \in Y$ and $(a,b,c) = \Phi(x,y,z)$. A computation in Magma shows that $f_c(a) = a$, so that a is a fixed point for f_c. Also, a computation shows that $f_c^5(a) = f_c^3(a)$, and that

$$f_c^5(b) - f_c^4(b) = -\frac{x^2+4x-1}{x^2-1} \ , \ f_c^4(b) - f_2(b) = \frac{4x}{x^2-1}.$$

Since the above expressions are nonzero by (5.2), it follows that b is a point of type 2_3. Therefore Φ maps Y into $Y_1^{\mathrm{dyn}}(G)$. It remains to prove surjectivity.

Let $(a,b,c) \in Y_1^{\mathrm{dyn}}(G)$. Since b is a point of type 2_3, we have from [32, p. 23] that there exists x for which

$$b^2 = \frac{2(x^3+x^2-x+1)}{(x^2-1)^2} \ , \ c = -\frac{x^4+2x^3+2x^2-2x+1}{(x^2-1)^2}.$$

This gives us the correct expression for c, and setting $y = b(x^2 - 1)$ gives us the first equation in (5.1) and the correct expression for b. Finally, since a is a fixed point, we have

$$f_c(a) - a = a^2 - a - \frac{x^4 + 2x^3 + 2x^2 - 2x + 1}{(x^2 - 1)^2}.$$

The discriminant of this polynomial (in a) is

$$\Delta := \frac{5x^4 + 8x^3 + 6x^2 - 8x + 5}{(x^2 - 1)^2}.$$

Thus $a = 1/2 + z/(2(x^2 - 1))$, where z satisfies

$$z^2 = \Delta(x^2 - 1)^2 = 5x^4 + 8x^3 + 6x^2 - 8x + 5,$$

which completes the proof. $\square$

Just as in §4.2.1, we are currently only able to find an upper bound for the number of quadratic points on $Y_1^{\mathrm{dyn}}(G)$.

Theorem 5.2. *Let X be the genus 5 affine curve defined by* (5.1)*. Then*

$$\begin{aligned} X(\mathbb{Q}, 2) &= X(\mathbb{Q}) \cup \{(x, \pm 2(2x+1), \pm 20x) : x^2 - 8x - 1 = 0\} \cup \mathcal{S} \\ &= \{(\pm 1, \pm 2, \pm 4)\} \cup \{(x, \pm 2(2x+1), \pm 20x) : x^2 - 8x - 1 = 0\} \cup \mathcal{S}, \end{aligned}$$

where $\#\mathcal{S} \in \{0, 4, 8, 12\}$. Moreover, if $(x, y, z) \in \mathcal{S}$, then $x \in \mathbb{Q}$, $y, z \notin \mathbb{Q}$, and $(x, \pm y, \pm z)$ are four distinct points in $\mathcal{S}$.

Proof. Let X_1 and X_2 denote the curves defined by $y^2 = 2(x^3 + x^2 - x + 1)$ and $z^2 = 5x^4 + 8x^3 + 6x^2 - 8x + 5$, respectively. Each of these curves has genus one, and Magma verifies that X_1 (resp., X_2) is birational to elliptic curve 11A3 (resp., 17A4) in [7], which has only five (resp., four) rational points. (As mentioned previously, the curve 11A3 is isomorphic to $X_1(11)$.) Hence $X_1(\mathbb{Q})$ consists of the four points $(\pm 1, \pm 2)$ (plus a point at infinity on the projective closure), and $X_2(\mathbb{Q})$ consists of the four points $(\pm 1, \pm 4)$. It follows that the rational points listed in the theorem are all of the rational points on X.

Now suppose (x, y, z) is a quadratic point on X. We will consider separately the two cases depending on whether $x \in \mathbb{Q}$.

Case 1: $x \in \mathbb{Q}$. If $x \in \{\pm 1\}$, then we already know that (x, y, z) is a rational point. On the other hand, if $x \in \mathbb{Q} \setminus \{\pm 1\}$, then it follows from our discussion about $X_1(\mathbb{Q})$ and $X_2(\mathbb{Q})$ that $y, z \notin \mathbb{Q}$. Since $y^2, z^2 \in \mathbb{Q}$, we see that setting $w := yz$ yields a rational solution to the equation

$$w^2 = 2(x^3 + x^2 - x + 1)(5x^4 + 8x^3 + 6x^2 - 8x + 5). \tag{5.4}$$

Let C denote the hyperelliptic curve of genus three defined by this equation. A quick search for points on C yields the five points

$$\{\infty, (\pm 1, \pm 8)\} \subseteq C(\mathbb{Q}).$$

Now let $J = \mathrm{Jac}(C)$. Applying Magma's `RankBound` function to J shows that $\mathrm{rk}\, J(\mathbb{Q}) \leq 1$. Since the point $[(1, 8) - \infty] \in J(\mathbb{Q})$ has infinite order we actually have $\mathrm{rk}\, J(\mathbb{Q}) = 1$. We may therefore apply Theorem 3.6 to bound $\#C(\mathbb{Q})$. The prime $p = 7$ is a prime of good reduction for C, and $\#C(\mathbb{F}_7) = 10$, so

$$\#C(\mathbb{Q}) \leq \#C(\mathbb{F}_7) + 2 \cdot \mathrm{rk}\, J(\mathbb{Q}) = 12.$$

Now observe that ∞ is a rational Weierstrass point on C, since (5.4) has odd degree. Also, since the cubic and quartic polynomials appearing in (5.4) are irreducible over $\mathbb{Q}$, C has no finite rational Weierstrass points. Non-Weierstrass points come in pairs (related via the hyperelliptic involution), so $\#C(\mathbb{Q})$ must be odd, and therefore $\#C(\mathbb{Q}) \leq 11$. Since we already have five rational points on C, there are at most six additional points. Since the map $X \to C$ given by $(x, y, z) \mapsto (x, yz)$ has degree two, this implies that there are at most 12 additional quadratic points on X with $x \in \mathbb{Q}$.

Case 2: $x \notin \mathbb{Q}$. Applying Lemma 3.1 to X_1, we see that there is a point $(x_0, y_0) \in X_1(\mathbb{Q}) \setminus \{\infty\}$ and a rational number v for which

$$p(t) = t^2 + \frac{2x_0 - v^2 + 2}{2} t + \frac{2x_0^2 + v^2 x_0 + 2x_0 - 2y_0 v - 2}{2}$$

is the minimal polynomial for x. As we have already mentioned, the only finite rational points on X_1 are the four points $(\pm 1, \pm 2)$, so the minimal polynomial must be expressed in one of the following forms (with $v \in \mathbb{Q}$):

$$\begin{aligned}
&t^2 - \frac{v^2 - 4}{2} t + \frac{v^2 - 4v + 2}{2}, \\
&t^2 - \frac{v^2 - 4}{2} t + \frac{v^2 + 4v + 2}{2}, \\
&t^2 - \frac{v^2}{2} t - \frac{v^2 + 4v + 2}{2}, \text{ or} \\
&t^2 - \frac{v^2}{2} t - \frac{v^2 - 4v + 2}{2}.
\end{aligned}$$

We observe that the second (resp., fourth) polynomial may be obtained by replacing the v in the first (resp., third) polynomial by $-v$. Therefore we can say that the minimal polynomial $p(t)$ of x may be written in one of the two forms

$$p(t) = t^2 - \frac{v^2 - 4}{2} t + \frac{v^2 + 4v + 2}{2} \text{ or} \tag{5.5}$$

$$p(t) = t^2 - \frac{v^2}{2} t - \frac{v^2 + 4v + 2}{2}. \tag{5.6}$$

We will now use X_2 to find different expressions for the minimal polynomial of x. In order to do so, we require a cubic model for X_2. Let E be the elliptic curve defined by the equation

$$Y^2 = X^3 - 3X^2 - 8X + 16.$$

The curve X_2 is birational to E via the maps

$$X = \frac{2(x^2+3-z)}{(x+1)^2}\ ,\ Y = \frac{2(3x^3+3x^2+9x-7-(x-3)z)}{(x+1)^3} \tag{5.7}$$

$$x = \frac{3X-4+Y}{X+4-Y}\ ,\ z = \frac{-4(X^3+8X-32+8Y)}{(X+4-Y)^2}. \tag{5.8}$$

Suppose $X \in \mathbb{Q}$. Using the expression for X found in (5.7) and solving for z, we find that

$$z = \frac{2(x^2+3) - X(x+1)^2}{2},$$

and therefore

$$z^2 - (5x^4+8x^3+6x^2-8x+5) = \frac{1}{4}(x+1)^2\left(x^2 + \frac{2X^2}{X^2-4X-16}x + \frac{X^2-12X+16}{X^2-4X-16}\right) = 0.$$

We cannot have $x+1=0$, since $x \notin \mathbb{Q}$, so the minimal polynomial for x has the form

$$p(t) = t^2 + \frac{2X^2}{X^2-4X-16}t + \frac{X^2-12X+16}{X^2-4X-16}.$$

Now suppose that $X \notin \mathbb{Q}$. Then applying Lemma 3.1 to E shows that there is a point $(X_0, Y_0) \in E(\mathbb{Q}) \setminus \{\infty\}$ and a rational number w for which

$$Y = Y_0 + w(X - X_0) \tag{5.9}$$

and

$$q(t) = t^2 + (X_0 - w^2 - 3)X + (X_0^2 + w^2 X_0 - 3X_0 - 2Y_0 w - 8)$$

is the minimal polynomial for X. We have already said that the affine curve X_2 has only four rational points $(\pm 1, \pm 4)$, and X_2 visibly has no rational points at infinity, so E must also have only four rational points. We can therefore see that $E(\mathbb{Q}) = \{\infty, (0, \pm 4), (4, 0)\}$.

If we take $(X_0, Y_0) = (0, 4)$, then combining (5.8) and (5.9) gives us

$$x = -\frac{w+3}{w-1},$$

which contradicts the fact that $w \in \mathbb{Q}$ and $x \notin \mathbb{Q}$.

If $(X_0, Y_0) = (0, -4)$, then (5.8) and (5.9) together give us

$$x = -\frac{(w+3)X - 8}{(w-1)X - 8},$$

and solving for X yields

$$X = \frac{8(x+1)}{(w-1)x + (w+3)}.$$

We may therefore write the equation $q(X) = 0$ as

$$-\frac{16\left((w^2-5)x^2 + 8(w-1)x - (w^2-5)\right)}{\left((w-1)x + (w+3)\right)^2} = 0.$$

Hence the minimal polynomial of x is of the form

$$p(t) = t^2 + \frac{8(w-1)}{w^2-5}t - 1.$$

Finally, if $(X_0, Y_0) = (4, 0)$, then (5.8) and (5.9) yield

$$x = -\frac{(w+3)X - 4(w+1)}{(w-1)X - 4(w+1)},$$

which implies that

$$X = \frac{4(w+1)(x+1)}{(w-1)x + (w+3)}.$$

Substituting this into the equation $q(X) = 0$ gives us

$$-\frac{8(w+1)\left((w^2-4w-1)x^2 - 8x - (w^2+4w-1)\right)}{\left((w-1)x + (w+3)\right)^2} = 0.$$

We cannot have $w+1 = 0$, since this would imply that $X = 0$, contradicting our assumption that $X \notin \mathbb{Q}$. Therefore the minimal polynomial of x is given by

$$p(t) = t^2 - \frac{8}{w^2-4w-1}t - \frac{w^2+4w-1}{w^2-4w-1}.$$

To summarize, applying Lemma 3.1 to the curve E has shown us that the minimal polynomial of x must be of one of the following forms:

$$p(t) = t^2 + \frac{2X^2}{X^2-4X-16}t + \frac{X^2-12X+16}{X^2-4X-16}, \tag{5.10}$$

$$p(t) = t^2 + \frac{8(w-1)}{w^2-5}t - 1, \tag{5.11}$$

$$p(t) = t^2 - \frac{8}{w^2-4w-1}t - \frac{w^2+4w-1}{w^2-4w-1}, \tag{5.12}$$

where $X, w \in \mathbb{Q}$. We now compare each of these three expressions with each of the two expressions (5.5) and (5.6). Equating coefficients of each pair of polynomials will yield six systems of equations whose rational solutions we must find. All calculations were done with Magma.

For (5.5) and (5.10), the system is

$$\begin{cases} 4X^2 &= -(v^2-4)(X^2-4X-16) \\ 2(X^2-12X+16) &= (v^2+4v+2)(X^2-4X-16). \end{cases}$$

This defines a zero-dimensional scheme which has only the single rational point $(v, X) = (-2, 0)$.

For (5.5) and (5.11), the system is

$$\begin{cases} 16(w-1) &= -(v^2-4)(w^2-5) \\ v^2+4v+2 &= -2. \end{cases}$$

Here one can easily verify by hand that the only rational solution is $(v, w) = (-2, 1)$.

For (5.5) and (5.12), the system is

$$\begin{cases} (v^2-4)(w^2-4w-1) &= 16 \\ 2(w^2+4w-1) &= (v^2+4v+2)(w^2-4w-1). \end{cases}$$

This defines a zero-dimensional scheme with no rational points.

For (5.6) and (5.10), the system is

$$\begin{cases} 4X^2 &= -v^2(X^2-4X-16) \\ 2(X^2-12X+16) &= -(v^2+4v+2)(X^2-4X-16). \end{cases}$$

This gives a zero-dimensional scheme whose only rational points are $(v, X) \in \{(-2, 4), (0, 0)\}$.

For (5.6) and (5.11), the system is

$$\begin{cases} 16(w-1) &= -v^2(w^2-5) \\ v^2+4v+2 &= 2. \end{cases}$$

One can easily check by hand that the only rational solutions to this system are $(v,w) \in \{(0,1),(-4,2),(-4,-3)\}$.

For (5.6) and (5.12), the system is

$$\begin{cases} v^2(w^2-4w-1) &= 16 \\ 2(w^2+4w-1) &= (v^2+4v+2)(w^2-4w-1). \end{cases}$$

This defines a zero-dimensional scheme whose only rational point is $(v,w)=(-2,-1)$.

For the comparison of (5.6) and (5.11), the points $(-4,2)$ and $(-4,-3)$ give a minimal polynomial of $p(t)=t^2-8t-1$ for x. In this case, the corresponding points on X are those quadratic points listed in the theorem. In every other case, the corresponding polynomial $p(t)$ is reducible over $\mathbb{Q}$ and cannot, therefore, be the minimal polynomial for $x \notin \mathbb{Q}$.

We conclude that if (x,y,z) is any unknown quadratic point on X, then $x \in \mathbb{Q}$, from which it follows by our discussion above that $y,z \notin \mathbb{Q}$ and $\#\mathcal{S} \le 12$. Since the cubic and quartic polynomials appearing in (5.1) are irreducible over $\mathbb{Q}$, no quadratic point $(x,y,z) \in X(\mathbb{Q},2)$ can have $yz=0$. Therefore $(x,\pm y,\pm z)$ are four distinct points on X, which in particular means that $\#\mathcal{S}$ is divisible by four. $\square$

Theorem 5.3. *Let G be the graph appearing in Figure 5.1. In addition to the pair $(K,c) = (\mathbb{Q}(\sqrt{17}), -21/16)$, which has $G(f_c,K) \cong 14(2,1,1)$ and is the unique quadratic pair satisfying $G(f_c,K) \supseteq 14(2,1,1) \supsetneq G$ (see [8, Cor. 3.52]), there are at most three quadratic pairs (K,c) for which $G(f_c,K)$ contains a subgraph isomorphic to G.*

Moreover, if (K,c) is one such pair, then $c \in \mathbb{Q}$. Letting a and b be points of period 1 and type 2_3, respectively, for f_c, then $a, b \notin \mathbb{Q}$ and $f_c(b) \in \mathbb{Q}$.

Proof. We first mention that $(\mathbb{Q}(\sqrt{17}), -21/16)$ is the unique quadratic pair (K,c) for which the graph $G(f_c,K)$ is isomorphic to graph 14(2,1,1) in Appendix A, as was proven in [8, Cor. 3.52]. Note that the graph G in Figure 5.1 is a proper subgraph of 14(2,1,1).

Now suppose (K,c) is another quadratic pair whose preperiodic graph contains a subgraph isomorphic to G. Let a be a K-rational fixed point for f_c, and let b be a K-rational point of type 2_3 for f_c. Letting X denote the curve defined by (5.1), Proposition 5.1 says that there is a point $(x,y,z) \in X(K)$ such that

$$a = \frac{x^2 - 1 + z}{2(x^2-1)} \ , \ b = \frac{y}{x^2-1} \ , \ c = -\frac{x^4 + 2x^3 + 2x^2 - 2x + 1}{(x^2-1)^2}.$$

Moreover, this point (x,y,z) must lie in $\mathcal{S}$, as must $(x, \pm y, \pm z)$. We now observe that all four points $(x, \pm y, \pm z)$ yield the same value of c, since c is defined only in terms of x. (Replacing y with $-y$ switches b and $-b$, the two points of type 2_3 that map to $f_c(b)$; replacing z with $-z$ has the effect of switching the two fixed points.) Since there are at most twelve points in $\mathcal{S}$ from Theorem 5.2, and since four such points yield the same value of c, it follows that there are at most three quadratic pairs (K,c) with $G \hookrightarrow G(f_c,K)$.

Theorem 5.2 also tells us that any point in $\mathcal{S}$ satisfies $x \in \mathbb{Q}$ and $y, z \notin \mathbb{Q}$. Then certainly we have $c \in \mathbb{Q}$ and $a, b \notin \mathbb{Q}$. However, since $b^2 = y^2/(x^2-1)^2 = 2(x^3+x^2-x+1)/(x^2-1) \in \mathbb{Q}$, we have $f_c(b) = b^2 + c \in \mathbb{Q}$. $\square$

5.2 Period 2 and type 1_2

Let G be the graph shown in Figure 5.2. We show that G cannot be realized as a subgraph of $G(f_c,K)$ for any quadratic pair (K,c).

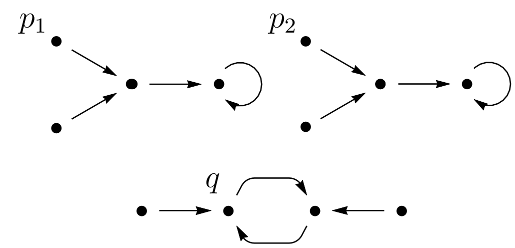


Figure 5.2: The graph generated by two points p_1 and p_2 of type 1_2 with disjoint orbits and a point q of period 2

We first observe that G is minimally generated by a point of period 2 and two points of type 1_2 that lie in distinct components of G. Therefore

$$Y_1^{\mathrm{dyn}}(G) = \{(p_1, p_2, q, c) \in \mathbb{A}^4 : p_1, p_2 \text{ are points of type } 1_2 \text{ and}$$
$$q \text{ is a point of period 2 for } f_c, \text{ with } f_c(p_1) \neq f_c(p_2)\}.$$

Proposition 5.4. *Let X be the affine curve of genus 5 defined by the equation*

$$\begin{cases} y^2 &= (5x^2 - 1)(x^2 + 3) \\ z^2 &= -(3x^2 + 1)(x^2 - 5), \end{cases} \tag{5.13}$$

and let Y be the open subset of X defined by

$$x(x - 1)(x + 1)(x^2 + 1)(x^2 + 3)(3x^2 + 1) \neq 0. \tag{5.14}$$

Consider the morphism $\Phi : Y \to \mathbb{A}^4$, $(x, y, z) \mapsto (p_1, p_2, q, c)$, given by

$$p_1 = \frac{y}{2(x^2 - 1)}\ ,\ p_2 = \frac{z}{2(x^2 - 1)}\ ,\ q = -\frac{x^2 - 4x - 1}{2(x^2 - 1)}\ ,\ c = -\frac{3x^4 + 10x^2 + 3}{4(x^2 - 1)^2}.$$

Let G be the graph shown in Figure 5.2. Then Φ maps Y isomorphically onto $Y_1^{\mathrm{dyn}}(G)$, with the inverse map given by

$$x = \frac{2f_c(p_1)+3}{2q+1} \ , \ y = 2p_1(x^2-1) \ , \ z = 2p_2(x^2-1). \tag{5.15}$$

Proof. The condition $(x-1)(x+1) \neq 0$ implies that Φ defines a morphism to $\mathbb{A}^4$. We verify using Magma that (5.15) provides a left inverse for Φ, so Φ is injective.

Suppose $(x, y, z) \in Y$ and $(p_1, p_2, q, c) = \Phi(x, y, z)$. It is easy to check that $f_c^2(q) = q$ and $f_c^3(p_i) = f_c^2(p_i)$ for each $i \in \{1, 2\}$, and that

$$\begin{aligned} f_c^2(p_1) - f_c(p_1) &= -\frac{x^2+3}{(x-1)(x+1)}, \\ f_c^2(p_2) - f_c(p_2) &= \frac{3x^2+1}{(x-1)(x+1)}, \\ f_c(q) - q &= -\frac{4x}{(x-1)(x+1)}, \end{aligned}$$

all of which are nonzero by (5.14). Hence p_1 and p_2 are points of type 1_2 and q is a point of period 2 for f_c. Finally, we see that

$$f_c(p_1) - f_c(p_2) = \frac{2(x^2+1)}{(x-1)(x+1)} \neq 0,$$

so p_1 and p_2 have disjoint orbits under f_c. Therefore Φ maps Y into $Y_1^{\mathrm{dyn}}(G)$.

It remains to show that Φ surjects onto $Y_1^{\mathrm{dyn}}(G)$. Let p_1 and p_2 be points of type 1_2 for f_c with disjoint orbits, and let q be a point of period 2 for f_c. Since $f_c(p_1)$ and $f_c(p_2)$ are distinct points of type 1_1, we know that $f_c(p_1)$ and $f_c(p_2)$ must be the negatives of distinct fixed points for f_c. Therefore, by Proposition 4.1, there exist r and s such that

$$f_c(p_1) = -\left(\frac{1}{2}+r\right) \ , \ f_c(p_2) = -\left(\frac{1}{2}-r\right) \ , \ q = -\frac{1}{2}+s \ , \ c = \frac{1}{4}-r^2 = -\frac{3}{4}-s^2. \tag{5.16}$$

Following the proof of [32, Thm. 2], we see that setting $x = (1-r)/s$ yields

$$r = -\frac{x^2+1}{x^2-1}\ ,\ s = \frac{2x}{x^2-1}\ ,\ c = -\frac{3x^4+10x^2+3}{4(x^2-1)^2}.$$

This gives the correct expression for c, and combining this with (5.16) yields

$$p_1^2 = \frac{(5x^2-1)(x^2+3)}{4(x^2-1)^2}\ ,\ p_2^2 = -\frac{(3x^2+1)(x^2-5)}{4(x^2-1)^2}\ ,\ q = -\frac{x^2-4x-1}{2(x^2-1)}.$$

Setting $y = 2p_1(x^2-1)$ and $z = 2p_2(x^2-1)$ gives a point $(x,y,z) \in Y$ for which $(p_1,p_2,q,c) = \Phi(x,y,z)$, completing the proof. □

We now show that the curve $Y_1^{\mathrm{dyn}}(G)$ has no quadratic points.

Theorem 5.5. *Let X be the affine curve of genus 5 defined by* (5.13)*. Then*

$$X(\mathbb{Q},2) = X(\mathbb{Q}) = \{(\pm1,\pm4,\pm4)\}.$$

Proof. We begin by observing that the affine curve X_1 defined by $y^2 = (5x^2-1)(x^2+3)$ is birational to the affine curve X_2 defined by $z^2 = -(3x^2+1)(x^2-5)$ via the map $(x,y) \mapsto (1/x, y/x^4)$. By a computation in Magma, one can verify that both curves are birational to the elliptic curve $X_1(15)$, labeled 15A8 in [7], which has only four rational points. Hence the points $(\pm1,\pm4)$ are the only four rational points on X_1 (resp., X_2), and therefore the only rational points on X are those described in the statement of the theorem.

Now suppose (x,y,z) is a quadratic point. We handle separately two cases depending on whether $x \in \mathbb{Q}$.

Case 1: $x \in \mathbb{Q}$. First, x cannot be equal to ±1, since then (x,y,z) is a rational point on X. It follows from our discussion about the rational points on X that if $x \neq \pm1$, then $y,z \notin \mathbb{Q}$, though certainly $y^2,z^2 \in \mathbb{Q}$. We may therefore write $y = u\sqrt{d}, z = v\sqrt{d}$ for some

$u, v \in \mathbb{Q}$ and $d \in \mathbb{Z}$ squarefree (with $d \neq 1$). We may then rewrite (5.13) as

$$\begin{cases} du^2 &= (5x^2-1)(x^2+3) \\ dv^2 &= -(3x^2+1)(x^2-5). \end{cases} \tag{5.17}$$

If p is a prime dividing d, then arguing in the usual way we find that $\mathrm{ord}_p(x)$, $\mathrm{ord}_p(y)$, and $\mathrm{ord}_p(z)$ are nonnegative, so we may reduce this equation modulo p to find

$$(5x^2-1)(x^2+3) \equiv -(3x^2+1)(x^2-5) \equiv 0 \mod p.$$

Hence p divides the resultant of these two quartic polynomials, which is equal to $2^{24} \cdot 3^2$. Thus $p \in \{2, 3\}$, and therefore $d \in \{-1, \pm 2, \pm 3, \pm 6\}$. One can verify that (5.17) has no 2-adic solutions for $d \in \{-1, \pm 2, 3, \pm 6\}$, so we need only consider the case $d = -3$.

Let $X_1^{(-3)}$ denote the twist of X_1 by -3. This curve is birational to curve 45A1 in [7], which has only two rational points. Therefore, the only rational points on $X_1^{(-3)}$ are the points $(0, \pm 1)$. However, when $x = 0$, we have $-3v^2 = 5$, which implies $v \notin \mathbb{Q}$. We conclude, therefore, that X can have no quadratic points with $x \in \mathbb{Q}$.

Case 2: $x \notin \mathbb{Q}$. In order to apply Lemma 3.1, we require cubic models for each of genus one curves X_1 and X_2. As mentioned above, X_1 is birational to X_2, so we will use the same cubic model for both. We find that X_1 and X_2 are birational to the elliptic curve E given by the equation

$$Y^2 = X^3 + \frac{5}{4}X^2 + \frac{1}{2}X + \frac{1}{4}.$$

Since E is birational to $X_1(15)$ (curve 15A8 in [7]), E has only four rational points, namely ∞, $(0, \pm 1/2)$, $(-1, 0)$. The birational maps between X_1 and E are given by

$$X=\frac{x^2-4x-1-y}{2(x+1)^2}\ ,\ Y=\frac{5x^3-7x^2+7x+3-3xy+y}{4(x+1)^3}\ ; \tag{5.18}$$

$$x=\frac{X-1+2Y}{3X+1-2Y}\ ,\ y=-\frac{8(2X^3-X-1+2Y)}{(3X+1-2Y)^2}, \tag{5.19}$$

and the birational maps between X_2 and E are given by

$$X'=-\frac{x^2-4x-1-z}{2(x-1)^2}\ ,\ Y'=\frac{3x^3-7x^2-7x-5-xz-3z}{4(x-1)^3}\ ; \tag{5.20}$$

$$x=-\frac{3X'+1-2Y'}{X'-1+2Y'}\ ,\ z=\frac{8(2X'^3-X'-1+2Y')}{(X'-1+2Y')^2}. \tag{5.21}$$

We must consider separately the different cases depending on whether X and X' are rational. First, suppose $X\in\mathbb{Q}$. By (5.18), we have

$$y=x^2-4x-1-2(x+1)^2X.$$

Substituting this expression for y into the first equation of (5.13) yields

$$(x+1)^2\left((X^2-X-1)x^2+2X(X+2)x+(X^2+X+1)\right)=0.$$

Since $x\notin\mathbb{Q}$, we must have $x+1\neq 0$, so the minimal polynomial for x is given by

$$p(t)=t^2+\frac{2X(X+2)}{X^2-X-1}t+\frac{X^2+X+1}{X^2-X-1}.$$

On the other hand, if $X\notin\mathbb{Q}$, then we apply Lemma 3.1 to see that there are $v\in\mathbb{Q}$ and $(X_0,Y_0)\in E(\mathbb{Q})\setminus\{\infty\}=\{(0,\pm1/2),(-1,0)\}$ such that

$$Y=Y_0+v(X-X_0) \tag{5.22}$$

and

$$q(t) = t^2 + (X_0 - v^2 + 5/4)t + (X_0^2 + v^2 X_0 + 5/4X_0 - 2Y_0 v + 1/2)$$

is the minimal polynomial of X.

If $(X_0, Y_0) = (0, 1/2)$, then (5.19) and (5.22) together yield

$$x = -\frac{v + 1/2}{v - 3/2},$$

contradicting the fact that $v \in \mathbb{Q}$ and $x \notin \mathbb{Q}$.

If $(X_0, Y_0) = (0, -1/2)$, then (5.19) and (5.22) give us

$$x = -\frac{(v + 1/2)X - 1}{(v - 3/2)X - 1},$$

and solving this equation for X gives

$$X = \frac{2(x + 1)}{(2v - 3)x + (2v + 1)}.$$

Thus the equation $q(X) = 0$ may be written

$$-\frac{(4v^2 - 8v - 1)x^2 - (4v^2 + 8v + 7)}{((2v - 3)x + (2v + 1))^2} = 0,$$

which means that

$$p(t) = t^2 - \frac{4v^2 + 8v + 7}{4v^2 - 8v - 1}$$

is the minimal polynomial of x.

If $(X_0, Y_0) = (-1, 0)$, then combining (5.19) and (5.22) gives us

$$x = -\frac{(v + 1/2)X + (v - 1/2)}{(v - 3/2)X + (v - 1/2)},$$

which then yields

$$X = -\frac{(2v-1)(x+1)}{(2v-3)x+(2v+1)}. \tag{5.23}$$

Substituting this expression for X into $q(X)=0$ gives us the equation

$$\frac{(2v-1)((4v^2-5)x^2+16vx-(4v^2+3))}{2((2v-3)x+(2v+1))^2} = 0.$$

Note that $(2v-1) \neq 0$ by (5.23), since $X \notin \mathbb{Q}$ is nonzero. Hence the minimal polynomial of x takes the form

$$p(t) = t^2 + \frac{16v}{4v^2-5}t - \frac{4v^2+3}{4v^2-5}. \tag{5.24}$$

We conclude, then, that the minimal polynomial $p(t)$ for $x \notin \mathbb{Q}$ is of one of the following forms, where $X \in \mathbb{Q}$ for the first expression and $v \in \mathbb{Q}$ for the second and third:

$$p(t) = t^2 + \frac{2X(X+2)}{X^2-X-1}t + \frac{X^2+X+1}{X^2-X-1}, \tag{5.25}$$
$$p(t) = t^2 - \frac{4v^2+8v+7}{4v^2-8v-1}, \tag{5.26}$$
$$p(t) = t^2 + \frac{16v}{4v^2-5}t - \frac{4v^2+3}{4v^2-5}. \tag{5.27}$$

Running the same argument with X' instead of X, we find that the minimal polynomial for x must also be of one of the following forms, where $X' \in \mathbb{Q}$ for the first expression and $w \in \mathbb{Q}$ for the second and third:

$$p(t) = t^2 - \frac{2X'(X'+2)}{X'^2+X'+1}t + \frac{X'^2-X'-1}{X'^2+X'+1}, \tag{5.28}$$
$$p(t) = t^2 - \frac{4w^2-8w-1}{4w^2+8w+7}, \tag{5.29}$$
$$p(t) = t^2 + \frac{16w}{4w^2+3}t - \frac{4w^2-5}{4w^2+3}. \tag{5.30}$$

We now compare each of the first three expressions with each of the last three; equating coefficients of each pair will give us nine systems of equations whose rational solutions we must find. Unless otherwise noted, all computations were done using Magma.

For (5.25) and (5.28), the system is

$$\begin{cases} 2X(X+2)(X'^2+X'+1) &= -2X'(X'+2)(X^2-X-1) \\ (X^2+X+1)(X'^2+X'+1) &= (X'^2-X'-1)(X^2-X-1). \end{cases}$$

This defines a zero-dimensional scheme whose only rational point is $(X, X') = (0, 0)$.

For (5.25) and (5.29), the system is

$$\begin{cases} 2X(X+2) &= 0 \\ (X^2+X+1)(4w^2+8w+7) &= -(4w^2-8w-1)(X^2-X-1). \end{cases}$$

Here one can check by hand that the only rational solution to this system is $(X, w) = (0, -1/2)$.

For (5.25) and (5.30), the system is

$$\begin{cases} 2X(X+2)(4w^2+3) &= 16w(X^2-X-1) \\ (X^2+X+1)(4w^2+3) &= -(X^2-X-1)(4w^2-5). \end{cases}$$

This defines a zero-dimensional scheme whose only rational point is $(X, w) = (-1, -1/2)$.

For (5.26) and (5.28), the system is

$$\begin{cases} 2X'(X'+2) &= 0 \\ (X'^2-X'-1)(4v^2-8v-1) &= -(X'^2+X'+1)(4v^2+8v+7), \end{cases}$$

which we have already seen has only one rational solution $(X', v) = (0, -1/2)$.

For (5.26) and (5.29), we get only the single equation

$$(4v^2 + 8v + 7)(4w^2 + 8w + 7) = (4v^2 - 8v - 1)(4w^2 - 8w - 1).$$

Subtracting the right side from the left and dividing by 16 yields

$$4v^2w + 4vw^2 + 2v^2 + 2w^2 + 3v + 3w + 3 = 0.$$

According to Magma, this equation defines a genus one curve C birational to the genus one modular curve $X_1(2, 12)$, labeled 24A4 in [7], which has only four rational points. One can see that C has at least one rational point, namely $(v, w) = (-1/2, -1/2)$. Let $\widetilde{C}$ be the projective closure of C in $\mathbb{P}^2$, given by the homogenous equation

$$4V^2W + 4VW^2 + 2V^2Z + 2W^2Z + 3VZ^2 + 3WZ^2 + 3Z^3 = 0.$$

Setting $Z = 0$, we get $VW(V + W) = 0$, so there are three points $[V : W : 0]$ at infinity, namely $[0 : 1 : 0]$, $[1 : 0 : 0]$, and $[1 : -1 : 0]$, and these are all rational. Having accounted for all four rational points on $\widetilde{C}$, we conclude that the only rational point on C is $(v, w) = (-1/2, -1/2)$.

For (5.26) and (5.30), the system is

$$\begin{cases} 16w &= 0 \\ (4v^2 + 8v + 7)(4w^2 + 3) &= (4v^2 - 8v - 1)(4w^2 - 5). \end{cases}$$

One can verify by hand that this system has no rational solutions.

For (5.27) and (5.28), the system is

$$\begin{cases} -2X'(X'+2)(4v^2-5) &= 16v(X'^2+X'+1) \\ (X'^2-X'-1)(4v^2-5) &= -(X'^2+X'+1)(4v^2+3). \end{cases}$$

This defines a zero-dimensional scheme whose only rational point is $(X', v) = (-1, -1/2)$.

For (5.27) and (5.29), the system is

$$\begin{cases} 16v &= 0 \\ (4v^2+3)(4w^2+8w+7) &= (4v^2-5)(4w^2-8w-1), \end{cases}$$

which we have already seen has no rational solutions.

For (5.27) and (5.30), the system is

$$\begin{cases} 16w(4v^2-5) &= 16v(4w^2+3) \\ (4v^2+3)(4w^2+3) &= (4v^2-5)(4w^2-5). \end{cases}$$

This defines a zero-dimensional scheme whose only rational points are the two points $(v, w) \in \{(\pm 1/2, \mp 1/2)\}$.

For all nine of these cases, the rational solutions to the systems of equations yield polynomials $p(t)$ that are reducible over $\mathbb{Q}$ and cannot, therefore, be minimal polynomials for $x \notin \mathbb{Q}$. We conclude that X cannot have any quadratic points with $x \notin \mathbb{Q}$. Since we have already ruled out quadratic points with $x \in \mathbb{Q}$, this completes the proof of the theorem. $\square$

Theorem 5.5 says that the curve X defined by (5.13) has no quadratic points, and that all rational points on X lie outside the open subset $Y \subset X$ defined by (5.14). In particular, $Y_1^{\mathrm{dyn}}(G)$ has no rational or quadratic points. We therefore conclude the following:

Theorem 5.6. *Let K be a quadratic field, let $c \in K$, and let G be the graph shown in Figure 5.2. Then $G(f_c, K)$ does not contain a subgraph isomorphic to G.*

5.3 Period 3 and type 1_2

Now let G be the graph shown in Figure 5.3. We show that G cannot be realized as a subgraph of $G(f_c, K)$ for any quadratic pair (K, c).

The graph G is minimally generated by a point of type 1_2 and a point of period 3. Therefore

$$Y_1^{\mathrm{dyn}}(G) = \{(a, b, c) \in \mathbb{A}^3 : \ a \text{ is of type } 1_2 \text{ and } b \text{ has period 3 for } f_c\}.$$

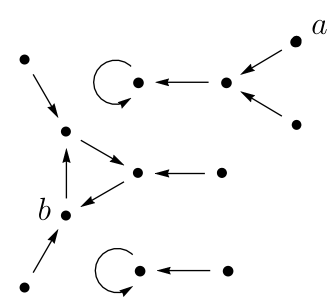


Figure 5.3: The graph generated by a point a of type 1_2 and a point b of period 3

Proposition 5.7. *Let X be the affine curve of genus 9 defined by the equation*

$$\begin{cases} y^2 &= t^6 + 2t^5 + 5t^4 + 10t^3 + 10t^2 + 4t + 1 \\ z^2 &= t^6 + 2t^5 + 2t^4 + 4t^3 + 7t^2 + 4t + 1 - 2t(t+1)y, \end{cases} \tag{5.31}$$

and let Y be the open subset of X defined by

$$t(t+1)(t^2+t+1)(y+t^2+t) \neq 0. \tag{5.32}$$

Consider the morphism $\Phi : Y \to \mathbb{A}^3$, $(t, y, z) \mapsto (a, b, c)$, *given by*

$$a = \frac{z}{2t(t+1)}\ , \quad b = \frac{t^3 + 2t^2 + t + 1}{2t(t+1)}\ , \quad c = -\frac{t^6 + 2t^5 + 4t^4 + 8t^3 + 9t^2 + 4t + 1}{4t^2(t+1)^2}.$$

Let G *be the graph shown in Figure 5.3. Then* Φ *maps* Y *isomorphically onto* $Y_1^{\text{dyn}}(G)$, *with the inverse map given by*

$$t = b + f_c(b)\ , \quad y = -t(t+1)(2f_c(a) + 1)\ , \quad z = 2t(t+1)a. \tag{5.33}$$

Proof. The condition $t(t+1) \neq 0$ implies that Φ is well-defined. One can verify (e.g., using Magma) that the relations in (5.33) provide a left inverse to Φ, which implies that Φ is injective.

Let (t, y, z) lie in Y. One can check that $f_c^3(a) = f_c^2(a)$ and $f_c^3(b) = b$. Furthermore,

$$f_c(a) - f_c^2(a) = -\frac{y + t^2 + t}{t(t+1)}\ , \quad f_c(b) - b = \frac{t^2 + t + 1}{t(t+1)},$$

which are nonzero by hypothesis, so a and b are points of type 1_2 and period 3, respectively. Hence Φ maps Y into $Y_1^{\text{dyn}}(G)$.

To prove surjectivity, suppose (a, b, c) lies on $Y_1^{\text{dyn}}(G)$. Since a is of type 1_2, $f_c(a)$ is of type 1_1, which means that $P = -f_c(a)$ is a fixed point for f_c. By Proposition 4.20(B), there exists (t, y) satisfying the first equation in (5.31) for which

$$P = \frac{t^2 + t + y}{2t(t+1)}\ , \quad b = \frac{t^3 + 2t^2 + t + 1}{2t(t+1)}\ , \quad c = -\frac{t^6 + 2t^5 + 4t^4 + 8t^3 + 9t^2 + 4t + 1}{4t^2(t+1)^2}.$$

Since $a^2 + c = -P$, we have

$$\begin{aligned}a^2 &= -\frac{t^2+t+y}{2t(t+1)} + \frac{t^6+2t^5+4t^4+8t^3+9t^2+4t+1}{4t^2(t+1)^2} \\ &= \frac{t^6+2t^5+2t^4+4t^3+7t^2+4t+1-2t(t+1)y}{4t^2(t+1)^2}.\end{aligned}$$

Setting $z := 2t(t+1)a$ completes the proof. □

Theorem 5.8. *Let X be the affine curve of genus 9 defined by* (5.31). *Then*

$$X(\mathbb{Q},2) = X(\mathbb{Q}) = \{(0,\pm1,\pm1),(-1,\pm1,\pm1)\}.$$

We will require the following lemma, which determines the set of rational points on a certain auxiliary curve.

Lemma 5.9. *Let C denote the hyperelliptic curve of genus 5 given by the equation*

$$w^2 = (t^3-3t-1)(t^3+2t^2-t-1)(t^6+2t^5+4t^4+8t^3+9t^2+4t+1). \tag{5.34}$$

Then $C(\mathbb{Q})$ consists only of the six points

$$\{(0,\pm1),(-1,\pm1),\infty^\pm\}.$$

Proof. Let J denote the Jacobian of C. Applying Magma's `RankBound` function, we find that $\operatorname{rk} J(\mathbb{Q}) \le 2$, so we may apply Theorem 3.6 to bound the number of rational points on C. The curve C has good reduction at the prime $p=5$, and a simple computation verifies that $\#C(\mathbb{F}_5) = 6$. By Theorem 3.6, we have

$$\#C(\mathbb{Q}) \le \#C(\mathbb{F}_5) + 2\cdot \operatorname{rk} J(\mathbb{Q}) + \left\lfloor \frac{2\cdot \operatorname{rk} J(\mathbb{Q})}{3} \right\rfloor \le 11.$$

We will now show that $\#C(\mathbb{Q})$ is divisible by six, which will then imply that $\#C(\mathbb{Q}) = 6$ and, therefore, $C(\mathbb{Q})$ is precisely the set listed in the lemma.

Considering the remark following Proposition 4.2, one might expect an automorphism of C that takes $t \mapsto -(t+1)/t$. This leads us to discover that

$$\sigma(t,w) := \left(-\frac{t+1}{t}, -\frac{w}{t^6}\right)$$

is an automorphism of C of order six, which we verify by describing σ^k for each $k = 1, \ldots, 6$:

$$\begin{aligned}
\sigma : (t,w) &\mapsto \left(-\frac{t+1}{t}, -\frac{w}{t^6}\right);\\
\sigma^2 : (t,w) &\mapsto \left(-\frac{1}{t+1}, \frac{w}{(t+1)^6}\right);\\
\iota = \sigma^3 : (t,w) &\mapsto (t, -w);\\
\sigma^4 : (t,w) &\mapsto \left(-\frac{t+1}{t}, \frac{w}{t^6}\right);\\
\sigma^5 : (t,w) &\mapsto \left(-\frac{1}{t+1}, -\frac{w}{(t+1)^6}\right);\\
\mathrm{id} = \sigma^6 : (t,w) &\mapsto (t,w).
\end{aligned}$$

We now claim that no rational point may be fixed by σ^k for any k, which implies that, for a rational point (t,w), the set

$$\{\sigma^k(t,w) : k \in \{0,\ldots,5\}\}$$

contains six distinct points. In particular, this would show that rational points come in multiples of six.

If (t,w) is fixed by any of σ, σ^2, σ^4, or σ^5, then comparing t-values shows that $t^2+t+1 = 0$, so that $t \notin \mathbb{Q}$. If (t,w) is fixed by $\sigma^3 = \iota$, then $w = 0$. Since the cubic and sextic polynomials appearing in (5.34) are irreducible over $\mathbb{Q}$, this implies that $t \notin \mathbb{Q}$. Therefore no rational point may be fixed by these automorphisms, completing the proof of the lemma. $\square$

Proof of Theorem 5.8. By Lemma 4.22(A), the only rational solutions to the equation $y^2 = t^6 + 2t^5 + 5t^4 + 10t^3 + 10t^2 + 4t + 1$ satisfy $t \in \{-1, 0\}$, so the points listed are the only rational points on X.

A simple calculation shows that if $t^2 + t + 1 = 0$, then z is not quadratic. It therefore follows from Lemma 4.22(A) that if (t, y, z) is a quadratic point on X, then $t \in \mathbb{Q} \setminus \{-1, 0\}$ and $y \notin \mathbb{Q}$. Let K be the quadratic field of definition of the point (t, x, y). If we take norms of both sides of the second equation in (5.31) and set $w := N_{K/\mathbb{Q}}(z) \in \mathbb{Q}$, we get

$$w^2 = (t^6 + 2t^5 + 2t^4 + 4t^3 + 7t^2 + 4t + 1)^2 - 4t^2(t + 1)^2y^2,$$

which we can rewrite as

$$\begin{aligned} w^2 &= (t^6 + 2t^5 + 2t^4 + 4t^3 + 7t^2 + 4t + 1)^2 \\ &\qquad - 4t^2(t + 1)^2(t^6 + 2t^5 + 5t^4 + 10t^3 + 10t^2 + 4t + 1) \\ &= (t^3 - 3t - 1)(t^3 + 2t^2 - t - 1)(t^6 + 2t^5 + 4t^4 + 8t^3 + 9t^2 + 4t + 1). \end{aligned}$$

Thus (t, w) is a rational point on the curve C defined by (5.34). By Lemma 5.9, the only finite rational points on C have $t \in \{-1, 0\}$. We have already shown that a quadratic point on X cannot have $t \in \{-1, 0\}$, so X has no quadratic points. $\square$

Theorem 5.8 says that the only points in $X(\mathbb{Q}, 2)$ lie outside the open subset $Y \subset X$ defined by (5.32). This implies that $Y_1^{\mathrm{dyn}}(G)$ has no rational or quadratic points, and we therefore have the following result:

Theorem 5.10. *Let K be a quadratic field, let $c \in K$, and let G be the graph shown in Figure 5.3. Then $G(f_c, K)$ does not contain a subgraph isomorphic to G.*

5.4 Period 3 and type 2_2

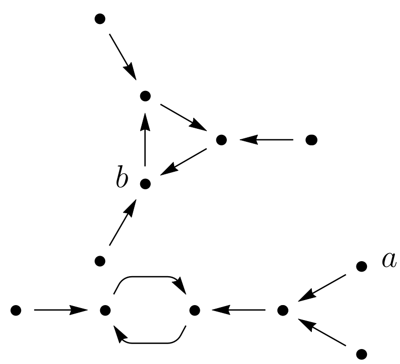


Figure 5.4: The graph generated by a point a of type 2_2 and a point b of period 3

Let G be the graph shown in Figure 5.4. We now show that G cannot be realized as a subgraph of $G(f_c, K)$ for any quadratic pair (K, c).

The graph G is minimally generated by a point of type 2_2 and a point of period 3, so we have

$$Y_1^{\mathrm{dyn}}(G) = \{(a, b, c) \in \mathbb{A}^3 : \ a \text{ is of type } 2_2 \text{ and } b \text{ has period 3 for } f_c\}.$$

Proposition 5.11. *Let X be the affine curve of genus 9 defined by the equation*

$$\begin{cases} y^2 &= t^6 + 2t^5 + t^4 + 2t^3 + 6t^2 + 4t + 1 \\ z^2 &= t^6 + 2t^5 + 6t^4 + 12t^3 + 11t^2 + 4t + 1 - 2t(t+1)y, \end{cases} \tag{5.35}$$

and let Y be the open subset of X defined by

$$t(t+1)(t^2 + t + 1)y(y - t^2 - t) \neq 0. \tag{5.36}$$

Consider the morphism $\Phi : Y \to \mathbb{A}^3$, $(t, y, z) \mapsto (a, b, c)$, given by

$$a = \frac{z}{2t(t+1)} \ , \ \ b = \frac{t^3 + 2t^2 + t + 1}{2t(t+1)} \ , \ \ c = -\frac{t^6 + 2t^5 + 4t^4 + 8t^3 + 9t^2 + 4t + 1}{4t^2(t+1)^2}.$$

Let G be the graph shown in Figure 5.4. Then Φ maps Y isomorphically onto $Y_1^{\mathrm{dyn}}(G)$, with the inverse map given by

$$t = b + f_c(b) \ , \ \ y = -t(t+1)(2f_c(a) - 1) \ , \ \ z = 2t(t+1)a. \tag{5.37}$$

Proof. The condition $t(t+1) \neq 0$ implies that Φ is well-defined. A Magma computation verifies that the relations given in (5.37) serve as a left inverse for Φ, so we have that Φ is injective.

If (t, y, z) lies on Y, then one can verify that $f_c^4(a) = f_c^2(a)$ and $f_c^3(b) = b$, and that

$$f_c^2(a) - f_c^3(a) = -\frac{y}{t(t+1)} \ , \ \ f_c(a) - f_c^3(a) = -\frac{y - t^2 - t}{t(t+1)} \ , \ \ b - f_c(b) = \frac{t^2 + t + 1}{t(t+1)},$$

which are all nonzero by hypothesis. Thus a and b are points of type 2_2 and period 3, respectively, for f_c, and therefore Φ maps Y into $Y_1^{\mathrm{dyn}}(G)$.

Finally, suppose (a, b, c) is a point on $Y_1^{\mathrm{dyn}}(G)$. Since $f_c(a)$ is a point of type 2_1, we must have that $P := -f_c(a)$ is a point of period 2. By Proposition 4.20, there exists a solution (t, y) to the first equation in (5.35) for which

$$P = -\frac{t^2 + t - y}{2t(t+1)} \ , \ \ b = \frac{t^3 + 2t^2 + t + 1}{2t(t+1)} \ , \ \ c = -\frac{t^6 + 2t^5 + 4t^4 + 8t^3 + 9t^2 + 4t + 1}{4t^2(t+1)^2}.$$

Since $a^2 + c = -P$, we may write

$$\begin{aligned}a^2 &= \frac{t^2+t-y}{2t(t+1)} + \frac{t^6+2t^5+4t^4+8t^3+9t^2+4t+1}{4t^2(t+1)^2} \\ &= \frac{t^6+2t^5+6t^4+12t^3+11t^2+4t+1-2t(t+1)y}{4t^2(t+1)^2}.\end{aligned}$$

Setting $z := 2t(t+1)a$ completes the proof. □

Theorem 5.12. *Let X be the affine curve of genus 9 defined by* (5.35)*. Then*

$$X(\mathbb{Q}, 2) = X(\mathbb{Q}) = \{(0, \pm 1, \pm 1), (-1, \pm 1, \pm 1)\}.$$

As we did in the previous section, we require a lemma that determines the complete set of rational points on an auxiliary curve.

Lemma 5.13. *Let C denote the hyperelliptic curve of genus 5 given by the equation*

$$w^2 = t^{12}+4t^{11}+12t^{10}+32t^9+82t^8+172t^7+250t^6+244t^5+169t^4+88t^3+34t^2+8t+1. \quad (5.38)$$

Then $C(\mathbb{Q})$ consists only of the six points

$$\{(0, \pm 1), (-1, \pm 1), \infty^{\pm}\}.$$

Proof. Let J denote the Jacobian of C. We apply Magma's `RankBound` function to see that $\operatorname{rk} J(\mathbb{Q}) \le 2$, so the method of Chabauty and Coleman applies. The curve C has good reduction at $p = 11$, and we find that $\#C(\mathbb{F}_{11}) = 6$, so Theorem 3.6 implies that

$$\#C(\mathbb{Q}) \le \#C(\mathbb{F}_{11}) + 2 \cdot \operatorname{rk} J(\mathbb{Q}) \le 10.$$

Just as in the proof of Lemma 5.9, it will suffice to show that $\#C(\mathbb{Q})$ is a multiple of six.

Motivated in the same way as in the proof of Lemma 5.9, we are led to consider the automorphism $\sigma \in \mathrm{Aut}(C)$ given by

$$\sigma(t, w) := \left(-\frac{t+1}{t}, -\frac{w}{t^6}\right).$$

We see that σ has order six:

$$\begin{aligned}
\sigma : (t, w) &\mapsto \left(-\frac{t+1}{t}, -\frac{w}{t^6}\right); \\
\sigma^2 : (t, w) &\mapsto \left(-\frac{1}{t+1}, \frac{w}{(t+1)^6}\right); \\
\iota = \sigma^3 : (t, w) &\mapsto (t, -w); \\
\sigma^4 : (t, w) &\mapsto \left(-\frac{t+1}{t}, \frac{w}{t^6}\right); \\
\sigma^5 : (t, w) &\mapsto \left(-\frac{1}{t+1}, -\frac{w}{(t+1)^6}\right); \\
\mathrm{id} = \sigma^6 : (t, w) &\mapsto (t, w).
\end{aligned}$$

In exactly the same way as in the proof of Lemma 5.9 (noting that the degree 12 polynomial in (5.38) is irreducible over $\mathbb{Q}$), we see that no rational point may be fixed by σ^k for any k. Therefore, rational points on C come in multiples of six, completing the proof of the lemma. $\square$

Proof of Theorem 5.12. By Lemma 4.22(B), the only rational solutions to the equation $y^2 = t^6 + 2t^5 + t^4 + 2t^3 + 6t^2 + 4t + 1$ are $(-1, \pm 1)$ and $(0, \pm 1)$. Since $t \in \{-1, 0\}$ also implies $z \in \{\pm 1\}$, we have found all rational points on X.

It also follows from Lemma 4.22(B) that if (t, y, z) is a quadratic point on X, then $t \in \mathbb{Q} \setminus \{-1, 0\}$ and $y \notin \mathbb{Q}$. Let K be the quadratic field generated by y. If we take norms of both sides of the second equation in (5.35) and set $w := N_{K/\mathbb{Q}}(z) \in \mathbb{Q}$, we get

$$w^2 = (t^6 + 2t^5 + 6t^4 + 12t^3 + 11t^2 + 4t + 1)^2 - 4t^2(t+1)^2y^2,$$

which we can rewrite as

$$\begin{aligned} w^2 &= (t^6 + 2t^5 + 6t^4 + 12t^3 + 11t^2 + 4t + 1)^2 - 4t^2(t+1)^2(t^6 + 2t^5 + t^4 + 2t^3 + 6t^2 + 4t + 1) \\ &= t^{12} + 4t^{11} + 12t^{10} + 32t^9 + 82t^8 + 172t^7 + 250t^6 + 244t^5 + 169t^4 + 88t^3 + 34t^2 + 8t + 1. \end{aligned}$$

We therefore have a rational point (t, w) on the curve C from Lemma 5.13. However, the only rational points on C have $t \in \{-1, 0\}$, and we have already shown that a quadratic point on X cannot have $t \in \{-1, 0\}$. Therefore X has no quadratic points. $\square$

By Theorem 5.12, no points in $X(\mathbb{Q}, 2)$ lie in the open subset $Y \subset X$ defined by (5.36). Therefore $Y_1^{\text{dyn}}(G)$ has no rational or quadratic points, a fact that we interpret as follows:

Theorem 5.14. *Let K be a quadratic field, let $c \in K$, and let G be the graph shown in Figure 5.4. Then $G(f_c, K)$ does not contain a subgraph isomorphic to G.*

5.5 Type 1_2 and 2_2

Let G be the graph appearing in Figure 5.5. Since G is generated by a point of type 1_2 and a point of type 2_2, we have

$$Y_1^{\text{dyn}}(G) = \{(a, b, c) \in \mathbb{A}^3 : a \text{ is of type } 1_2 \text{ and } b \text{ is of type } 2_2 \text{ for } f_c\}.$$

Proposition 5.15. *Let X be the affine curve of genus 5 defined by the equation*

$$\begin{cases} y^2 &= (5q^2 - 1)(q^2 + 3) \\ z^2 &= 5q^4 - 8q^3 + 6q^2 + 8q + 5, \end{cases} \tag{5.39}$$

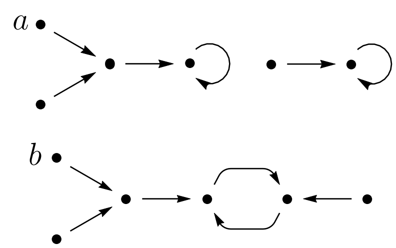


Figure 5.5: The graph generated by a point a of type 1_2 and a point b of type 2_2

and let Y be the open subset of X defined by

$$q(q-1)(q+1)(q^2+3)(q^2-4q-1) \neq 0. \tag{5.40}$$

Consider the morphism $\Phi : Y \to \mathbb{A}^3$, $(q, y, z) \mapsto (a, b, c)$, given by

$$a = \frac{y}{2(q^2-1)} \ , \quad b = \frac{z}{2(q^2-1)} \ , \quad c = -\frac{3q^4+10q^2+3}{4(q^2-1)^2}.$$

Then Φ maps Y isomorphically onto $Y_1^{\mathrm{dyn}}(G)$, with the inverse map given by

$$q = -\frac{3+2f_c(a)}{1+2f_c^2(b)} \ , \quad y = 2a(q^2-1) \ , \quad z = 2b(q^2-1). \tag{5.41}$$

Proof. The condition $(q-1)(q+1) \neq 0$ shows that Φ is well-defined, and since (5.41) provides a left inverse to Φ, Φ must be injective.

Now let $(a, b, c) = \Phi(q, y, z)$ for some point (q, y, z) on Y. Then a simple computation verifies that $f_c^3(a) = f_c^2(a)$, so $f_c^2(a)$ is a fixed point, and

$$f_c^2(a) - f_c(a) = -\frac{q^2+3}{q^2-1} \neq 0,$$

so $f_c(a)$ is not a fixed point. Hence a is of type 1_2.

One can also verify that $f_c^4(b) = f_c^2(b)$ and

$$f_c^3(b) - f_c^2(b) = \frac{4q}{q^2 - 1} \neq 0,$$

so $f_c^2(b)$ is a point of period 2, and

$$f_c^3(b) - f_c(b) = -\frac{q^2 - 4q - 1}{q^2 - 1} \neq 0,$$

so $f_c(b)$ is not a point of period 2. Therefore b is of type 2_2. It follows that Φ maps Y into $Y_1^{\mathrm{dyn}}(G)$.

Finally, suppose (a, b, c) lies on $Y_1^{\mathrm{dyn}}(G)$. By Proposition 4.20(A), we may write

$$c = -\frac{3q^4 + 10q^2 + 3}{4(q^2 - 1)^2} \ , \ \ r = -\frac{q^2 + 1}{q^2 - 1} \ , \ \ s = \frac{2q}{q^2 - 1}, \tag{5.42}$$

where r and s are the parameters appearing in Proposition 4.1. Since a and b are of types 1_2 and 2_2, respectively, for f_c, then $f_c(a)$ (resp. $f_c(b)$) is a point of type 1_1 (resp. 2_1). This means that $f_c(a)$ is the negative of a fixed point, and $f_c(b)$ is the negative of a point of period 2. We may therefore assume without loss of generality that

$$a^2 + c = -\left(\frac{1}{2} + r\right) \ , \ \ b^2 + c = -\left(-\frac{1}{2} + s\right).$$

Substituting the expressions for c, r, and s from (5.42) gives the equations

$$\begin{cases} a^2 & = \dfrac{(5q^2 - 1)(q^2 + 3)}{4(q - 1)^2(q + 1)^2} \\ \\ b^2 & = \dfrac{5q^4 - 8q^3 + 6q^2 + 8q + 5}{4(q - 1)^2(q + 1)^2}. \end{cases}$$

Finally, setting $y = 2a(q-1)(q+1)$ and $z = 2b(q-1)(q+1)$ gives us (5.39). $\square$

Theorem 5.16. *Let X be the affine curve of genus 5 defined by* (5.39)*. Then*

$$X(\mathbb{Q}, 2) = X(\mathbb{Q}) = \{(\pm 1, \pm 4, \pm 4)\}.$$

Proof. Let X_1 be the affine curve defined by $y^2 = (5q^2-1)(q^2+3)$, and let X_2 be the affine curve defined by $z^2 = 5q^4 - 8q^3 + 6q^2 + 8q + 5$. As noted in [32, pp. 21–22], the curves X_1 and X_2 are birational to elliptic curves 15A8 and 17A4, respectively, in [7]. (As mentioned previously, curve 15A8 is isomorphic to $X_1(15)$.) Each of these two elliptic curves has four rational points, so it follows that the only rational points on X_1 (resp., X_2) are $(\pm 1, \pm 4)$. Therefore the only rational points on X are the eight points listed in the theorem. It remains to show that X admits no quadratic points.

We handle the proof in two cases, depending on whether $q \in \mathbb{Q}$.

Case 1: $q \in \mathbb{Q}$. We have already handled the case $q = \pm 1$, and we have said that if $q \in \mathbb{Q} \setminus \{\pm 1\}$, then $y, z \notin \mathbb{Q}$. Thus y and z generate the same quadratic field $K = \mathbb{Q}(\sqrt{d})$ with $d \neq 1$ squarefree. Writing $y = u\sqrt{d}$ and $z = v\sqrt{d}$ with $u, v \in \mathbb{Q}$, we rewrite (5.39) as

$$\begin{cases} du^2 &= f(q) := (5q^2-1)(q^2+3) \\ dv^2 &= g(q) := 5q^4 - 8q^3 + 6q^2 + 8q + 5 \end{cases} \tag{5.43}$$

Observe that if $p \neq 5$ is a prime dividing d, then each of q, u, and v is integral at p. We may therefore reduce modulo p to see that

$$f(q) \equiv g(q) \equiv 0 \mod p.$$

Therefore p divides the resultant of the polynomials f and g. Since $\mathrm{Res}(f,g) = 2^{24} \cdot 5$, we must have $p = 2$ or $p = 5$. Therefore $d \in \{-1, \pm 2, \pm 5, \pm 10\}$. Since $g(q)$ is positive for all real

q, we must have $d \in \{2, 5, 10\}$. Furthermore, for $d \in \{2, 10\}$, (5.43) has no 2-adic solutions. We are left, therefore, with $d = 5$.

Let $X_1^{(5)}$ denote the twist of X_1 by $d = 5$. A Magma computation verifies that $X_1^{(5)}$ is birational to the elliptic curve labeled 75B1 in [7], which has only two rational points. Since the normalization of the projective closure of $X_1^{(5)}$ has two rational points at infinity, the affine curve $X_1^{(5)}$ cannot have rational points. We conclude that X has no quadratic points with $q \in \mathbb{Q}$.

Case 2: $q \notin \mathbb{Q}$. We now require cubic models for each of the curves $y^2 = f(q)$ and $z^2 = g(q)$, so that we may apply Lemma 3.1 to find the possible minimal polynomials of such q. First, a computation in Magma shows that the curve X_1 is birational to the elliptic curve E_1 given by

$$Y^2 = X^3 + 5X^2 + 8X + 16, \tag{5.44}$$

via the maps given by

$$X = \frac{2\,(q^2 - 4q - 1 - y)}{(q+1)^2}\ ,\ Y = \frac{2\,(5q^3 - 7q^2 + 7q + 3 - (3q-1)y)}{(q+1)^3}\ ; \tag{5.45}$$

$$q = \frac{X - 4 + Y}{3X + 4 - Y}\ ,\ y = -\frac{4\,(X^3 - 8X - 32 + 8Y)}{(3X + 4 - Y)^2}. \tag{5.46}$$

Since E_1 is birational to X_1, and therefore to curve 15A8, E_1 has only four rational points, and a quick search for points finds that

$$E_1(\mathbb{Q}) = \{\infty, (0, \pm 4), (-4, 0)\}.$$

Another Magma computation shows that X_2 is birational to the elliptic curve E_2 given by

$$Z^2 = W^3 - 3W^2 - 8W + 16, \tag{5.47}$$

with the birational maps given by

$$W = \frac{2\,(3q^2+1-z)}{(q+1)^2}\ ,\ Z = \frac{2\,(7q^3-9q^2-3q-3-(3q-1)z)}{(q+1)^3}\ ; \tag{5.48}$$

$$q = \frac{W+4+Z}{3W-4-Z}\ ,\ z = -\frac{4\,(W^3+8W-32-8Z)}{(3W-4-Z)^2}. \tag{5.49}$$

The curve E_2 is birational to curve 17A4, which has only four rational points. One can therefore see that

$$E_2(\mathbb{Q}) = \{\infty, (0,\pm 4), (4,0)\}.$$

As in previous sections, we must consider separate cases, depending on whether X and W are rational.

Suppose $X \in \mathbb{Q}$. By (5.45), we can write

$$y = q^2 - 4q - 1 - \frac{1}{2}(q+1)^2 X,$$

and substituting this expression into the equation defining X_1 (the first equation of (5.39)) gives us

$$\frac{1}{4}(q+1)^2\left((X^2-4X-16)q^2 + 2X(X+8)q + (X^2+4X+16)\right) = 0.$$

Since $q \notin \mathbb{Q}$, we must have $q+1 \neq 0$, so the minimal polynomial of q is given by

$$p(t) = t^2 - \frac{2X(X+8)}{X^2-4X-16}t + \frac{X^2+4X+16}{X^2-4X-16}.$$

Now suppose instead that $X \notin \mathbb{Q}$. By Lemma 3.1 there is a rational point $(X_0, Y_0) \in E_1(\mathbb{Q}) \setminus \{\infty\} = \{(0,\pm 4), (-4,0)\}$ and a rational number v such that

$$Y = Y_0 + v(X - X_0) \tag{5.50}$$

and

$$m(t) = t^2 + (X_0 - v^2 + 5)t + (X_0^2 + v^2 X_0 + 5X_0 - 2Y_0 v + 8)$$

is the minimal polynomial of X.

If $(X_0, Y_0) = (0, 4)$, then combining (5.46) and (5.50) gives us

$$q = -\frac{v+1}{v-3},$$

which contradicts the fact that $v \in \mathbb{Q}$ and $q \notin \mathbb{Q}$.

If $(X_0, Y_0) = (0, -4)$, then (5.46) and (5.50) yield

$$q = -\frac{(v+1)X - 8}{(v-3)X - 8}.$$

Solving for X gives us

$$X = \frac{8(q+1)}{(v-3)q + (v+1)},$$

and we may therefore write the equation $m(X) = 0$ as

$$-\frac{16\left((v^2 - 4v - 1)q^2 - (v^2 + 4v + 7)\right)}{\left((v-3)q + (v+1)\right)^2} = 0.$$

Therefore, the minimal polynomial of q is given by

$$p(t) = t^2 - \frac{v^2 + 4v + 7}{v^2 - 4v - 1}.$$

If instead we take $(X_0, Y_0) = (-4, 0)$, then (5.46) and (5.50) give us

$$q = -\frac{(v+1)X + 4(v-1)}{(v-3)X + 4(v-1)},$$

from which we find that

$$X = -\frac{4(v-1)(q+1)}{(v-3)q+(v+1)},$$

and therefore the equation $m(X) = 0$ becomes

$$\frac{8(v-1)\left((v^2-5)q^2+8vq-(v^2+3)\right)}{\left((v-3)q+(v+1)\right)^2} = 0.$$

We must have $v - 1 \neq 0$, since $X \notin \mathbb{Q}$ is nonzero. Therefore the minimal polynomial of q has the form

$$p(t) = t^2 + \frac{8v}{v^2-5}t - \frac{v^2+3}{v^2-5}.$$

To summarize, we now know from applying Lemma 3.1 to the curve E_1 that the minimal polynomial for q must be of one of the following forms, where $X \in \mathbb{Q}$ for the first equation and $v \in \mathbb{Q}$ for the second and third:

$$p(t) = t^2 - \frac{2X(X+8)}{X^2-4X-16}t + \frac{X^2+4X+16}{X^2-4X-16}, \tag{5.51}$$

$$p(t) = t^2 - \frac{v^2+4v+7}{v^2-4v-1}, \tag{5.52}$$

$$p(t) = t^2 + \frac{8v}{v^2-5}t - \frac{v^2+3}{v^2-5}. \tag{5.53}$$

Applying the same arguments with E_2 shows that the minimal polynomial of q must also be expressed in one of the following forms, where $W \in \mathbb{Q}$ for the first equation and $w \in \mathbb{Q}$ for the second and third:

$$p(t) = t^2 + \frac{2W^2}{W^2-12W+16}t + \frac{W^2-4W-16}{W^2-12W+16}, \tag{5.54}$$

$$p(t) = t^2 + \frac{8(w+1)}{w^2-5}t - 1, \tag{5.55}$$

$$p(t) = t^2 + \frac{8}{w^2-4w-1}t - \frac{w^2+4w-1}{w^2-4w-1}. \tag{5.56}$$

Therefore the minimal polynomial of q must be equal to one of the first three expressions as well as one of the last three expressions. Comparing the coefficients yields nine systems of equations, and we must find all rational solutions to each system. All computations were carried out using Magma.

For (5.51) and (5.54), the system is

$$\begin{cases} -2X(X+8)(W^2-12W+16) & = 2W^2(X^2-4X-16) \\ (X^2+4X+16)(W^2-12W+16) & = (W^2-4W-16)(X^2-4X-16). \end{cases}$$

This defines a zero-dimensional scheme whose only rational point is $(X, W) = (0, 0)$.

For (5.51) and (5.55), the system is

$$\begin{cases} -2X(X+8)(w^2-5) & = 8(w+1)(X^2-4X-16) \\ X^2+4X+16 & = -(X^2-4X-16). \end{cases}$$

One can verify by hand that the only rational solution to this system is $(X, w) = (0, -1)$.

For (5.51) and (5.56), the system is

$$\begin{cases} -2X(X+8)(w^2-4w-1) & = 8(X^2-4X-16) \\ (X^2+4X+16)(w^2-4w-1) & = -(w^2+4w-1)(X^2-4X-16). \end{cases}$$

This defines a zero-dimensional scheme whose only rational point is $(X, w) = (-4, -1)$.

For (5.52) and (5.54), the system is

$$\begin{cases} 2W^2 & = 0 \\ -(v^2+4v+7)(W^2-12W+16) & = (W^2-4W-16)(v^2-4v-1). \end{cases}$$

One can easily check that the only rational solution to this system is $(v, W) = (-1, 0)$.

For (5.52) and (5.55), the system is

$$\begin{cases} 8(w+1) &= 0 \\ v^2 + 4v + 7 &= v^2 - 4v - 1. \end{cases}$$

Here, one can check by hand that the only rational solution to this system is $(v, w) = (-1, -1)$.

For (5.52) and (5.56), the system is

$$\begin{cases} 8 &= 0 \\ (v^2 + 4v + 7)(w^2 - 4w - 1) &= (w^2 + 4w - 1)(v^2 - 4v - 1), \end{cases}$$

which clearly has no solutions.

For (5.53) and (5.54), the system is

$$\begin{cases} 8v(W^2 - 12W + 16) &= 2W^2(v^2 - 5) \\ -(v^2 + 3)(W^2 - 12W + 16) &= (W^2 - 4W - 16)(v^2 - 5). \end{cases}$$

This defines a zero-dimensional scheme whose only rational point is $(v, W) = (1, 4)$.

For (5.53) and (5.55), the system is

$$\begin{cases} 8v(w^2 - 5) &= 8(w+1)(v^2 - 5) \\ v^2 + 3 &= v^2 - 5, \end{cases}$$

which clearly has no solutions.

For (5.53) and (5.56), the system is

$$\begin{cases} 8v(w^2-4w-1) &= 8(v^2-5) \\ (v^2+3)(w^2-4w-1) &= (w^2+4w-1)(v^2-5). \end{cases}$$

This defines a zero-dimensional scheme whose only rational points are the two points $(v, w) = (\pm 1, \pm 1)$.

For every rational solution to one of these nine systems, the corresponding polynomial $p(t)$ is reducible over $\mathbb{Q}$ and cannot, therefore, be the minimal polynomial for $q \notin \mathbb{Q}$. We conclude that X cannot have any quadratic points with $q \notin \mathbb{Q}$, which completes the proof of the theorem. $\square$

Theorem 5.16 says that the only points on X of degree at most two lie outside the open subset $Y \subset X$ defined by (5.40). Therefore, $Y_1^{\mathrm{dyn}}(G)$ has no rational or quadratic points, which implies the following:

Theorem 5.17. *Let K be a quadratic field, let $c \in K$, and let G be the graph shown in Figure 5.5. Then $G(f_c, K)$ does not contain a subgraph isomorphic to G.*

5.6 Type 1_3

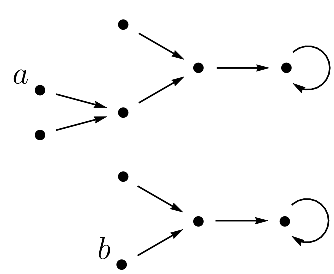


Figure 5.6: The graph generated by a point a of type 1_3 and a point b of type 1_2 with disjoint orbits

In this section, we show that the graph G shown in Figure 5.6 may never be realized as the subgraph of $G(f_c, K)$ for any quadratic pair (K, c). To prove this, we show that the curve

$$Y_1^{\mathrm{dyn}}(G) = \{(a, b, c) : \ a \text{ is of type } 1_3 \text{ and } b \text{ is of type } 1_2 \text{ for } f_c, \text{ with } b \neq \pm f_c(a)\}$$

has no quadratic points.

Proposition 5.18. *Let X be the affine curve of genus 5 defined by the equation*

$$\begin{cases} y^2 & = -(x^2-3)(x^2+1) \\ z^2 & = -2(x^3-x^2-x-1). \end{cases} \tag{5.57}$$

and let Y be the open subset of X defined by

$$(x-1)(x+1)(x^2+1)(x^2+3) \neq 0. \tag{5.58}$$

Consider the morphism $\Phi : Y \to \mathbb{A}^3$, $(x, y, z) \mapsto (a, b, c)$, given by

$$a = \frac{z}{x^2-1}\ , \quad b = \frac{y}{x^2-1}\ , \quad c = \frac{-2(x^2+1)}{(x^2-1)^2}.$$

Then Φ maps Y isomorphically onto $Y_1^{\mathrm{dyn}}(G)$, with the inverse map given by

$$x = -\frac{f_c(a)}{f_c^2(a)}\ , \quad y = b(x^2-1)\ , \quad z = a(x^2-1). \tag{5.59}$$

Proof. The condition $(x-1)(x+1) \neq 0$ implies that Φ is well-defined, and one can check that (5.59) provides a left inverse for Φ, hence Φ is injective.

Let (x, y, z) be a point in Y, and let $(a, b, c) = \Phi(x, y, z)$. A Magma computation shows that $f_c^4(a) = f_c^3(a)$ and

$$f_c^3(a) - f_c^2(a) = -\frac{4}{(x^2-1)} \neq 0,$$

so a is a point of type 1_3 for f_c. Similarly, we find that $f_c^3(b) = f_c^2(b)$ and

$$f_c^2(b) - f_c(b) = \frac{2(x^2+1)}{x^2-1},$$

which is nonzero by hypothesis, so b is a point of type 1_2 for f_c. Finally, we check that

$$(f_c(a) - b)(f_c(a) + b) = \frac{x^2+3}{x^2-1},$$

which is also nonzero by hypothesis. Hence $b \neq \pm f_c(a)$; hence Φ maps Y into $Y_1^{\mathrm{dyn}}(G)$. It remains now to show that Φ maps Y onto $Y_1^{\mathrm{dyn}}(G)$.

Let (a, b, c) be a point on $Y_1^{\mathrm{dyn}}(G)$. By [32, p. 19], since $f_c(a) \neq \pm b$ is a point of type 1_2, there exists $x \notin \{\pm 1\}$ such that

$$c = \frac{-2(x^2+1)}{(x^2-1)^2}\ , \quad f_c(a) = -\frac{2x}{x^2-1}\ , \quad b^2 = -\frac{(x^2-3)(x^2+1)}{(x^2-1)^2}. \tag{5.60}$$

Setting $y = b(x^2-1)$ yields the first equation in (5.57); writing $f_c(a) = a^2 + c$ and using the relations in (5.60) gives us

$$a^2 = -\frac{2(x^3-x^2-x-1)}{(x^2-1)^2},$$

and setting $z = a(x^2-1)$ gives us the second equation in (5.57). $\square$

Theorem 5.19. *Let X be the genus 5 affine curve defined by* (5.57). *Then*

$$X(\mathbb{Q}, 2) = X(\mathbb{Q}) = \{(\pm 1, \pm 2, \pm 2)\}.$$

Proof. Let X_1 be the affine curve defined by $y^2 = -(x^2-3)(x^2+1)$, and let X_2 be the affine curve defined by $z^2 = -2(x^3 - x^2 - x - 1)$. As indicated in [32, p. 19], the only rational points on X_1 are the four points $(\pm 1, \pm 2)$. Setting $x = \pm 1$ gives $z = \pm 2$ on X_2, so the only rational points on X are those listed in the theorem.

We now show that X has no quadratic points. We will again consider separately the cases $x \in \mathbb{Q}$ and $x \notin \mathbb{Q}$.

Case 1: $x \in \mathbb{Q}$. The curve X_1 (resp., X_2) is birational to elliptic curves 24A4 (resp., 11A3) from [7], which has only four (resp., five) rational points. (As mentioned previously, the curves 11A3 and 24A4 are isomorphic to the modular curves $X_1(11)$ and $X_1(2,12)$, respectively.) It follows that the only rational points on X_1 are $(x,y) \in \{(\pm 1, \pm 2)\}$, and the only rational points on X_2 are $(x,z) \in \{(\pm 1, \pm 2)\}$ (plus the rational point at infinity). Therefore, if $x \notin \{\pm 1\}$, then $y, z \notin \mathbb{Q}$. In other words, if (x,y,z) is a quadratic point on X with $x \in \mathbb{Q}$, then y and z generate the same quadratic extension $K = \mathbb{Q}(\sqrt{d})$ with $d \neq 1$ squarefree. Since $y^2, z^2 \in \mathbb{Q}$, we may write $y = u\sqrt{d}$ and $z = v\sqrt{d}$ for some $u, v \in \mathbb{Q}$. We may therefore rewrite (5.57) as

$$\begin{cases} du^2 & = f(x) := -(x^2-3)(x^2+1) \\ dv^2 & = g(x) := -2(x^3 - x^2 - x - 1). \end{cases}$$

Suppose p divides d. From the expression $du^2 = -(x^2-3)(x^2+1)$, we see that $\mathrm{ord}_p(x) \geq 0$, and therefore $\mathrm{ord}_p(u)$ and $\mathrm{ord}_p(v)$ are nonnegative as well. We may therefore reduce modulo p to get

$$f(x) \equiv g(x) \equiv 0 \mod p.$$

Hence p divides the resultant of the polynomials f and g. Since $\mathrm{Res}(f,g) = -2^8$, we find that $d \in \{-1, \pm 2\}$. However, one can verify using Magma that the curves $\pm 2u^2 = f(x)$ do not have points over $\mathbb{Q}_2$, so we must have $d = -1$.

Let $X_2^{(-1)}$ denote the twist of X_2 by $d = -1$. Magma verifies that $X_2^{(-1)}$ is birational to elliptic curve 176B1 in [7], which has only a single rational point. The projective closure of $X_2^{(-1)}$ has a point at infinity, which implies that the affine curve $X_2^{(-1)}$ has no rational points. Therefore X cannot have any quadratic points with $x \in \mathbb{Q}$.

Case 2: $x \notin \mathbb{Q}$. We would like to apply Lemma 3.1 to the curves X_1 and X_2, so we require a cubic model for the curve X_1. We find that the curve X_1 is birational to the curve E given by

$$Y^2 = X(X^2 - X + 1),$$

with

$$X = -\frac{y-2}{(x+1)^2}\ ,\ Y = \frac{2y - (x^3 + x^2 - x + 3)}{(x+1)^3}\ ; \tag{5.61}$$

$$x = -\frac{Y + 2X - 1}{Y + 1}. \tag{5.62}$$

Since the curve X_1 has only four rational points, the only rational points on E are $\{(0,0),(1,\pm 1)\}$ (plus the point at infinity).

Since we are assuming that x is quadratic, then by Lemma 3.1 there exist a rational point (x_0, z_0) on X_2 and a rational number v such that the minimal polynomial of x is given by

$$p(t) = t^2 + \frac{2x_0 + v^2 - 2}{2}t + \frac{2x_0^2 - v^2x_0 - 2x_0 + 2z_0v - 2}{2}. \tag{5.63}$$

We handle separately the cases when X is rational and when X is quadratic.

Case 2a: $X \in \mathbb{Q}$. Substituting

$$x = -\frac{Y + 2X - 1}{Y + 1}\ ,\ y = -X(x+1)^2 + 2 \tag{5.64}$$

into the equation $y^2 = f(x)$ yields

$$(x+1)^2\left((X^2+1)x^2+2(X+1)(X-1)x+(X^2-4X+1)\right)=0.$$

Since $x \notin \mathbb{Q}$, we have $(x+1) \neq 0$, so x must have minimal polynomial

$$p(t)=t^2+\frac{2(X+1)(X-1)}{X^2+1}t+\frac{X^2-4X+1}{X^2+1}.$$

Equating the coefficients of this expression and (5.63) yields the system

$$\begin{cases} (2x_0+v^2-2)(X^2+1) & =4(X+1)(X-1) \\ (2x_0^2-v^2x_0-2x_0+2y_0v-2)(X^2+1) & =2(X^2-4X+1). \end{cases}$$

For each pair $(x_0,z_0) \in \{(\pm 1, \pm 2)\}$, this system defines a zero-dimensional scheme, for which Magma can compute all rational points (X,v). For $(x_0,z_0)=(1,2)$, the only rational point is $(X,v)=(0,2)$, which yields $p(t)=(t-1)^2$. Similarly, for $(x_0,z_0)=(1,-2)$, the only rational point is $(X,v)=(0,-2)$, which again yields $p(t)=(t-1)^2$. For $(x_0,z_0)\in\{(-1,\pm 2)\}$ the scheme has no rational points. Since the only polynomials which have arisen in this way are reducible, we conclude that we have no quadratic points with $X \in \mathbb{Q}$.

Case 2b: $X \notin \mathbb{Q}$. Again using Lemma 3.1, there exist a rational point (X_0,Y_0) on E and a rational number w such that

$$\begin{gathered} X^2+(X_0-w^2-1)X+(X_0^2+w^2X_0-X_0-2Y_0w+1)=0\ , \\ Y=Y_0+w(X-X_0). \end{gathered} \tag{5.65}$$

The idea is to combine (5.65) with (5.64) to find an expression for the minimal polynomial of x, and then to compare that expression with the one given in (5.63). We must do this for each of the points $(X_0,Y_0)\in\{(0,0),(1,\pm 1)\}$.

For $(X_0,Y_0)=(0,0)$, (5.65) becomes

$$X^2 - (w^2+1)X + 1 = 0\ ,\ Y = wX.$$

Combining this with (5.64) tells us that

$$(w+1)\left((w^2+1)x^2 - 4wx - (w^2-3)\right) = 0.$$

If $w = -1$, then we have from (5.65) that $X^2 - 2X + 1 = 0$; but this is not irreducible over $\mathbb{Q}$, so we cannot have $w = -1$. Therefore, the minimal polynomial of x is given by

$$p(t) = t^2 + \frac{4w}{w^2+1}t - \frac{w^2-3}{w^2+1} = 0.$$

Equating coefficients of this polynomial with those in (5.63) yields the following system:

$$\begin{cases} 8w &= (2x_0 + v^2 - 2)(w^2+1) \\ -2(w^2-3) &= (2x_0^2 - v^2x_0 - 2x_0 + 2z_0v - 2)(w^2+1), \end{cases}$$

where $(x_0, z_0) \in \{(\pm 1, \pm 2)\}$. For each pair (x_0, z_0), the system defines a zero-dimensional scheme, and Magma can compute the set of rational points (v, w) on each scheme. In each case, there is exactly one rational point. For $(x_0, z_0) = (1, \pm 2)$, we have $(v, w) = (\pm 2, 1)$, and each of $(x_0, z_0) = (-1, \pm 2)$ yields $(v, w) = (0, -1)$. In any case, the corresponding polynomial $p(t)$ is equal to $t^2 \pm 2t + 1$ and is therefore reducible over $\mathbb{Q}$.

We perform a similar computation for $(X_0, Y_0) = (1, 1)$, and we again find only the polynomials $p(t) = t^2 \pm 2t + 1$. For $(X_0, Y_0) = (1, -1)$, (5.64) and (5.65) combine to give the equation $w + 1 = 0$, which we have already ruled out. Therefore there are no quadratic points on X with $x \notin \mathbb{Q}$, which concludes our proof. $\square$

We therefore conclude the main result of the section.

Theorem 5.20. *Let K be a quadratic field, let $c \in K$, and let G be the graph shown in Figure 5.6. Then $G(f_c, K)$ does not contain a subgraph isomorphic to G.*

Chapter 6

Proof of the main theorem

We now combine the results of the previous two chapters with the results found in [8] to prove the main theorem stated in the introduction.

In order to properly state and prove the main theorem, we provide labels in Figure 6.1 for a number of admissible graphs that were not found in the search carried out in [8], so that we may refer to them throughout the proof of Theorem 6.1. Any graph that was found in the search from [8] appears in §A.1 and will be identified by the label given there.

Theorem 6.1. *Let K be a quadratic field, and let $c \in K$. Suppose f_c does not admit K-rational points of period greater than four, and suppose that $G(f_c, K)$ is not isomorphic to one of the 46 graphs shown in Appendix A. Then one of the following must be true:*

(A) $G(f_c, K)$ properly contains one of the following graphs: 10(1,1)b, 10(2), 10(3)a, 10(3)b, 12(2,1,1)b, 12(4), 12(4,2);

(B) $G(f_c, K)$ contains the graph G_2;

(C) $G(f_c, K)$ contains the graph G_4; or

(D) $G(f_c, K)$ contains the graph G_{11}.

Moreover,

(a) there are at most 4 quadratic pairs (K, c) for which $G(f_c, K)$ properly contains the graph 12(2,1,1)b, at most 5 pairs for which $G(f_c, K)$ properly contains 12(4), and at most 1 pair for which $G(f_c, K)$ properly contains 12(4,2);

(b) there are at most 4 quadratic pairs (K, c) for which $G(f_c, K)$ contains G_2;

(c) there are at most 3 quadratic pairs (K, c) for which $G(f_c, K)$ contains G_4; and

(d) there is at most 1 quadratic pair (K, c) for which $G(f_c, K)$ contains G_{11}.

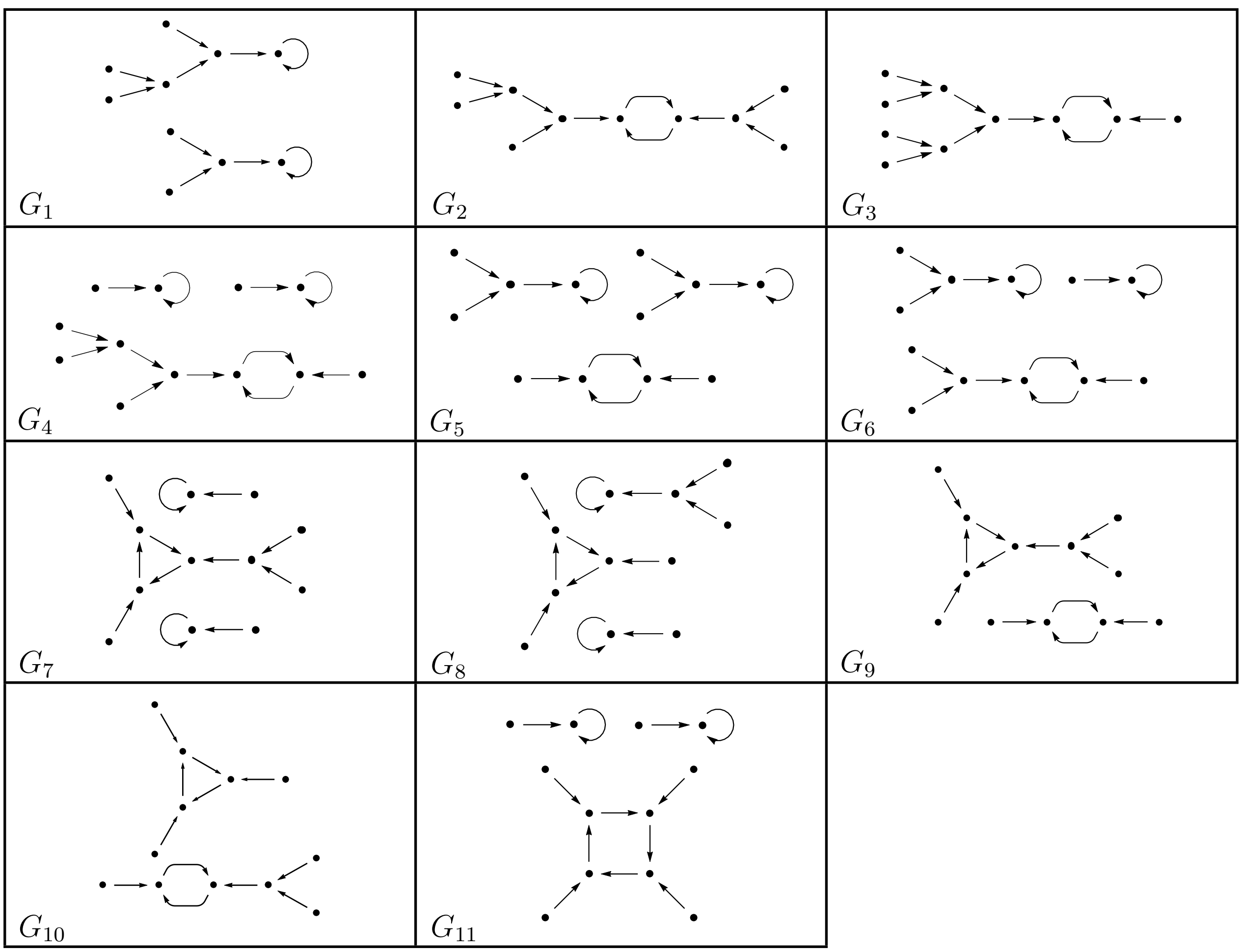


Figure 6.1: Some admissible graphs not appearing in Appendix A

Remark 6.2. Before we prove Theorem 6.1, we make a remark about some of the dynamical modular curves used in [8]. In some cases, while we were studying a certain graph G that we found in our search for preperiodic graphs (i.e., a graph appearing in Appendix A), we actually constructed and studied the quadratic points on a curve birational to $X_1^{\text{dyn}}(H)$ for a proper subgraph $H \subsetneq G$ which did *not* appear in our search. The idea is that if one determines (or at least bounds the number of) all occurrences of a particular graph H, then one has also done so for any graph G properly containing H. In Table 6.1, we list all instances in [8] where we studied a graph G by instead looking at the dynamical modular curve for a proper subgraph $H \subsetneq G$. Therefore, the statements made in [8] about 12(2), 14(2,1,1), 14(3,1,1), and 14(3,2) imply certain similar results about the strictly smaller graphs G_2, G_3, G_7, and G_9, and we will use this information in the proof of Theorem 6.1.

Table 6.1: Graphs G studied in [8] via a proper subgraph $H \subsetneq G$

Graph of interest (G)	Proper subgraph ($H \subsetneq G$)
12(2)	G_2
14(2,1,1)	G_3
14(3,1,1)	G_7
14(3,2)	G_9

Proof of Theorem 6.1. Let (K, c) be any quadratic pair. In [8, §5], we determined all instances of quadratic pairs (K, c) for which $G(f_c, K)$ is critically or fixed-point degenerate. We may therefore assume that $G(f_c, K)$ is admissible, which implies that $G(f_c, K)$ has either zero or two fixed points and an even number of vertices.

We prove the theorem in a number of cases depending on what cycle lengths are present in the graph $G(f_c, K)$. Since we are assuming that $G(f_c, K)$ contains no cycles of length greater than 4, the results of Chapter 4 imply that the only possible combinations of cycle lengths are (1,1), (2), (3), (4), (2,1,1), (3,1,1), (3,2), (4,1,1), and (4,2). (Actually, we have not ruled out the case of (4,2,1,1), but such a graph would necessarily contain 12(4,2) and G_{11}, which are already listed in the theorem.)

First, suppose the only cycles in $G(f_c, K)$ are fixed points. Among such admissible graphs, only 4(1,1) has four vertices, only 6(1,1) has six vertices, only 8(1,1)a and 8(1,1)b have eight vertices, and only 10(1,1)a, 10(1,1)b, and G_1 have ten vertices. By [8, Cor. 3.19], there is precisely one quadratic pair $(K, c) = (\mathbb{Q}(-\sqrt{7}), 3/16)$ for which $G(f_c, K)$ contains 10(1,1)a, and by Theorem 5.20, $G(f_c, K)$ cannot contain the graph G_1. The only remaining possibility, therefore, is that $G(f_c, K)$ contains 10(1,1)b, which is listed in part (A) of the theorem.

Next, suppose that the only cycle in $G(f_c, K)$ is a cycle of length two. Among such admissible graphs, only 4(2) has four vertices, only 6(2) has six vertices, only 8(2)a and 8(2)b have eight vertices, and only 10(2), G_2, and G_3 have ten vertices. As mentioned in Remark 6.2, the relevant information about graph G_2 is found in [8, §3.13], where it is shown that there are at most four quadratic pairs (K, c) for which $G(f_c, K)$ contains G_2 and is different from 12(2), which gives us part (b) of the theorem. It is proven in [8, §3.18] that the only quadratic pair for which $G(f_c, K)$ contains G_3 is $(K, c) = (\mathbb{Q}(\sqrt{17}), -21/16)$, where actually $G(f_c, K) \cong 14(2, 1, 1)$. The only remaining possibility for $G(f_c, K)$ is 10(2), which is listed in part (A) of the theorem.

Now suppose that the only cycles in $G(f_c, K)$ are cycles of length three. By Theorem 4.4, $G(f_c, K)$ may contain only a single cycle of length three. Among such admissible graphs, only 6(3) has six vertices, only 8(3) has eight vertices, and only 10(3)a and 10(3)b have ten vertices. Therefore, if $G(f_c, K)$ does not appear in §A.1, then $G(f_c, K)$ properly contains either 10(3)a or 10(3)b, both of which are listed in part (A) of the theorem.

Next, suppose the only cycles in $G(f_c, K)$ have length four. By Theorem 4.11, $G(f_c, K)$ may contain only a single 4-cycle. Among such admissible graphs, only 8(4) has eight vertices. Recall from Proposition 4.15 that if $G(f_c, K)$ contains a 4-cycle, then $c \in \mathbb{Q}$, and if

σ generates $\mathrm{Gal}(K/\mathbb{Q})$, then $\sigma P = f_c^2(P)$ for each point P of period 4. Now suppose $G(f_c, K)$ contains a point Q of type 4_2, with $f_c^2(Q) = P$ a point of period 4. Then the action of σ on $G(f_c, K)$ requires the existence of another point Q' of type 4_2, with $f_c^2(Q') = f_c^2(P)$. We conclude that if $G(f_c, K)$ contains 8(4) with just one additional vertex, then, by admissibility and the action of σ, $G(f_c, K)$ must contain 12(4). In [8, Cor. 3.44], we showed that there are at most five unknown quadratic pairs (K, c) for which $G(f_c, K)$ contains 12(4), which is one of the statements from part (a) of the theorem.

We now consider situations in which $G(f_c, K)$ contains cycles of two different lengths. Suppose first that $G(f_c, K)$ contains only cycles of lengths one and two. The minimal such admissible graph is 8(2,1,1), which has eight vertices. Only 10(2,1,1)a and 10(2,1,1)b have ten vertices, and only 12(2,1,1)a, 12(2,1,1)b, G_4, G_5, and G_6 have twelve vertices. It is shown in [8, Cor. 3.36] that there is only one quadratic pair $(K, c) = (\mathbb{Q}(\sqrt{-7}), -5/16)$ for which $G(f_c, K)$ contains 12(2,1,1)a, and [8, Cor. 3.40] says that there are at most four unknown quadratic pairs for which $G(f_c, K)$ contains 12(2,1,1)b, giving us another statement from part (a) of the theorem. Theorem 5.3 says that there are at most three quadratic pairs for which $G(f_c, K)$ contains G_4 and is different from 14(2,1,1), giving us part (c); Theorem 5.5 says that $G(f_c, K)$ cannot contain G_5; and Theorem 5.17 says that $G(f_c, K)$ cannot contain G_6.

Next, suppose $G(f_c, K)$ contains cycles only of lengths one and three. The minimal such admissible graph is 10(3,1,1), which has ten vertices, and the only such graphs that contain twelve vertices are G_7 and G_8. It is shown in [8, §3.19] (see also Remark 6.2) that $(K, c) = (\mathbb{Q}(\sqrt{33}), -29/16)$ is the only quadratic pair for which $G(f_c, K)$ contains G_7, and for that pair we actually have $G(f_c, K) \cong 14(3, 1, 1)$. Theorem 5.10 says that there are no quadratic pairs (K, c) for which $G(f_c, K)$ contains G_8.

Now suppose $G(f_c, K)$ only has cycles of lengths two and three. The minimal such admissible graph is 10(3,2), which has ten vertices, and the only such graphs containing twelve vertices are G_9 and G_{10}. By [8, §3.20] (again, see also Remark 6.2), $(K, c) = (\mathbb{Q}(\sqrt{-17}), -29/16)$ is the only quadratic pair for which $G(f_c, K)$ contains G_9, and for that pair we actually have $G(f_c, K) \cong 14(3, 2)$. Theorem 5.14 says that there are no quadratic pairs (K, c) for which $G(f_c, K)$ contains G_{10}.

We showed in Theorem 4.24 that there is at most one quadratic pair (K, c) for which $G(f_c, K)$ has points of period 1 and 4, proving (d), and Theorem 4.28 (also [8, Cor. 3.48]) says that there is at most one unknown quadratic pair (K, c) for which $G(f_c, K)$ has a points of period 2 and 4, thus containing 12(4,2), proving the last statement from part (a).

The theorem now follows from combining all of the cases described in this proof. $\square$

Chapter 7

Dynamical modular curves of small genus

For the purposes of this chapter, we will be considering the dynamical modular curves as curves defined over $\mathbb{Q}$. We are interested in studying those dynamical modular curves $Y_1^{\mathrm{dyn}}(G)$ of genus at most two. Every such curve necessarily has infinitely many quadratic points, so the corresponding graphs G occur as subgraphs of $G(f_c, K)$ for infinitely many quadratic pairs (K, c). However, over a given quadratic field K, $Y_1^{\mathrm{dyn}}(G)(K)$ may be no larger than $Y_1^{\mathrm{dyn}}(G)(\mathbb{Q})$. For applications in §8, we would like to answer the following question:

Question 7.1. *For a given dynamical modular curve $Y = Y_1^{\mathrm{dyn}}(G)$ of genus at most two, can we give a sufficient condition on the quadratic field K for $Y(K)$ to equal $Y(\mathbb{Q})$?*

Since every genus zero dynamical modular curve is isomorphic over $\mathbb{Q}$ to $\mathbb{P}^1$, and since $\mathbb{P}^1(K) \supsetneq \mathbb{P}^1(\mathbb{Q})$ for any number field $K \neq \mathbb{Q}$, we will restrict our attention to curves of genus one or two. For such curves, we will be interested in the following question, which we will use to give an answer to Question 7.1:

Question 7.2. *Given a dynamical modular curve $X_1^{\mathrm{dyn}}(G)$ of genus one or two, what are the torsion subgroups $J_1^{\mathrm{dyn}}(G)(K)_{\mathrm{tors}}$ as K ranges over all quadratic extensions $K/\mathbb{Q}$?*

In order to answer this question, we require a list of all dynamical modular curves of genus one or two.

Proposition 7.3.

(A) The curve $X_1^{\mathrm{dyn}}(G)$ has genus 0 if and only if G is isomorphic to one of the following graphs:

$$4(1,1)\ ,\ \ 4(2)\ ,\ \ 6(1,1)\ ,\ \ 6(2)\ ,\ \ 6(3)\ ,\ \ 8(2,1,1).$$

(B) The curve $X_1^{\mathrm{dyn}}(G)$ has genus 1 if and only if G is isomorphic to one of the following graphs:

$$8(1,1)a\ ,\ \ 8(1,1)b\ ,\ \ 8(2)a\ ,\ \ 8(2)b\ ,\ \ 10(2,1,1)a\ ,\ \ 10(2,1,1)b.$$

(C) The curve $X_1^{\mathrm{dyn}}(G)$ has genus 2 if and only if G is isomorphic to one of the following graphs:

$$8(3)\ ,\ \ 8(4)\ ,\ \ 10(3,1,1)\ ,\ \ 10(3,2).$$

That the genera of the curves listed in Proposition 7.3 are correct has been verified in [32] and [22] (see also [8]). It therefore suffices to show that if G is any admissible graph other than those listed above, then $X_1^{\mathrm{dyn}}(G)$ has genus greater than two.

Recall from Proposition 2.18 that if G and H are admissible graphs with $G \supsetneq H$, then there is a morphism $X_1^{\mathrm{dyn}}(G) \to X_1^{\mathrm{dyn}}(H)$ of degree at least two. Moreover, if $X_1^{\mathrm{dyn}}(G)$ is irreducible and $X_1^{\mathrm{dyn}}(H)$ has genus $g \geq 2$, then the genus of $X_1^{\mathrm{dyn}}(G)$ is strictly greater than g. On the other hand, even if $X_1^{\mathrm{dyn}}(G)$ is reducible, the genus of an irreducible component of $X_1^{\mathrm{dyn}}(G)$ can be no smaller than the genus of the irreducible component of $X_1^{\mathrm{dyn}}(H)$ onto which it maps.

We now describe a general method for finding the set $\mathcal{L}(\Gamma)$ of all admissible graphs G such that $X_1^{\mathrm{dyn}}(G)$ has genus $g \leq \Gamma$, where Γ is an arbitrary positive integer, *under the* *assumption of Conjecture 2.34.* The main idea for this method is the same as for proving Proposition 2.18: if $G \supsetneq H$, then G may be built from H via a sequence of minimal additions, as defined in §2.1. In particular, if G contains a cycle of length N, then G may be built, by a sequence of minimal additions, from the graph generated by a single point of period N.

For each $N \in \mathbb{N}$, denote by g_N the genus of the curve $X_1^{\mathrm{dyn}}(N)$. For each $\Gamma \in \mathbb{N}$, let $S(\Gamma) := \{N \in \mathbb{N} : g_N \le \Gamma\}$.

Algorithm 7.4 (Determining the set $\mathcal{L}(\Gamma)$)**.**

Input: A positive integer Γ.

Output: The set L of all admissible graphs G for which $X_1^{\mathrm{dyn}}(G)$ has genus at most Γ.

(1) Set $L := \emptyset$, set L' to be the set of all graphs G generated by a single point of period N with $N \in S(\Gamma)$, and set $L'' := L'$.

(2) While L' is nonempty:

  (a) Let G be the first element of L', and let g be the genus of $X_1^{\mathrm{dyn}}(G)$.

    (i) If $\frac{\Gamma+1}{2} < g \le \Gamma$, add G to L.

    (ii) If $g \le \frac{\Gamma+1}{2}$:

    - Add G to L.
    - Set S to be the set of graphs that may be obtained from G via a minimal addition. (For addition of a cycle, the length of the cycle must lie in $S(\Gamma)$.)
    - Add to L' any elements of S that are not already in L'', and replace L'' with $L'' \cup S$.

  (b) Remove G from L'.

(3) Return L.

Remark 7.5. The bound $\frac{\Gamma+1}{2}$ that distinguishes the two cases in step (2a) comes from the Riemann-Hurwitz formula: Suppose $X_1^{\text{dyn}}(G)$ has genus $g > \frac{\Gamma+1}{2}$. If G' is any graph properly containing G, then Proposition 2.18 says that there is a map of degree $d \geq 2$ from $X_1^{\text{dyn}}(G')$ to $X_1^{\text{dyn}}(G)$, and Riemann-Hurwitz then implies that the genus of $X_1^{\text{dyn}}(G')$ is larger than Γ — again, we are assuming Conjecture 2.34, which states that all dynamical modular curves are irreducible over $\mathbb{C}$. In this case, if $g \leq \Gamma$, then we conclude that G belongs to $\mathcal{L}(\Gamma)$, but any graph G' that properly contains G does not.

In order for the set $\mathcal{L}(\Gamma)$ to be finite, it must certainly be the case that $S(\Gamma)$ is finite. This is well known, and Bousch [2] has provided a formula for g_N for all $N \in \mathbb{N}$, which we record in Lemma 7.6. We also provide a lower bound γ_N for g_N that is more clearly seen to be an (eventually) increasing function of N. First, we record Bousch's formula for g_N.

Lemma 7.6 ([2, Thm. 2])**.** *Let N be a positive integer. Then*

$$g_N = 1 + \frac{N-3}{2}\nu(N) - \frac{1}{2}\sum_{\substack{d|N \\ d<N}} d\nu(d)\varphi(N/d), \tag{7.1}$$

where φ is the Euler phi function, μ is the Möbius function, and ν is given by

$$\nu(n) := \frac{1}{2}\sum_{e|n} \mu(n/e)2^e.$$

We use Lemma 7.6 to give a rough lower bound for g_N.

Proposition 7.7. *For $N \in \mathbb{N}$, define*

$$\gamma_N := 2^{N/2-3}\left(2^{N/2+1}(N-3) - 3(N^2-4)\right) + \frac{(N-1)(N+2)}{4}.$$

(A) For all $N \in \mathbb{N}$,

$$g_N \geq \gamma_N.$$

(B) The function $N \mapsto \gamma_N$ is strictly increasing when $N \geq 7$.

Remark 7.8. It is clear that $\gamma_N \to \infty$ as $N \to \infty$, since the main term is on the order of 2^{N-2}. However, we require the precise statement (B) for our application below.

Proof. First, note that each of the divisors d appearing in (7.1) is at most $\lfloor N/2 \rfloor$. Hence, for any $n \in \mathbb{N}$, we have

$$2^{n-1} - 2^{\lfloor n/2 \rfloor} + 1 = 2^{n-1} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} 2^k \leq \nu(n) \leq 2^{n-1} + \sum_{k=0}^{\lfloor n/2 \rfloor - 1} 2^k = 2^{n-1} + 2^{\lfloor n/2 \rfloor} - 1.$$

We therefore obtain the following inequalities, where we again use the fact that each proper divisor of N is at most $\lfloor N/2 \rfloor$:

$$\begin{aligned} g_N &\geq 1 + \frac{N-3}{2}\left(2^{N-1} - 2^{N/2} + 1\right) - \frac{1}{2}\sum_{\substack{d \mid N \\ d<N}} d \cdot \varphi(N/d) \cdot \left(2^{d-1} + 2^{d/2} - 1\right) \\ &\geq 1 + \frac{N-3}{2}\left(2^{N-1} - 2^{N/2} + 1\right) - \frac{1}{2} \cdot \frac{N}{2} \cdot \left(2^{N/2-1} + 2^{N/4} - 1\right) \sum_{\substack{d \mid N \\ d<N}} \varphi(N/d) \\ &= 1 + \frac{N-3}{2}\left(2^{N-1} - 2^{N/2} + 1\right) - \frac{1}{2} \cdot \frac{N(N-1)}{2} \cdot \left(2^{N/2-1} + 2^{N/4} - 1\right). \end{aligned} \tag{7.2}$$

Finally, we observe that the expression appearing in (7.2) will only be made smaller if we replace the exponent $N/4$ with $N/2$; rearranging the result yields part (A).

Part (B) is a calculus exercise. $\square$

As mentioned above, Algorithm 7.4 works if the curves in question are irreducible over $\mathbb{C}$. Unfortunately, we have thus far been unable to show that $X_1^{\mathrm{dyn}}(G)$ is irreducible over $\mathbb{C}$ for all admissible graphs G. However, it is the case that each of the individual dynamical modular curves explicitly constructed in this thesis as well as in [32] and [8] is irreducible over $\mathbb{C}$, a fact that can be verified using Magma's `IsAbsolutelyIrreducible` function on each of the curves in question. We record this fact for reference later. We refer to graphs by the labels given in Appendix A and Chapter 6.

Lemma 7.9. *Let X be one of the following curves:*

(A) The dynamical modular curves described in [32] or [8], which are those curves $X_1^{\mathrm{dyn}}(G)$ with G in the list

$$4(1,1),\ 4(2),\ 6(1,1),\ 6(2),\ 6(3),\ 8(1,1)a/b,\ 8(2)a/b,\ 8(2,1,1),\ 8(3),\ 8(4),$$
$$10(1,1)a,\ 10(2,1,1)a/b,\ 10(3,1,1),\ 10(3,2),\ 12(2,1,1)a/b,\ 12(4),\ 12(4,2),$$
$$G_2,\ G_3,\ G_7,\ G_9.$$

(B) The additional dynamical modular curves described in Chapters 4 and 5, which are the curves

$$X_1^{\mathrm{dyn}}(3^{(2)}),\ X_1^{\mathrm{dyn}}(1,4),\ X_1^{\mathrm{dyn}}(3,4),\ X_1^{\mathrm{dyn}}(1,2,3),$$

and those curves $X_1^{\mathrm{dyn}}(G)$ with G in the list

$$G_1,\ G_4,\ G_5,\ G_6,\ G_8,\ G_{10}.$$

Then X is irreducible over $\mathbb{C}$.

To prove Proposition 7.3, we apply Algorithm 7.4 in the case $\Gamma = 2$. In order to do so, we must find all $N \in \mathbb{N}$ for which $g_N \leq 2$.

Lemma 7.10. *The set of all $N \in \mathbb{N}$ for which $g_N \leq 2$ is*

$$S(2) = \{1, 2, 3, 4\}.$$

Proof. Let γ_N be as defined in Proposition 7.7. Since $\gamma_9 > 136$, it follows from the proposition that $g_N > 136$ for all $N \geq 9$. Therefore, if $N \in \mathbb{N}$ is such that $g_N \leq 2$, then $N \leq 8$. The values of g_N for $N \leq 8$ may be found in [34, p. 164]; we record these values in Table 7.1. In particular, we see that the only $N \leq 8$ for which $g_N \leq 2$ are $N \in \{1, 2, 3, 4\}$. $\square$

Table 7.1: Genera of $X_1^{\text{dyn}}(N)$ for small values of N

N	1	2	3	4	5	6	7	8
g_N	0	0	0	2	14	34	124	285

Corollary 7.11. *If $X_1^{\text{dyn}}(G)$ has genus at most two, then G cannot contain a cycle of length greater than four.*

Even among graphs G with cycles of length at most four, we can say more. For example, $X_1^{\text{dyn}}(3^{(2)})$ has genus 4, so G can contain at most one 3-cycle. Also, $X_1^{\text{dyn}}(1, 2, 3)$ has genus 9, so G cannot contain a 1-cycle, a 2-cycle, *and* a 3-cycle. Finally, $X_1^{\text{dyn}}(4)$ has genus 2, so for any graph G containing 8(4) as a *proper* subgraph, the curve $X_1^{\text{dyn}}(G)$ has genus greater than two.

Applying Algorithm 7.4 with $\Gamma = 2$ yields the set $\mathcal{L}(2)$, which we illustrate as a directed system in Figure 7.1, with arrows $G \to H$ if and only if $G \supsetneq H$. Graphs are identified by the labels given in Appendix A. The number of boxes around a given graph indicates the genus of the corresponding curve. Since the graphs appearing in $\mathcal{L}(2)$ are precisely those that were claimed in Proposition 7.3, we have finished the proof of the proposition.

Remark 7.12. Although *in general* we must assume Conjecture 2.34 to completely determine the set $\mathcal{L}(\Gamma)$ for arbitrary $\Gamma \in \mathbb{N}$, the correctness of $\mathcal{L}(2)$ — and hence the validity of Proposition 7.3 — does not depend on Conjecture 2.34. For each of the graphs H appearing in Figure 7.1 (and therefore also in Proposition 7.3), any minimal extension G of H by either appending a pair of preimages to an endpoint of H or by adding a cycle of length at

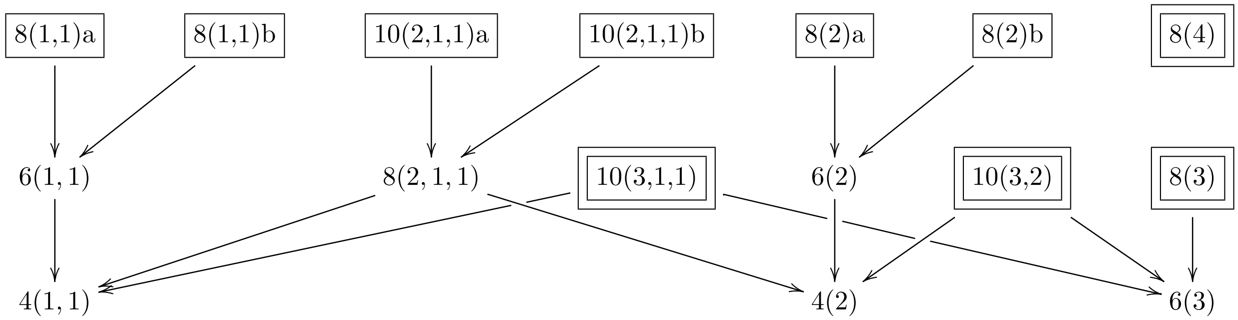


Figure 7.1: The directed system of graphs G with genus of $X_1^{\mathrm{dyn}}(G)$ at most two

most 4 to H has a dynamical modular curve $X_1^{\mathrm{dyn}}(G)$ that is irreducible over $\mathbb{C}$. This has been verified either by Magma's `IsAbsolutelyIrreducible` function (see Lemma 7.9) or by Theorem 2.32. Therefore, if such a minimal extension does not appear in this list, it is because it is irreducible of genus strictly larger than two. It follows that for any graph G' not appearing in Figure 7.1, even if the curve $X_1^{\mathrm{dyn}}(G')$ is reducible, each of its components has genus strictly larger than two.

7.1 Genus one

Each of the dynamical modular curves of genus one has rational points and therefore defines an elliptic curve over $\mathbb{Q}$. All of these curves have small conductor, so we may refer to Cremona's tables [7] for their torsion subgroups over $\mathbb{Q}$. We give in Table 7.2 the Cremona labels (found in [32]) and $\mathbb{Q}$-rational torsion subgroups for each of the genus one dynamical modular curves.

We now determine, for each of the curves X listed in Figure 7.2, the torsion subgroup $X(K)_{\mathrm{tors}}$ over all quadratic fields K.

Table 7.2: Dynamical modular curves of genus one

G	Cremona label for $X_1^{\text{dyn}}(G)$	$X_1^{\text{dyn}}(G)(\mathbb{Q})_{\text{tors}}$
8(1,1)a	24A4	$\mathbb{Z}/4\mathbb{Z}$
8(1,1)b	11A3	$\mathbb{Z}/5\mathbb{Z}$
8(2)a	40A3	$\mathbb{Z}/4\mathbb{Z}$
8(2)b	11A3	$\mathbb{Z}/5\mathbb{Z}$
10(2,1,1)a	17A4	$\mathbb{Z}/4\mathbb{Z}$
10(2,1,1)b	15A8	$\mathbb{Z}/4\mathbb{Z}$

Theorem 7.13. *Let K be a quadratic field.*

(A) (11A3) If $G \cong 8(1,1)b$ or $G \cong 8(2)b$, then

$$X_1^{\text{dyn}}(G)(K)_{\text{tors}} \cong \mathbb{Z}/5\mathbb{Z}.$$

(B) (15A8) If $G \cong 10(2,1,1)b$, then

$$X_1^{\text{dyn}}(G)(K)_{\text{tors}} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, & \text{if } K = \mathbb{Q}(\sqrt{-15}) \\ \mathbb{Z}/8\mathbb{Z}, & \text{if } K = \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{5}) \\ \mathbb{Z}/4\mathbb{Z}, & \text{otherwise.} \end{cases}$$

(C) (17A4) If $G \cong 10(2,1,1)a$, then

$$X_1^{\text{dyn}}(G)(K)_{\text{tors}} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, & \text{if } K = \mathbb{Q}(\sqrt{17}) \\ \mathbb{Z}/4\mathbb{Z}, & \text{otherwise.} \end{cases}$$

(D) (24A4) If $G \cong 8(1,1)a$*, then*

$$X_1^{\mathrm{dyn}}(G)(K)_{\mathrm{tors}} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, & \text{if } K = \mathbb{Q}(\sqrt{-3}) \\ \mathbb{Z}/8\mathbb{Z}, & \text{if } K = \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{3}) \\ \mathbb{Z}/4\mathbb{Z}, & \text{otherwise.} \end{cases}$$

(E) (40A3) If $G \cong 8(2)a$*, then*

$$X_1^{\mathrm{dyn}}(G)(K)_{\mathrm{tors}} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, & \text{if } K = \mathbb{Q}(\sqrt{5}) \\ \mathbb{Z}/4\mathbb{Z}, & \text{otherwise.} \end{cases}$$

Proof. The curves with Cremona labels 11A3, 15A8, and 24A4 are birational to the classical modular curves $X_1(11)$, $X_1(15)$, and $X_1(2,12)$, respectively. Theorem 7.13 was proven for these curves by Rabarison [33], so parts (A), (B), and (D) of the theorem hold. Rabarison's proof relies on the extension of Mazur's theorem to quadratic fields, due to Kenku-Momose [14] and Kamienny [13]. Our method of proof for the remaining curves — 17A4 and 40A3 — is more elementary and, though we do not do so here, may be used to give an alternative proof for the curves 11A3, 15A8, and 24A4.

It remains for us to prove parts (C) and (E) of the theorem. Let E_{17} and E_{40} denote the curves 17A4 and 40A3, respectively, given in [7] by the equations

$$E_{17} : y^2 + xy + y = x^3 - x^2 - x$$
$$E_{40} : y^2 = (x-1)(x^2+x-1)$$

The primes 3 and 5 (resp., 3 and 17) are primes of good reduction for E_{17} (resp., E_{40}). If $\mathfrak{p}$ is a prime in $\mathcal{O}_K$ lying above the rational prime p, then the residue field $k_{\mathfrak{p}}$ embeds into

$\mathbb{F}_{p^2}$; computing in Magma, we find that

$$E_{17}(\mathbb{F}_{3^2}) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}\ , \quad E_{17}(\mathbb{F}_{5^2}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/16\mathbb{Z};$$

$$E_{40}(\mathbb{Z}_{3^2}) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}\ , \quad E_{40}(\mathbb{F}_{17^2}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/160\mathbb{Z},$$

from which it follows that both $E_{17}(K)_{\rm tors}$ and $E_{40}(K)_{\rm tors}$ embed into $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Since both $E = E_{17}$ and $E = E_{40}$ have $E(\mathbb{Q})_{\rm tors} \cong \mathbb{Z}/4\mathbb{Z}$, it follows that if E gains additional torsion points after base change to K, then E necessarily gains a K-rational 2-torsion point, and therefore the full torsion subgroup $E(K)_{\rm tors}$ is precisely $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. It remains, then, to find the fields of definition for $E_{17}[2]$ and $E_{40}[2]$.

One can easily verify that $E_{17}[2]$ consists of the points ∞, $(1, -1)$, and the two points $(x, -1/2(x+1))$ with $4x^2+x-1 = 0$. Therefore E_{17} attains full 2-torsion over $K = \mathbb{Q}(\sqrt{17})$. The 2-torsion on E_{40} is perhaps more apparent: $E_{40}[2]$ consists of the points ∞, $(1, 0)$, and $(x, 0)$ with $x^2 + x - 1 = 0$. Hence E_{40} attains full 2-torsion over $K = \mathbb{Q}(\sqrt{5})$. ☐

As a result of the classification of the torsion subgroups of these curves over quadratic fields, we are able to make the following statement.

Proposition 7.14. *Let G be an admissible graph for which $X_1^{\rm dyn}(G)$ has genus one, and let K be a quadratic field. If $\operatorname{rk} X_1^{\rm dyn}(G)(K) = 0$, then G does not occur as a subgraph of $G(f_c, K)$ for any $c \in K$, unless $G \cong 10(2,1,1)b$, $K = \mathbb{Q}(\sqrt{-15})$, and $c = 3/16$.*

Proof. First, we make the basic observation that if K is a field for which $\operatorname{rk} X_1^{\rm dyn}(G)(K) = 0$ and $X_1^{\rm dyn}(G)(K)_{\rm tors} = X_1^{\rm dyn}(G)(\mathbb{Q})_{\rm tors}$, then necessarily $X_1^{\rm dyn}(G)(K) = X_1^{\rm dyn}(G)(\mathbb{Q})$. For each of these genus one dynamical modular curves, the set $Y_1^{\rm dyn}(G)(\mathbb{Q})$ is empty, so it follows that $Y_1^{\rm dyn}(G)(K)$ is empty as well. Therefore the graph G never occurs as a subgraph of $G(f_c, K)$ for any $c \in K$.

For a given graph G whose dynamical modular curve has genus one, it remains to consider those quadratic fields K for which $X_1^{\mathrm{dyn}}(G)(K)_{\mathrm{tors}} \supsetneq X_1^{\mathrm{dyn}}(G)(\mathbb{Q})_{\mathrm{tors}}$ and $\mathrm{rk}\, X_1^{\mathrm{dyn}}(G)(K) = 0$.

We consider the graphs in the same order as in Theorem 7.13, where all torsion subgroups are described.

First, suppose $G \cong 8(1,1)b$ or $G \cong 8(2)b$. Then we actually have $X_1^{\mathrm{dyn}}(G)(K)_{\mathrm{tors}} = X_1^{\mathrm{dyn}}(G)(\mathbb{Q})_{\mathrm{tors}}$ for all quadratic fields K, so the proposition holds for these graphs.

Now let $G \cong 10(2,1,1)b$. The only quadratic fields over which $X_1^{\mathrm{dyn}}(G)$ gains torsion points are $K = \mathbb{Q}(\sqrt{d})$ with $d \in \{-15, -3, 5\}$. The curve $X_1^{\mathrm{dyn}}(G)$ has rank zero over all three of these fields. The four additional points on $X_1^{\mathrm{dyn}}(\mathbb{Q}(\sqrt{-15}))$ correspond to $c = 3/16$, in which case $G(f_c, K) = G(f_{3/16}, \mathbb{Q}(\sqrt{-15})) \cong 10(2,1,1)b$. The four additional points on $X_1^{\mathrm{dyn}}(\mathbb{Q}(\sqrt{-3}))$ lie outside of $Y_1^{\mathrm{dyn}}(G)$. Two of the additional points on $X_1^{\mathrm{dyn}}(\mathbb{Q}(\sqrt{5}))$ lie outside of $Y_1^{\mathrm{dyn}}(G)$, and the other two correspond to $c = -2$, in which case $G(f_c, K) = G(f_{-2}, \mathbb{Q}(\sqrt{5})) \cong 9(2,1,1)$ is a critical degeneration of 10(2,1,1)b.

In the case $G \cong 10(2,1,1)a$, the only quadratic field over which $X_1^{\mathrm{dyn}}(G)$ gains torsion points is $\mathbb{Q}(\sqrt{17})$. However, $X_1^{\mathrm{dyn}}(G)$ has rank one over $\mathbb{Q}(\sqrt{17})$.

We now consider $G \cong 8(1,1)a$. In this case, $X_1^{\mathrm{dyn}}(G)(K)_{\mathrm{tors}}$ is strictly larger than $X_1^{\mathrm{dyn}}(\mathbb{Q})_{\mathrm{tors}}$ only for $K = \mathbb{Q}(\sqrt{d})$ with $d \in \{-3, -1, 3\}$. For $d \in \{-3, -1\}$, the four additional torsion points lie outside of $Y_1^{\mathrm{dyn}}(G)$. When $d = 3$, the extra torsion points on $X_1^{\mathrm{dyn}}(G)$ correspond to $c = -2$, in which case $G(f_c, K) = G(f_{-2}, \mathbb{Q}(\sqrt{3})) \cong 7(1,1)b$ is a critical degeneration of 8(1,1)a.

Finally, let $G \cong 8(2)a$. The elliptic curve $X_1^{\mathrm{dyn}}(G)$ only gains additional torsion over $\mathbb{Q}(\sqrt{5})$. In this case, however, all additional torsion points lie outside of $Y_1^{\mathrm{dyn}}(G)$. $\square$

7.2 Genus two

In this section, we consider the Jacobians of each of the four genus two dynamical modular curves listed in Proposition 7.3. As we did for the genus one curves in the previous section, we explicitly determine the torsion subgroups of these four Jacobians over all quadratic extensions $K/\mathbb{Q}$.

Three of the four genus two dynamical modular curves are also classical modular curves:

$$X_1^{\rm dyn}(4) \cong X_1(16): \qquad y^2 = f_{16}(x) := -x(x^2+1)(x^2-2x-1) \tag{7.3}$$

$$X_1^{\rm dyn}(1,3) \cong X_1(18): \qquad y^2 = f_{18}(x) := x^6 + 2x^5 + 5x^4 + 10x^3 + 10x^2 + 4x + 1 \tag{7.4}$$

$$X_1^{\rm dyn}(2,3) \cong X_1(13): \qquad y^2 = f_{13}(x) := x^6 + 2x^5 + x^4 + 2x^3 + 6x^2 + 4x + 1. \tag{7.5}$$

These curves correspond to the graphs 8(4), 10(3,1,1), and 10(3,2), respectively. For the remainder of this section, we will let $\mathbf{G}$ refer to the graph 8(3). It was shown in [32] that $X_1^{\rm dyn}(\mathbf{G})$ is given by the equation

$$y^2 = x^6 - 2x^4 + 2x^3 + 5x^2 + 2x + 1.$$

The torsion subgroups of the classical modular Jacobians $J_1(N)(\mathbb{Q})_{\rm tors}$ with $N \in \{13, 16, 18\}$ are well known, and the torsion subgroup $J_1^{\rm dyn}(\mathbf{G})(\mathbb{Q})_{\rm tors}$ was determined in [32]:

$$\begin{aligned}
J_1^{\rm dyn}(4)(\mathbb{Q}) &= J_1(16)(\mathbb{Q})_{\rm tors} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}\\
J_1^{\rm dyn}(1,3)(\mathbb{Q}) &= J_1(18)(\mathbb{Q})_{\rm tors} \cong \mathbb{Z}/21\mathbb{Z}\\
J_1^{\rm dyn}(2,3)(\mathbb{Q}) &= J_1(13)(\mathbb{Q})_{\rm tors} \cong \mathbb{Z}/19\mathbb{Z}\\
J_1^{\rm dyn}(\mathbf{G})(\mathbb{Q})_{\rm tors} &\cong 0.
\end{aligned}$$

We also mention that the Jacobians $J_1^{\mathrm{dyn}}(4)$, $J_1^{\mathrm{dyn}}(1,3)$, and $J_1^{\mathrm{dyn}}(2,3)$ have rank zero over $\mathbb{Q}$, and $J_1^{\mathrm{dyn}}(\mathbf{G})$ has rank one over $\mathbb{Q}$. (This is well known for the classical modular Jacobians, and the rank of $J_1^{\mathrm{dyn}}(\mathbf{G})$ over $\mathbb{Q}$ was shown to be one by Poonen [32].)

We now determine over which quadratic fields K these curves gain new torsion points.

Theorem 7.15. *Let K be a quadratic field.*

(A)

$$J_1^{\mathrm{dyn}}(4)(K)_{\mathrm{tors}} \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/10\mathbb{Z}, & \text{if } K = \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{2}) \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}, & \text{otherwise.} \end{cases}$$

(B)

$$J_1^{\mathrm{dyn}}(1,3)(K)_{\mathrm{tors}} \cong \begin{cases} \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/21\mathbb{Z}, & \text{if } K = \mathbb{Q}(\sqrt{-3}) \\ \mathbb{Z}/21\mathbb{Z}, & \text{otherwise.} \end{cases}$$

(C)

$$J_1^{\mathrm{dyn}}(2,3)(K)_{\mathrm{tors}} \cong \mathbb{Z}/19\mathbb{Z}.$$

(D)

$$J_1^{\mathrm{dyn}}(\mathbf{G})(K)_{\mathrm{tors}} \cong 0.$$

The most difficult case is (B). In order to prove (B), we will require two lemmas concerning the curve $X_1^{\mathrm{dyn}}(1,3)$.

Lemma 7.16. *Let $X = X_1^{\mathrm{dyn}}(1,3)$, given by the model from (7.4). The linear system $|3\infty^+|$ on X contains no divisors of the form $3P$, except for $3\infty^+$ itself.*

Proof. Let $P \in X(\overline{\mathbb{Q}})$ be such that $3P \in |3\infty^+|$, and suppose for the sake of contradiction that $P \neq \infty^+$.

First, one can verify in Magma that the point $[\infty^- - \infty^+]$ is a point of order 21, which means that $[3\infty^- - 3\infty^+] = 3[\infty^- - \infty^+] \neq [0]$, so $3\infty^- \notin |3\infty^+|$ and therefore $P \neq \infty^-$.

Next, we observe that if $P = (x_0, 0)$ is a finite Weierstrass point, then

$$[3P - 3\infty^+] = [P + \infty^- - 2\infty^+] + [2P - \infty^+ - \infty^-] = [P + \infty^- - 2\infty^+].$$

However, by the Riemann-Roch theorem, the linear system $|2\infty^+|$ contains only the divisor $2\infty^+$, so $P + \infty^- \notin |2\infty^+|$. This implies that $[3P - 3\infty^+] \neq 0$, so $3P \notin |3\infty^+|$.

Finally, suppose $P = (x_0, y_0)$ with $y_0 \neq 0$. Let g be a function on X with zero divisor $3P$ and pole divisor $3\infty^+$. In particular, g lies in the Riemann-Roch space $\mathcal{L}(3\infty^+)$, which has dimension two by the Riemann-Roch theorem. The constant functions certainly lie in $\mathcal{L}(3\infty^+)$, and we claim that the function $h := y + x^3 + x^2 + 2x$ also lies in $\mathcal{L}(3\infty^+)$, so that $\mathcal{L}(3\infty^+) = \langle 1, h\rangle$. In other words, we claim that the only pole of h is a triple pole at ∞^+. Certainly h has no finite poles. To understand the behavior of h at infinity, we cover X by the affine patches $y^2 = f_{18}(x)$ and $v^2 = u^6 \cdot f_{18}(1/u)$, with the identifications $x = 1/u$ and $y = v/u^3$ (as described in §3.2). The two points $\infty^\pm$ on X correspond to the points $(u, v) = (0, \pm 1)$. We rewrite h in terms of u and v to get

$$h = \frac{v + 2u^2 + u + 1}{u^3}. \tag{7.6}$$

Clearly h has a triple pole at $(u, v) = (0, 1)$. On the other hand, multiplying each of the numerator and denominator of (7.6) by $v - (2u^2 + u + 1)$ yields

$$h = \frac{u^3 + 4u^2 + 6u + 6}{v - (2u^2 + u + 1)},$$

which now visibly does not have a pole at $(u, v) = (0, -1)$. Therefore $h \in \mathcal{L}(3\infty^+)$, as claimed.

It follows that the function g may be written as $a + b(y + x^3 + x^2 + 2x)$ for some scalars a and b. Since g is nonconstant, we must have $b \neq 0$; scaling by $1/b$, we may assume g is of the form

$$g = y + x^3 + x^2 + 2x + A,$$

which we rewrite as

$$g = \frac{y^2 - (x^3 + x^2 + 2x + A)^2}{y - (x^3 + x^2 + 2x + A)} = -\frac{p(x)}{y - (x^3 + x^2 + 2x + A)},$$

where

$$p(x) := 2(A-3)x^3 + 2(A-3)x^2 + 4(A-1)x + (A+1)(A-1).$$

Since P is not a Weierstrass point, $x - x_0$ is a uniformizer at P; since g vanishes to order three at P, this means that $(x - x_0)^3$ must divide $p(x)$. Thus each of $p(x)$ and $p'(x)$ has a multiple root, so

$$\begin{aligned} \operatorname{disc}(p) &= -4(A-1)(A-3)\left(27A^4 - 118A^3 + 180A^2 - 42A + 17\right) = 0, \\ \operatorname{disc}(p') &= -16(A-3)(5A-3) = 0. \end{aligned}$$

This forces $A = 3$, which contradicts the fact that $p(x)$ must have degree three. Having exhausted all possibilities for P, the result now follows. □

Lemma 7.17. *Let $X = X_1^{\mathrm{dyn}}(1,3)$, and let $J = J_1^{\mathrm{dyn}}(1,3)$ be the Jacobian of X. The 3-torsion subgroup $J[3]$ contains only nine points of degree at most two over $\mathbb{Q}$, all of which are defined over $\mathbb{Q}(\sqrt{-3})$.*

Proof. Suppose $\{P, Q\}$ is a point of order 3 on J. This means that, on the level of divisor classes,

$$3[P + Q - \infty^+ - \infty^-] = [3P + 3Q - (3\infty^+ + 3\infty^-)] = [0], \tag{7.7}$$

so there is a function g on X whose divisor is $(g) = 3P + 3Q - (3\infty^+ + 3\infty^-)$.

We first show that neither P nor Q may be a point at infinity. Suppose to the contrary that $Q = \infty^-$. (There is no loss of generality here: The pair $\{P, Q\}$ is unordered, so we are free to switch P and Q. Also, if $\{P, \infty^+\}$ is a 3-torsion point, then so is $-\{P, \infty^+\} = \{\iota P, \infty^-\}$.) Then $[0] = [3P + 3Q - (3\infty^+ + 3\infty^-)] = [3P - 3\infty^+]$. However, by Lemma 7.16 this implies $P = \infty^+$, which means that $\{P, Q\} = \{\infty^+, \infty^-\}$ is the trivial class and therefore does not have order three.

Since neither P nor Q is a point at infinity, there is no cancellation in the difference $3P + 3Q - 3\infty^+ - 3\infty^-$, so the function g must have zero divisor equal to $3P + 3Q$ and pole divisor equal to $D := 3\infty^+ + 3\infty^-$. The Riemann-Roch theorem implies that the dimension of $\mathcal{L}(D)$ is 5; since the set

$$\{1, x, x^2, x^3, y\}$$

is a linearly independent set of elements of $\mathcal{L}(D)$, it must be a basis. Therefore there exist scalars $a, b, c, d, e \in \overline{\mathbb{Q}}$ for which

$$g = ay + bx^3 + cx^2 + dx + e.$$

We claim that $a \neq 0$. Indeed, if $a = 0$, then the set of points on X for which $g = 0$ is

$$S := \{(x, \pm\sqrt{f_{18}(x)}) : bx^3 + cx^2 + dx + e = 0\}.$$

Since $(g) = 3(P + Q - \infty^+ - \infty^-)$, then S is equal to the set $\{P, Q\}$. Thus either $g = 0$ has a single solution x_0, or $g = 0$ has two distinct solutions x_1 and x_2 with $f_{18}(x_1) = f_{18}(x_2) = 0$. In the former case, P and Q are related by the hyperelliptic involution, so $\{P, Q\} = \mathcal{O}$; in the latter, P and Q are distinct Weierstrass points, so $\{P, Q\}$ is a point of order 2. In either case, $\{P, Q\}$ is not a point of order 3, so we must have $a \neq 0$; dividing by a if necessary, we take g to be of the form

$$g = y - (Ax^3 + Bx^2 + Cx + D),$$

which we rewrite as

$$\begin{aligned} g &= \frac{y^2 - (Ax^3 + Bx^2 + Cx + D)^2}{y + (Ax^3 + Bx^2 + Cx + D)} \\ &= -\frac{q(x)}{y + (Ax^3 + Bx^2 + Cx + D)}, \end{aligned}$$

where

$$\begin{aligned} q(x) = (A+1)(A-1)x^6 + 2(AB-1)x^5 + (2AC + B^2 - 5)x^4 \\ + 2(AD + BC - 5)x^3 + (2BD + C^2 - 10)x^2 \\ + 2(CD - 2)x + (D+1)(D-1). \end{aligned} \tag{7.8}$$

The function $q(x)$ must vanish to order at least 3 at each of $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. Since P and Q are not Weierstrass points (as pointed out in our argument that $a \neq 0$), $(x - x_1)$ and $(x - x_2)$ are uniformizers at P and Q, respectively. Thus the polynomial $(x - x_1)^3(x - x_2)^3$ must divide $q(x)$; since q has degree at most six, this means we must have

$$q(x) = (A+1)(A-1)(x^2 - tx + n)^3, \tag{7.9}$$

where $t = x_1 + x_2$ and $n = x_1 x_2$. Equating the coefficients of the expressions for $q(x)$ given in (7.8) and (7.9) yields the following system of equations:

$$\begin{cases} 2(AB-1) &= -3(A+1)(A-1)t \\ 2AC+B^2-5 &= 3(A+1)(A-1)(t^2+n) \\ 2(AD+BC-5) &= -(A+1)(A-1)t(t^2+6n) \\ 2BD+C^2-10 &= 3(A+1)(A-1)n(t^2+n) \\ 2(CD-2) &= -3(A+1)(A-1)tn^2 \\ (D+1)(D-1) &= (A+1)(A-1)n^3 \end{cases} \tag{7.10}$$

The system (7.10) defines a zero-dimensional scheme $S \subseteq \mathbb{A}^6$. A computation in Magma finds all 80 points $(A,B,C,D,t,n) \in S(\overline{\mathbb{Q}})$, each of which corresponds to a point of order 3 in $J(\overline{\mathbb{Q}})$.

Now, in order for $\{P,Q\}$ to be a *quadratic* point on J, say defined over the quadratic field K, P and Q must either both be defined over K, or P and Q must be Galois conjugates defined over some quadratic extension L/K. In either case, the parameters t and n must both lie in K. The only points in $S(\overline{\mathbb{Q}})$ with $[\mathbb{Q}(t,n):\mathbb{Q}] \le 2$ are the eight points satisfying $(t,n) \in \{(-1,1),(-2(\zeta_3+1),\zeta_3),(-2(\zeta_3^2+1),\zeta_3^2)\}$, where $\zeta_3 \in \mathbb{Q}(\sqrt{-3})$ is a primitive cube root of unity. Therefore the only quadratic field over which J gains additional 3-torsion is $K = \mathbb{Q}(\sqrt{-3})$, and over this field there are a total of eight points (two of which are $\mathbb{Q}$-rational) of order three. The lemma now follows. $\square$

We are now ready to prove Theorem 7.15.

Proof of Theorem 7.15. For parts (C) and (D), we observe that 3 and 5 are both primes of good reduction for $J_1^{\text{dyn}}(2,3)$ and $J_1^{\text{dyn}}(\mathbf{G})$, and that

$$\#J_1^{\text{dyn}}(2,3)(\mathbb{F}_{3^2}) = 3\cdot 19\ ,\quad \#J_1^{\text{dyn}}(2,3)(\mathbb{F}_{5^2}) = 19^2;$$

$$\#J_1^{\text{dyn}}(\mathbf{G})(\mathbb{F}_{3^2}) = 3^4\ ,\quad \#J_1^{\text{dyn}}(\mathbf{G})(\mathbb{F}_{5^2}) = 19\cdot 43.$$

Therefore $J_1^{\mathrm{dyn}}(2,3)(K)_{\mathrm{tors}} \hookrightarrow \mathbb{Z}/19\mathbb{Z}$ and $J_1^{\mathrm{dyn}}(G)(K)_{\mathrm{tors}} = 0$. Since $J_1^{\mathrm{dyn}}(2,3)(\mathbb{Q}) \cong \mathbb{Z}/19\mathbb{Z}$, this proves (C) and (D).

For (A), we note that 3 is a prime of good reduction for $J_1^{\mathrm{dyn}}(4)$, and that

$$J_1^{\mathrm{dyn}}(4)(\mathbb{F}_{3^2}) \cong (\mathbb{Z}/2\mathbb{Z})^3 \oplus \mathbb{Z}/10\mathbb{Z}.$$

Therefore, if $P \in J_1^{\mathrm{dyn}}(4)(K)$ is a torsion point of order n, then n must be divisible by 2, 3, or 5. Moreover, since 5 is a prime of good reduction and $\# J_1^{\mathrm{dyn}}(4)(\mathbb{F}_{5^2}) = 2^7 \cdot 5$, there can be no 3-torsion. Hence

$$J_1^{\mathrm{dyn}}(4)(K)_{\mathrm{tors}} \hookrightarrow (\mathbb{Z}/2\mathbb{Z})^3 \oplus \mathbb{Z}/10\mathbb{Z}.$$

Since $J_1^{\mathrm{dyn}}(4)(\mathbb{Q})_{\mathrm{tors}} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$, the only way for $J_1^{\mathrm{dyn}}(4)(K)_{\mathrm{tors}}$ to be strictly larger than $J_1^{\mathrm{dyn}}(4)(\mathbb{Q})_{\mathrm{tors}}$ is for $J_1^{\mathrm{dyn}}(4)$ to gain a 2-torsion point upon base change from $\mathbb{Q}$ to K. As mentioned in §3.2, the 2-torsion points are the points supported on the Weierstrass locus of $X_1^{\mathrm{dyn}}(4)$; in other words, they are the points $\{P, Q\}$ where each of P and Q is either the point at infinity of a point of the form $(x, 0)$. The Weierstrass points are the point ∞ and the points

$$P := (0,0)\ ,\ Q^{\pm} := (\pm\sqrt{-1}, 0)\ ,\ R^{\pm} := (1 \pm \sqrt{2}, 0).$$

The sixteen points in $J_1^{\mathrm{dyn}}(4)[2]$ are therefore those appearing in Table 7.3. Hence the only quadratic fields over which $J_1^{\mathrm{dyn}}(4)$ gains additional torsion are $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{2})$, and over each of these fields the torsion subgroup is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/10\mathbb{Z}$.

Table 7.3: The 2-torsion points on $J_1^{\rm dyn}(4)$

Field of definition	Points			
$\mathbb{Q}$	$\mathcal{O}$	$\{\infty, P\}$	$\{Q^+, Q^-\}$	$\{R^+, R^-\}$
$\mathbb{Q}(\sqrt{-1})$	$\{\infty, Q^+\}$	$\{\infty, Q^-\}$	$\{P, Q^+\}$	$\{P, Q^-\}$
$\mathbb{Q}(\sqrt{2})$	$\{\infty, R^+\}$	$\{\infty, R^-\}$	$\{P, R^+\}$	$\{P, R^-\}$
$\mathbb{Q}(\sqrt{-1}, \sqrt{2})$	$\{Q^+, R^+\}$	$\{Q^+, R^-\}$	$\{Q^-, R^+\}$	$\{Q^-, R^-\}$

Finally, we consider part (B). We observe that $J_1^{\rm dyn}(1,3)$ has good reduction at the primes 5 and 11, and we compute

$$J_1^{\rm dyn}(1,3)(\mathbb{F}_{5^2}) \cong (\mathbb{Z}/3\mathbb{Z})^2 \oplus (\mathbb{Z}/7\mathbb{Z})^2 \ ,$$
$$J_1^{\rm dyn}(1,3)(\mathbb{F}_{11^2}) \cong (\mathbb{Z}/4\mathbb{Z})^2 \oplus (\mathbb{Z}/3\mathbb{Z})^2 \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/13\mathbb{Z}.$$

Therefore

$$J_1^{\rm dyn}(1,3)(K)_{\rm tors} \hookrightarrow (\mathbb{Z}/3\mathbb{Z})^2 \oplus \mathbb{Z}/7\mathbb{Z} = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/21\mathbb{Z}.$$

Since $J_1^{\rm dyn}(1,3)(\mathbb{Q}) \cong \mathbb{Z}/21\mathbb{Z}$, the only way for $J_1^{\rm dyn}(1,3)$ to gain torsion points over a quadratic field K is to gain a point of order 3, in which case the full torsion subgroup is $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/21\mathbb{Z}$. We know from Lemma 7.17 that the only quadratic field over which $J_1^{\rm dyn}(1,3)$ admits additional K-rational points of order 3 is $K = \mathbb{Q}(\sqrt{-3})$, and (B) now follows. $\square$

Proposition 7.18. *Let G be an admissible graph for which $X_1^{\rm dyn}(G)$ has genus two, and let K be a quadratic field. Suppose* $\operatorname{rk} X_1^{\rm dyn}(G)(K) = \operatorname{rk} X_1^{\rm dyn}(G)(\mathbb{Q})$.

(A) If G is isomorphic to 8(4), 10(3,1,1), or 10(3,2), then G does not occur as a subgraph of $G(f_c, K)$ for any $c \in K$.

(B) If G is isomorphic to 8(3), then the only $c \in K$ for which $G(f_c, K)$ contains a subgraph isomorphic to G is $c = -29/16$, in which case $G(f_c, \mathbb{Q}) = G(f_{-29/16}, \mathbb{Q}) \cong 8(3)$.

Proof. We begin with statement (A). If G is one of the graphs 8(4), 10(3,1,1), or 10(3,2), then $\operatorname{rk} J_1^{\text{dyn}}(G)(\mathbb{Q}) = 0$, in which case the conditions

$$\operatorname{rk} J_1^{\text{dyn}}(G)(K) = \operatorname{rk} J_1^{\text{dyn}}(G)(\mathbb{Q}) \quad \text{and} \quad J_1^{\text{dyn}}(G)(K)_{\text{tors}} = J_1^{\text{dyn}}(G)(\mathbb{Q})_{\text{tors}}$$

automatically imply that $J_1^{\text{dyn}}(G)(K) = J_1^{\text{dyn}}(G)(\mathbb{Q})$. By Proposition 3.11 (or the proof thereof, for the case of $X_1^{\text{dyn}}(4)$), this implies that

$$X_1^{\text{dyn}}(G)(K) = X_1^{\text{dyn}}(G)(\mathbb{Q}) \cup \{K\text{-rational Weierstrass points}\}.$$

Of these three curves, only $X_1^{\text{dyn}}(4)$ has quadratic Weierstrass points, and those are defined over the fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{2})$. These are also the only fields over which $J_1^{\text{dyn}}(4)$ gains additional torsion, so we conclude that statement (A) holds for each graph G and each quadratic field K over which $J_1^{\text{dyn}}(G)$ does not gain additional torsion. It therefore remains only to consider those quadratic fields K for which $J_1^{\text{dyn}}(G)(K)_{\text{tors}} \supsetneq J_1^{\text{dyn}}(G)(\mathbb{Q})_{\text{tors}}$.

We begin by studying the case $G \cong 8(4)$. As mentioned above, $J_1^{\text{dyn}}(G)$ only gains torsion points over $K = \mathbb{Q}(\sqrt{-1})$ and $K = \mathbb{Q}(\sqrt{2})$. The curve $X_1^{\text{dyn}}(G)$ has Weierstrass points over each of these fields, but the Weierstrass points do not lie in the open subset $Y_1^{\text{dyn}}(G)$. If P is a non-Weierstrass quadratic point on $X_1^{\text{dyn}}(G)(K)$, then we would find a nonzero point $\{P, P\} \in J_1^{\text{dyn}}(G)(K)$. However, over each of these two fields K, we can explicitly determine all forty elements of $J_1^{\text{dyn}}(G)(K)$, and we find none of the form $\{P, P\}$ with P a quadratic point on $X_1^{\text{dyn}}(G)$.

Now let $G \cong 10(3,1,1)$. The Jacobian $J_1^{\text{dyn}}(G)$ only gains additional torsion over the quadratic field $K = \mathbb{Q}(\sqrt{-3})$. As we did in the previous case, we can explicitly find all 63 points on $J_1^{\mathrm{dyn}}(G)(K)$. In this case, the only points of the form $\{P, P\}$ with $P = (x, y)$ a quadratic point in $X_1^{\mathrm{dyn}}(G)(K)$ satisfy $x^2 + x + 1 = 0$. Therefore the only new K-rational points on $X_1^{\mathrm{dyn}}(G)$ lie outside of $Y_1^{\mathrm{dyn}}(G)$.

For $G \cong 10(3, 2)$, the torsion subgroup is unchanged upon base change to any quadratic field K. Therefore the conclusion of (A) holds for all quadratic fields.

We now prove (B). In this case, we have $G \cong 8(3)$ and $\operatorname{rk} X_1^{\mathrm{dyn}}(G)(\mathbb{Q}) = 1$. Since $J_1^{\mathrm{dyn}}(G)(K)$ is trivial for all quadratic fields K, Proposition 3.11 tells us that $X_1^{\mathrm{dyn}}(G)(K) = X_1^{\mathrm{dyn}}(G)(\mathbb{Q})$ — and therefore $Y_1^{\mathrm{dyn}}(G)(K) = Y_1^{\mathrm{dyn}}(G)(\mathbb{Q})$ — whenever $\operatorname{rk} J_1^{\mathrm{dyn}}(G)(K) = 1$. As shown in [32, §4], the only points on $Y_1^{\mathrm{dyn}}(G)(\mathbb{Q})$ correspond to $c = -29/16$, which completes the proof of (B), noting from [32] that $G(f_{-29/16}, \mathbb{Q}) \cong 8(3)$. $\square$

7.3 The curve $X_0^{\mathrm{dyn}}(5)$

It is shown in [10] that if $c \in \mathbb{Q}$, then f_c may not admit rational points of period 5. Rather than attempting to directly find all rational points on the curve $X_1^{\mathrm{dyn}}(5)$, the authors of [10] work with the quotient curve $X_0^{\mathrm{dyn}}(5)$, for which they find the model

$$y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1. \tag{7.11}$$

They show that $\operatorname{rk} J_0^{\mathrm{dyn}}(5)(\mathbb{Q}) = 1$, and then they determine the complete set of rational points

$$X_0^{\mathrm{dyn}}(5)(\mathbb{Q}) = \{(0, \pm 1), (-3, \pm 1), \infty^{\pm}\}$$

using a version of the Chabauty-Coleman method for genus two curves due to Flynn [9].

Since $X_0^{\mathrm{dyn}}(5)$ has genus two, we may apply the methods used in the previous section to compute the torsion subgroup of $J_0^{\mathrm{dyn}}(5)(K)$ for quadratic fields K. From this information, we will deduce a sufficient condition for a quadratic field K to contain no elements c for which f_c admits K-rational points of period 5.

Theorem 7.19. *Let K be a quadratic field. Then*

$$J_0^{\mathrm{dyn}}(5)(K)_{\mathrm{tors}} = 0.$$

Proof. The primes $p = 3$ and $p = 5$ are primes of good reduction for the curve X given in (7.11), which is birational to $X_0^{\mathrm{dyn}}(5)$. Letting $J := \mathrm{Jac}(X)$, a Magma computation shows that

$$\#J(\mathbb{F}_{3^2}) = 3^4 \ , \ \#J(\mathbb{F}_{5^2}) = 29 \cdot 41.$$

Recall that if $\mathfrak{p}$ is any prime in $\mathcal{O}_K$ lying above the rational prime p, then $\mathbb{F}_{\mathfrak{p}} \hookrightarrow \mathbb{F}_{p^2}$ and, therefore, $J(\mathbb{F}_{\mathfrak{p}}) \hookrightarrow J(\mathbb{F}_{p^2})$. Since $\#J(\mathbb{F}_{3^2})$ and $\#J(\mathbb{F}_{5^2})$ are coprime, we conclude that $J(K)_{\mathrm{tors}} = 0$. □

Corollary 7.20. *Let K be a quadratic field. If $\mathrm{rk}\, J_0^{\mathrm{dyn}}(5)(K) = 1$, then there is no element $c \in K$ for which f_c admits a K-rational point of period 5.*

Proof. Let $X := X_0^{\mathrm{dyn}}(5)$ and $J := J_0^{\mathrm{dyn}}(5)$. If $\mathrm{rk}\, J(K) = 1$, then we have $\mathrm{rk}\, J(K) = \mathrm{rk}\, J(\mathbb{Q})$ and $J(K)_{\mathrm{tors}} = 0$, so Proposition 3.11 says that the only points in $X(K) \setminus X(\mathbb{Q})$ are the K-rational Weierstrass points on X. However, the sextic polynomial in (7.11) is irreducible over $\mathbb{Q}$, so X has no quadratic Weierstrass points. Therefore $X(K) = X(\mathbb{Q})$, which means that there are no additional K-rational points on $Y_0^{\mathrm{dyn}}(5)$. □

Chapter 8

Preperiodic points over cyclotomic quadratic fields

We now move from making general statements that hold over arbitrary quadratic fields to giving results over certain specific quadratic fields. The result is, in each case, a classification theorem like that of Poonen [32], but over a quadratic extension of $\mathbb{Q}$ rather than over $\mathbb{Q}$ itself.

We will rely on the results of Chapter 7. For many of the arguments, we require that the Jacobians of certain dynamical modular curves have rank zero over various quadratic fields. In every such case, the relevant rank computations were performed using the method of two-descent implemented in Magma's `RankBound` function.

In [28, 27], Najman classifies all possible torsion subgroups of elliptic curves over the cyclotomic quadratic fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ by studying the classical modular curves over those fields. We now give a dynamical analogue of Najman's results by giving a classification of all preperiodic graphs $G(f_c, K)$ which can occur over these two fields. Graphs appearing in Appendix A will be indicated by the labels found there, and any other admissible graph will be referred to by the name given in Figure 6.1.

Theorem 8.1.

(A) Let $K = \mathbb{Q}(\sqrt{-1})$, and let $c \in K$. Suppose f_c does not admit K-rational points of period greater than 4. Then $G(f_c, K)$ is isomorphic to one of the following 14 graphs:

0, 3(2), 4(1,1), 4(2), 5(1,1)a, 5(1,1)b, 5(2)a, 6(1,1), 6(2), 6(2,1), 6(3), 8(2,1,1), 8(3), 10(2,1,1)a.

(B) Let $K = \mathbb{Q}(\sqrt{-3})$, and let $c \in K$. Suppose f_c does not admit K-rational points of period greater than 5. Then $G(f_c, K)$ is isomorphic to one of the following 13 graphs:

0, 3(2), 4(1), 4(1,1), 4(2), 5(1,1)a, 6(1,1), 6(2), 6(3), 7(2,1,1)a, 8(2)a, 8(2,1,1), 8(3).

Proof. We first consider $K = \mathbb{Q}(\sqrt{-1})$. Each of the graphs listed in part (A) is realized as $G(f_c, K)$ for some $c \in K$, as found in our search in [8]. By [8, §5], we know that the only critically or fixed-point degenerate preperiodic graphs that occur over K are 3(2), 5(1,1)a, 5(1,1)b, 5(2)a, and 6(2,1). It therefore suffices to show that if $G(f_c, K)$ is *admissible* with cycles of length at most 4, then $G(f_c, K)$ is isomorphic to one of the following graphs:

4(1,1), 4(2), 6(1,1), 6(2), 6(3), 8(2,1,1), 8(3), 10(2,1,1)a.

A Magma computation shows that $\operatorname{rk} J_1^{\mathrm{dyn}}(G)(K) = 0$ whenever G is among the following graphs:

8(1,1)a, 8(1,1)b, 8(2)a, 8(2)b, 8(4), 10(2,1,1)b, 10(3,1,1), 10(3,2).

By Propositions 7.14 and 7.18, these graphs may therefore not appear as $G(f_c, K)$ for any $c \in K$. Also, for $G \cong 8(3)$, we have $\operatorname{rk} J_1^{\mathrm{dyn}}(G)(K) = 1$; by Proposition 7.18, this means that the only $c \in K$ for which $G(f_c, K)$ contains 8(3) is $c = -29/16$, and in this case $G(f_c, K)$ is isomorphic to 8(3). Therefore, by considering the portion of the directed system of admissible graphs illustrated in Figure 7.1, we see that if $G(f_c, K)$ is not among those graphs listed in the statement of the theorem, then $G(f_c, K)$ strictly contains 10(2,1,1)a.

Note that, because $G(f_c, K)$ cannot contain 10(3,1,1) nor 8(4), if $G(f_c, K)$ strictly contains 10(2,1,1)a, then $G(f_c, K)$ has only fixed points and a two-cycle. The minimal such graphs strictly containing 10(2,1,1)a are the graphs 12(2,1,1)a, G_4, and G_6. By [8, §3.14], the graph 12(2,1,1)a cannot occur over K, and the same is true for G_6 by Theorem 5.17. To see that G_4 cannot be achieved over K, recall from §5.1 that $Y_1^{\mathrm{dyn}}(G_4)$ has a model of the form

$$\begin{cases} y^2 &= 2(x^3 + x^2 - x + 1) \\ z^2 &= 5x^4 + 8x^3 + 6x^2 - 8x + 5, \end{cases}$$

and that any quadratic point on $Y_1^{\mathrm{dyn}}(G_4)$ has $x \in \mathbb{Q}$ and $y, z \notin \mathbb{Q}$. Therefore a K-rational point on $Y_1^{\mathrm{dyn}}(G_4)$ would yield a *rational* point on the twist of this curve by -1:

$$\begin{cases} -y^2 &= 2(x^3 + x^2 - x + 1) \\ -z^2 &= 5x^4 + 8x^3 + 6x^2 - 8x + 5. \end{cases} \tag{8.1}$$

The curve C defined by $-y^2 = 2(x^3 + x^2 - x + 1)$ is birational to elliptic curve 176B1 in [7], which has only one rational point. Since C has a rational point at infinity, C must have no finite rational points, so the affine curve defined by (8.1) has no rational points. Therefore, G_4 cannot occur as a subgraph of $G(f_c, K)$ for any $c \in K$, completing the proof of (A).

Now we take $K = \mathbb{Q}(\sqrt{-3})$, and we prove part (B) in a similar manner. We begin by noting that each of the graphs listed in the statement of (B) was found over K in the search conducted in [8].

According to [8, §5], the only critically or fixed-point degenerate graphs occuring over K are 3(2), 4(1), 5(1,1)a, and 7(2,1,1)a. We must therefore show that if $G(f_c, K)$ is admissible, then it is isomorphic to one of the following graphs:

$$4(1,1),\ 4(2),\ 6(1,1),\ 6(2),\ 6(3),\ 8(2)\text{a},\ 8(2,1,1),\ 8(3).$$

A Magma computation shows that $\operatorname{rk} J_1^{\text{dyn}}(G) = 0$ for all of the following graphs G:

$$8(1,1)\text{a},\ 8(1,1)\text{b},\ 8(2)\text{b},\ 8(4),\ 10(2,1,1)\text{a},\ 10(2,1,1)\text{b},\ 10(3,1,1),\ 10(3,2).$$

By Propositions 7.14 and 7.18, these graphs can not occur as subgraphs of $G(f_c, K)$ for any $c \in K$. Also, if $G \cong 8(3)$, we have $\operatorname{rk} J_1^{\text{dyn}}(G)(K) = 1$, so the only occurrence of 8(3) as a subgraph of $G(f_c, K)$ when $c = -29/16$, in which case $G(f_c, K)$ is isomorphic to 8(3). Furthermore, Magma verifies that $\operatorname{rk} J_0^{\text{dyn}}(5)(K) = 1$, so f_c cannot admit a K-rational point of period 5 for any $c \in K$.

Once again considering Figure 7.1, we see that if there is some $c \in K$ for which $G(f_c, K)$ does not appear in the list given in (B), then $G(f_c, K)$ must strictly contain the graph 8(2)a. We have already shown that $G(f_c, K)$ cannot contain points of period 4 or 5, and we have also shown that $G(f_c, K)$ cannot contain both a point of period 2 and a point of period 3. Therefore, if $G(f_c, K)$ properly contains 8(2)a, then $G(f_c, K)$ may only contain points of period 1 and 2. The minimal such admissible graphs properly containing 8(2)a are 12(2,1,1)a and G_2. By [8, §3.14], the graph 12(2,1,1)a cannot occur over K. The curve $Y_1^{\text{dyn}}(G_2)$ has a model given in [8, §3.13] by the equations

$$\begin{cases} y^2 &= 2(x^4 + 2x^3 - 2x + 1) \\ z^2 &= 2(x^3 + x^2 - x + 1). \end{cases}$$

(See also Remark 6.2.) We show in [8] that any quadratic point (x, y, z) on $Y_1^{\text{dyn}}(G_2)$, defined over a field different from $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{57})$, must have $x \in \mathbb{Q}$ and $y, z \notin \mathbb{Q}$. Therefore, if $Y_1^{\text{dyn}}(G_2)$ has a quadratic point defined over $\mathbb{Q}(\sqrt{-3})$, then we must have a *rational* point on the twist of $Y_1^{\text{dyn}}(G_2)$ by -3:

$$\begin{cases} -3y^2 &= 2(x^4 + 2x^3 - 2x + 1) \\ -3z^2 &= 2(x^3 + x^2 - x + 1). \end{cases} \tag{8.2}$$

The curve C defined by $-3z^2 = 2(x^3 + x^2 - x + 1)$ is birational to elliptic curve 99D1 in [7], which has a single rational point. Since C has a rational point at infinity, C cannot have any affine rational points, and therefore there are no rational points on the curve defined by (8.2). Therefore G_2 cannot occur as a subgraph of $G(f_c, K)$ for any $c \in K$. This completes the proof of the proposition. $\square$

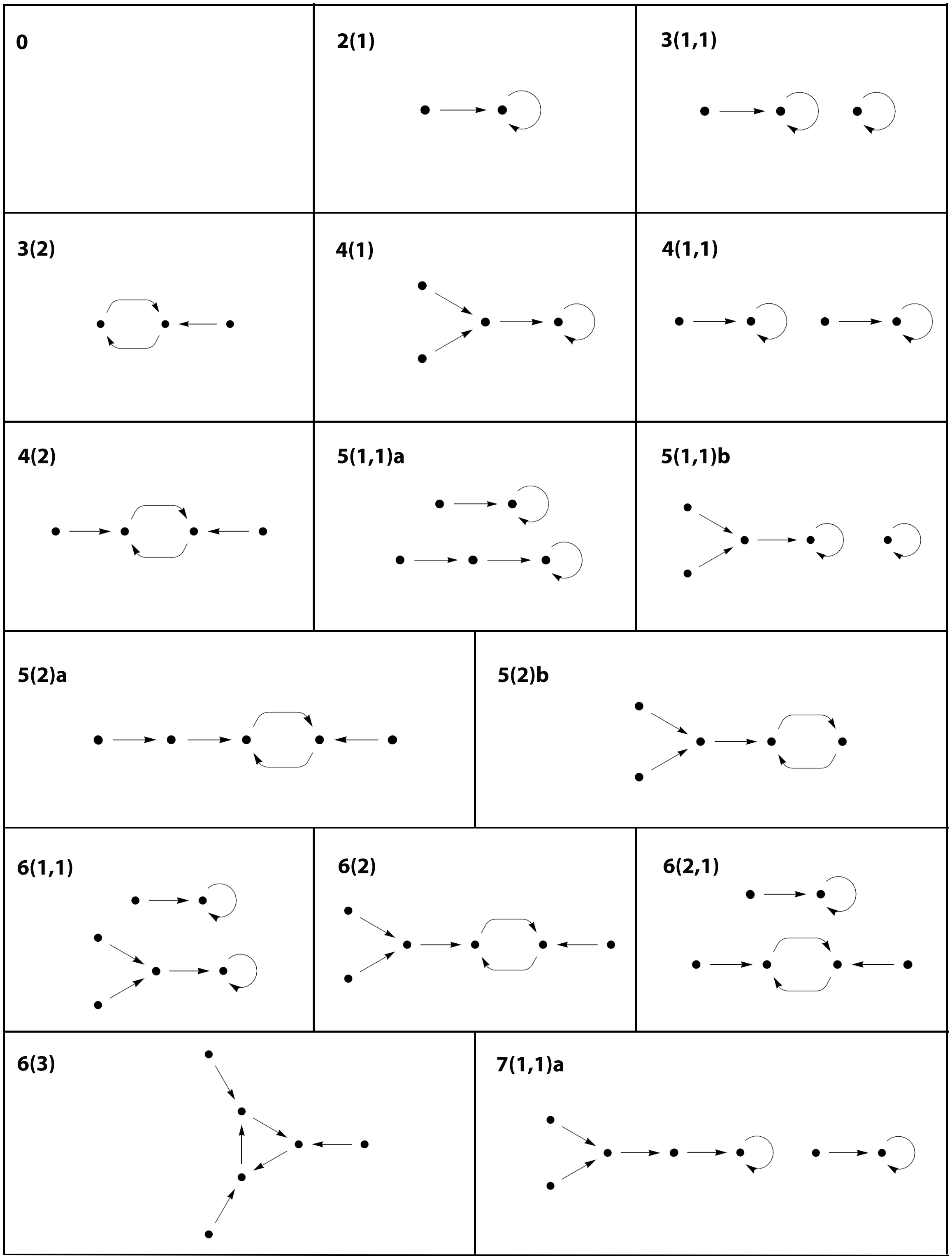
Appendix A

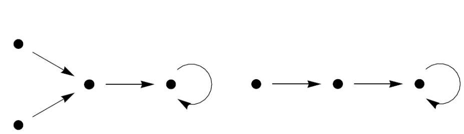
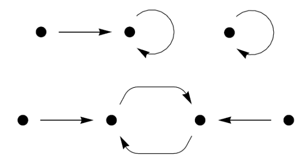
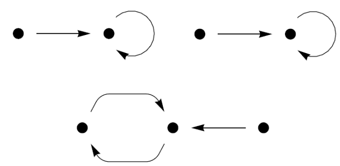
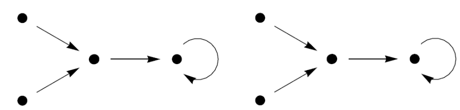
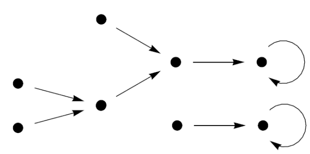
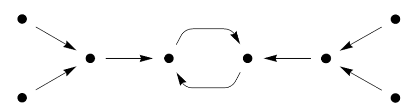
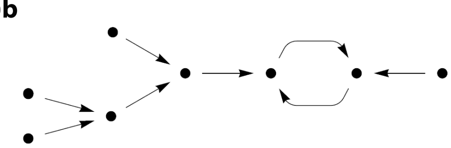
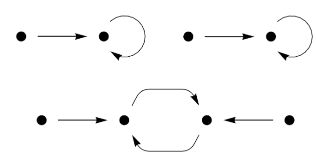
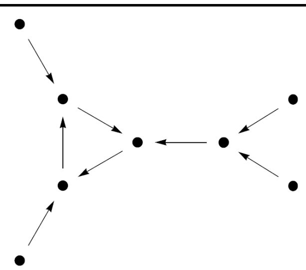
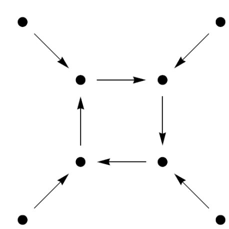
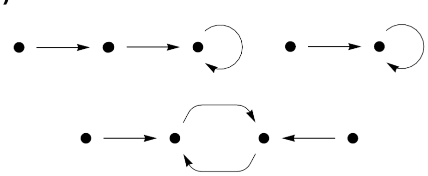
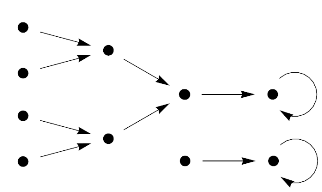
Preperiodic graphs over quadratic extensions

In this appendix, we provide a summary of the data obtained in the search for preperiodic graphs for quadratic polynomials over quadratic fields conducted in [8].

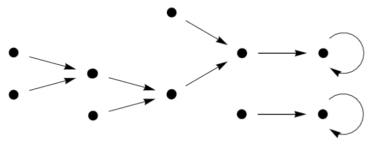
A.1 List of known graphs

We list here the 46 preperiodic graphs discovered in [8]. The label of each graph is in the form $N(\ell_1, \ell_2, \ldots)$, where N denotes the number of vertices in the graph and $\ell_1, \ell_2, \ldots$ are the lengths of the directed cycles in the graph in nonincreasing order. If more than one isomorphism class of graphs with this data was observed, we add a lowercase Roman letter to distinguish them. For example, the labels 5(1,1)a and 5(1,1)b correspond to the two isomorphism classes of graphs observed that have five vertices and two fixed points. In all figures below we omit the connected component corresponding to the point at infinity.

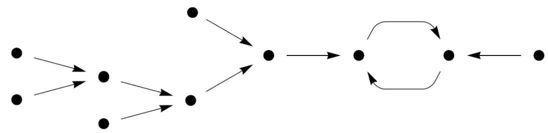

0
2(1)
3(1,1)
3(2)
4(1)
4(1,1)
4(2)
5(1,1)a
5(1,1)b
5(2)a
5(2)b
6(1,1)
6(2)
6(2,1)
6(3)
7(1,1)a

7(1,1)b 	**7(2,1,1)a** 
7(2,1,1)b 	**8(1,1)a** 
8(1,1)b 	**8(2)a** 
8(2)b 	**8(2,1,1)** 
8(3) 	**8(4)** 
9(2,1,1) 	**10(1,1)a** 

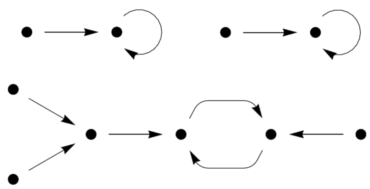
10(1,1)b



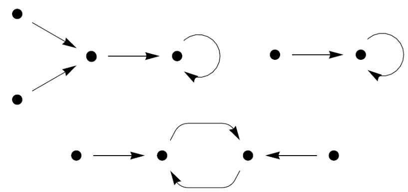
10(2)



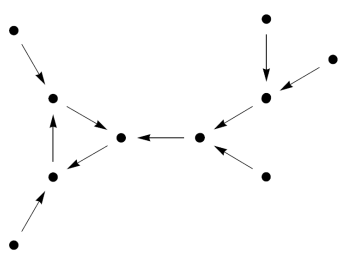
10(2,1,1)a



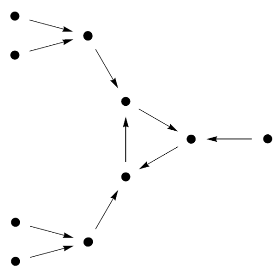
10(2,1,1)b



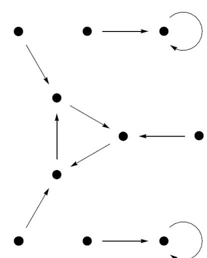
10(3)a



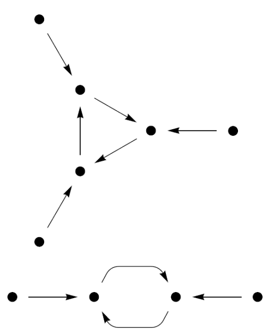
10(3)b



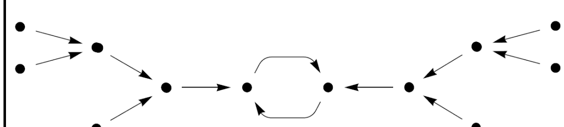
10(3,1,1)



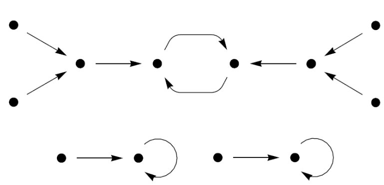
10(3,2)



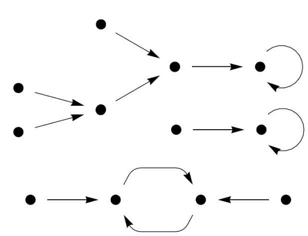
12(2)

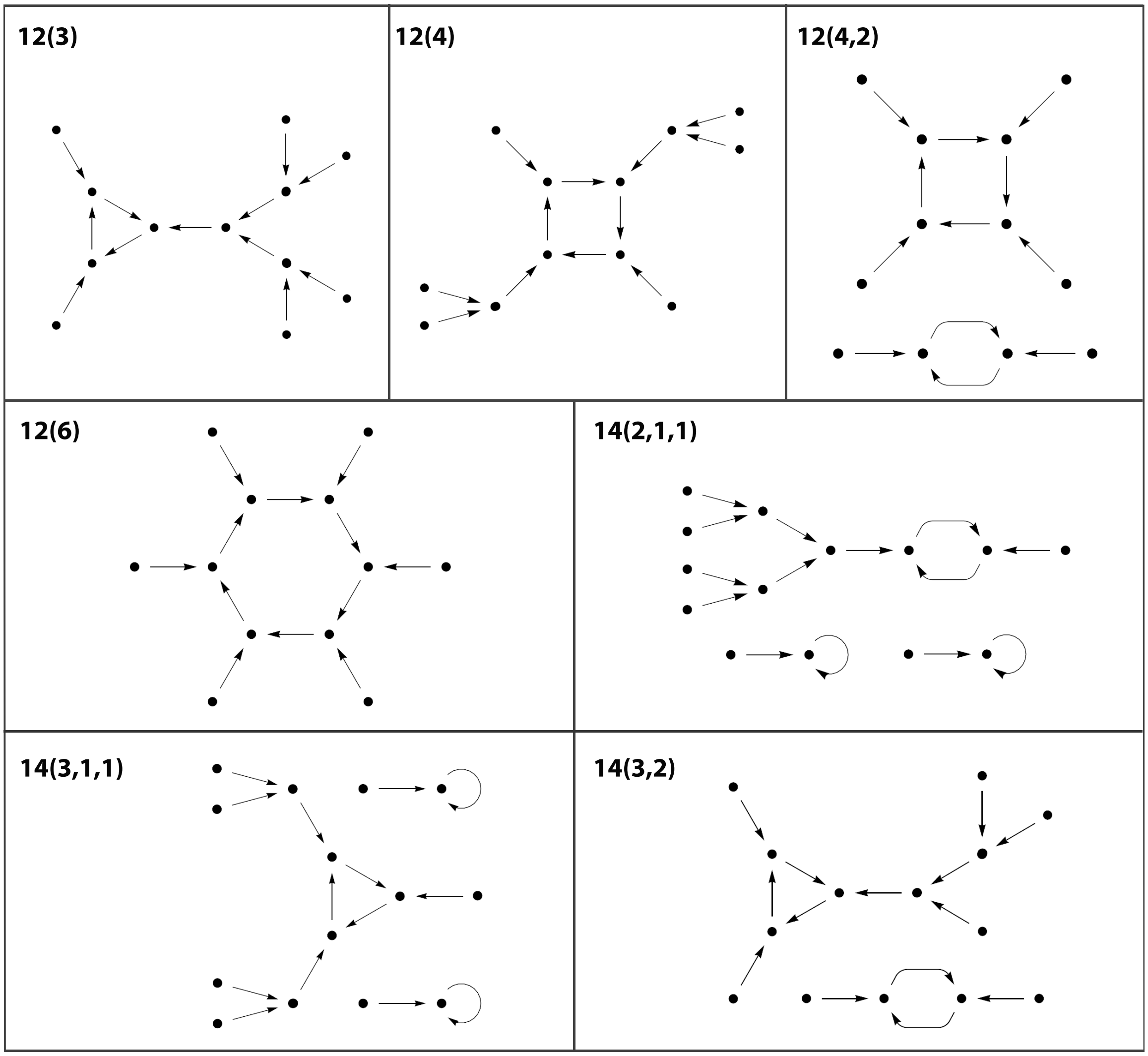


12(2,1,1)a



12(2,1,1)b





12(3)
12(4)
12(4,2)
12(6)
14(2,1,1)
14(3,1,1)
14(3,2)

A.2 Representative data

We give here a representative set of data for each graph in §A.1. Each item in the list below includes the following information:

$$K, p(t), c, \mathrm{PrePer}(f_c, K)'.$$

Here $K = \mathbb{Q}(\sqrt{D})$ is a quadratic field over which this preperiodic structure was observed; $p(t)$ is a defining polynomial for K with a root $g \in K$; c is an element of K such that the set $\mathrm{PrePer}(f_c, K) \setminus \{\infty\}$, when endowed with the structure of a directed graph, is isomorphic to the given graph; and $\mathrm{PrePer}(f_c, K)'$ is an abbreviated form of the full set of finite K-rational preperiodic points for f_c: since $x \in \mathrm{PrePer}(f_c, K)$ if and only if $-x \in \mathrm{PrePer}(f_c, K)$, we list only one of x and $-x$ in the set $\mathrm{PrePer}(f_c, K)'$. We do not make explicit the correspondence between individual elements of this set and vertices of the graph. If a particular graph was observed over both real and imaginary quadratic fields, we give a representative set of data for each case.

0: $\mathbb{Q}(\sqrt{5}),\ \ t^2 - t - 1,\ \ 1,\ \ \emptyset$

$\mathbb{Q}(\sqrt{-3}),\ \ t^2 - t + 1,\ \ 2,\ \ \emptyset$

2(1): $\mathbb{Q}(\sqrt{5}),\ \ t^2 - t - 1,\ \ \frac{1}{4},\ \ \left\{\frac{1}{2}\right\}$

$\mathbb{Q}(\sqrt{-7}),\ \ t^2 - t + 2,\ \ \frac{1}{4},\ \ \left\{\frac{1}{2}\right\}$

3(1,1): $\mathbb{Q}(\sqrt{5}),\ \ t^2 - t - 1,\ \ 0,\ \ \{0, 1\}$

$\mathbb{Q}(\sqrt{-7}),\ \ t^2 - t + 2,\ \ 0,\ \ \{0, 1\}$

3(2): $\mathbb{Q}(\sqrt{3}),\ \ t^2 - 3,\ \ -1,\ \ \{0, 1\}$

$\mathbb{Q}(\sqrt{-3}),\ \ t^2 - t + 1,\ \ -1,\ \ \{0, 1\}$

4(1): $\mathbb{Q}(\sqrt{-3}),\ \ t^2 - t + 1,\ \ \frac{1}{4},\ \ \left\{\frac{1}{2}, g - \frac{1}{2}\right\}$

4(1,1): $\mathbb{Q}(\sqrt{5})$, t^2-t-1, $\frac{1}{5}$, $\left\{\frac{1}{5}g+\frac{2}{5}, \frac{1}{5}g-\frac{3}{5}\right\}$

$\mathbb{Q}(\sqrt{-3})$, t^2-t+1, 1, $\{g, g-1\}$

4(2): $\mathbb{Q}(\sqrt{5})$, t^2-t-1, $-\frac{4}{5}$, $\left\{\frac{1}{5}g+\frac{2}{5}, \frac{1}{5}g-\frac{3}{5}\right\}$

$\mathbb{Q}(\sqrt{-3})$, t^2-t+1, $-\frac{2}{3}$, $\left\{\frac{1}{3}g-\frac{2}{3}, \frac{1}{3}g+\frac{1}{3}\right\}$

5(1,1)a: $\mathbb{Q}(\sqrt{13})$, t^2-t-3, -2, $\{0,2,1\}$

$\mathbb{Q}(\sqrt{-3})$, t^2-t+1, -2, $\{0,2,1\}$

5(1,1)b: $\mathbb{Q}(\sqrt{-1})$, t^2+1, 0, $\{0,1,g\}$

5(2)a: $\mathbb{Q}(\sqrt{-1})$, t^2+1, g, $\{0,g,g-1\}$

5(2)b: $\mathbb{Q}(\sqrt{2})$, t^2-2, -1, $\{0,1,g\}$

6(1,1): $\mathbb{Q}(\sqrt{5})$, t^2-t-1, $-\frac{3}{4}$, $\left\{\frac{1}{2}, g-\frac{1}{2}, \frac{3}{2}\right\}$

$\mathbb{Q}(\sqrt{-3})$, t^2-t+1, $-\frac{3}{4}$, $\left\{\frac{1}{2}, \frac{3}{2}, g-\frac{1}{2}\right\}$

6(2): $\mathbb{Q}(\sqrt{5})$, t^2-t-1, -3, $\{1,2,2g-1\}$

$\mathbb{Q}(\sqrt{-3})$, t^2-t+1, $-\frac{13}{9}$, $\left\{\frac{1}{3}, \frac{4}{3}, \frac{5}{3}\right\}$

6(2,1): $\mathbb{Q}(\sqrt{-1})$, t^2+1, $\frac{1}{4}$, $\left\{\frac{1}{2}, g-\frac{1}{2}, g+\frac{1}{2}\right\}$

6(3): $\mathbb{Q}(\sqrt{33})$, t^2-t-8, $-\frac{301}{144}$, $\left\{\frac{5}{12}, \frac{19}{12}, \frac{23}{12}\right\}$

$\mathbb{Q}(\sqrt{-67})$, t^2-t+17, $-\frac{301}{144}$, $\left\{\frac{5}{12}, \frac{19}{12}, \frac{23}{12}\right\}$

7(1,1)a: $\mathbb{Q}(\sqrt{2})$, t^2-2, -2, $\{0,1,2,g\}$

7(1,1)b: $\mathbb{Q}(\sqrt{3})$, t^2-3, -2, $\{0,1,2,g\}$

7(2,1,1)a: $\mathbb{Q}(\sqrt{-3})$, t^2-t+1, 0, $\{0,1,g,g-1\}$

7(2,1,1)b: $\mathbb{Q}(\sqrt{5})$, t^2-t-1, -1, $\{0,1,g,g-1\}$

8(1,1)a: $\mathbb{Q}(\sqrt{13})$, t^2-t-3, $-\frac{289}{144}$, $\{\frac{5}{6}g+\frac{1}{12},\frac{1}{2}g-\frac{13}{12},\frac{1}{2}g+\frac{7}{12},\frac{5}{6}g-\frac{11}{12}\}$

$\mathbb{Q}(\sqrt{-15})$, t^2-t+4, $-\frac{5}{16}$, $\{\frac{1}{4},\frac{3}{4},\frac{5}{4},\frac{1}{2}g-\frac{1}{4}\}$

8(1,1)b: $\mathbb{Q}(\sqrt{13})$, t^2-t-3, $-\frac{40}{9}$, $\{\frac{4}{3},\frac{8}{3},\frac{5}{3},\frac{4}{3}g-\frac{2}{3}\}$

$\mathbb{Q}(\sqrt{-2})$, t^2+2, $-\frac{10}{9}$, $\{\frac{2}{3},\frac{1}{3}g,\frac{4}{3},\frac{5}{3}\}$

8(2)a: $\mathbb{Q}(\sqrt{10})$, t^2-10, $-\frac{13}{9}$, $\{\frac{1}{3},\frac{1}{3}g,\frac{4}{3},\frac{5}{3}\}$

$\mathbb{Q}(\sqrt{-3})$, t^2-t+1, $-\frac{5}{12}$, $\{\frac{2}{3}g-\frac{5}{6},\frac{2}{3}g+\frac{1}{6},\frac{1}{3}g+\frac{5}{6},\frac{1}{3}g-\frac{7}{6}\}$

8(2)b: $\mathbb{Q}(\sqrt{13})$, t^2-t-3, $-\frac{37}{9}$, $\{\frac{4}{3},\frac{5}{3},\frac{7}{3},\frac{4}{3}g-\frac{2}{3}\}$

$\mathbb{Q}(\sqrt{-7})$, t^2-t+2, $-\frac{13}{16}$, $\{\frac{1}{4},\frac{3}{4},\frac{1}{2}g-\frac{1}{4},\frac{5}{4}\}$

8(2,1,1): $\mathbb{Q}(\sqrt{5})$, t^2-t-1, -12, $\{3,3g-1,3g-2,4\}$

$\mathbb{Q}(\sqrt{-3})$, t^2-t+1, $\frac{7}{12}$, $\{\frac{2}{3}g+\frac{1}{6},\frac{2}{3}g-\frac{5}{6},\frac{4}{3}g-\frac{7}{6},\frac{4}{3}g-\frac{1}{6}\}$

8(3): $\mathbb{Q}(\sqrt{5})$, t^2-t-1, $-\frac{29}{16}$, $\{\frac{1}{4},\frac{5}{4},\frac{3}{4},\frac{7}{4}\}$

$\mathbb{Q}(\sqrt{-3})$, t^2-t+1, $-\frac{29}{16}$, $\{\frac{1}{4},\frac{5}{4},\frac{3}{4},\frac{7}{4}\}$

8(4): $\mathbb{Q}(\sqrt{10})$, t^2-10, $-\frac{155}{72}$, $\{\frac{1}{4}g-\frac{1}{6},\frac{1}{4}g+\frac{1}{6},\frac{1}{12}g-\frac{3}{2},\frac{1}{12}g+\frac{3}{2}\}$

$\mathbb{Q}(\sqrt{-455})$, $t^2-t+114$, $\frac{199}{720}$, $\{\frac{1}{10}g+\frac{17}{60},\frac{1}{15}g-\frac{47}{60},\frac{1}{10}g-\frac{23}{60},\frac{1}{15}g+\frac{43}{60}\}$

9(2,1,1): $\mathbb{Q}(\sqrt{5})$, t^2-t-1, -2, $\{0,1,2,g,g-1\}$

10(1,1)a: $\mathbb{Q}(\sqrt{-7})$, t^2-t+2, $\frac{3}{16}$, $\{\frac{1}{4},\frac{1}{2}g+\frac{1}{4},\frac{1}{2}g-\frac{1}{4},\frac{1}{2}g-\frac{3}{4},\frac{3}{4}\}$

10(1,1)b: $\mathbb{Q}(\sqrt{17})$, t^2-t-4, $-\frac{1}{2}g-\frac{13}{16}$, $\{\frac{1}{4},\frac{1}{2}g+\frac{3}{4},\frac{3}{4},\frac{1}{2}g-\frac{1}{4},\frac{1}{2}g+\frac{1}{4}\}$

10(2): $\mathbb{Q}(\sqrt{73})$, t^2-t-18, $\frac{1}{9}g-\frac{205}{144}$,

$$\left\{\tfrac{1}{6}g+\tfrac{1}{12}, \tfrac{1}{6}g-\tfrac{11}{12}, \tfrac{1}{6}g+\tfrac{7}{12}, \tfrac{1}{3}g-\tfrac{7}{12}, \tfrac{1}{3}g-\tfrac{1}{12}\right\}$$

$\mathbb{Q}(\sqrt{-7})$, t^2-t+2, $-\frac{1}{2}g-\frac{5}{16}$, $\left\{\frac{1}{4}, \frac{1}{2}g-\frac{1}{4}, \frac{1}{2}g+\frac{1}{4}, \frac{3}{4}, \frac{1}{2}g+\frac{3}{4}\right\}$

10(2,1,1)a: $\mathbb{Q}(\sqrt{17})$, t^2-t-4, $-\frac{273}{64}$, $\left\{\frac{11}{8}, \frac{13}{8}, \frac{19}{8}, \frac{5}{4}g-\frac{5}{8}, \frac{21}{8}\right\}$

$\mathbb{Q}(\sqrt{-1})$, t^2+1, $\frac{3}{8}g-\frac{1}{4}$, $\left\{\frac{3}{4}g+\frac{1}{4}, \frac{3}{4}g-\frac{3}{4}, \frac{1}{4}g-\frac{1}{4}, \frac{1}{4}g+\frac{3}{4}, \frac{1}{4}g-\frac{5}{4}\right\}$

10(2,1,1)b: $\mathbb{Q}(\sqrt{13})$, t^2-t-3, $-\frac{10}{9}$, $\left\{\frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{1}{3}g-\frac{2}{3}, \frac{1}{3}g+\frac{1}{3}\right\}$

$\mathbb{Q}(\sqrt{-7})$, t^2-t+2, $-\frac{21}{16}$, $\left\{\frac{1}{4}, \frac{7}{4}, \frac{1}{2}g-\frac{1}{4}, \frac{3}{4}, \frac{5}{4}\right\}$

10(3)a: $\mathbb{Q}(\sqrt{41})$, t^2-t-10, $-\frac{29}{16}$, $\left\{\frac{1}{4}, \frac{5}{4}, \frac{3}{4}, \frac{1}{2}g-\frac{1}{4}, \frac{7}{4}\right\}$

10(3)b: $\mathbb{Q}(\sqrt{57})$, t^2-t-14, $-\frac{29}{16}$, $\left\{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{1}{2}g-\frac{1}{4}\right\}$

10(3,1,1) $\mathbb{Q}(\sqrt{337})$, t^2-t-84, $-\frac{301}{144}$, $\left\{\frac{5}{12}, \frac{19}{12}, \frac{23}{12}, \frac{1}{6}g+\frac{5}{12}, \frac{1}{6}g-\frac{7}{12}\right\}$

10(3,2): $\mathbb{Q}(\sqrt{193})$, t^2-t-48, $-\frac{301}{144}$, $\left\{\frac{5}{12}, \frac{19}{12}, \frac{23}{12}, \frac{1}{6}g+\frac{5}{12}, \frac{1}{6}g-\frac{7}{12}\right\}$

12(2): $\mathbb{Q}(\sqrt{2})$, t^2-2, $-\frac{15}{8}$, $\left\{\frac{3}{4}g+\frac{1}{2}, \frac{3}{4}g-\frac{1}{2}, \frac{1}{4}g+\frac{1}{2}, \frac{1}{4}g-\frac{3}{2}, \frac{1}{4}g-\frac{1}{2}, \frac{1}{4}g+\frac{3}{2}\right\}$

12(2,1,1)a: $\mathbb{Q}(\sqrt{17})$, t^2-t-4, $-\frac{13}{16}$, $\left\{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{1}{2}g+\frac{1}{4}, \frac{1}{2}g-\frac{3}{4}, \frac{1}{2}g-\frac{1}{4}\right\}$

12(2,1,1)b: $\mathbb{Q}(\sqrt{33})$, t^2-t-8, $-\frac{45}{16}$, $\left\{\frac{3}{4}, \frac{9}{4}, \frac{5}{4}, \frac{1}{2}g-\frac{3}{4}, \frac{1}{2}g+\frac{1}{4}, \frac{1}{2}g-\frac{1}{4}\right\}$

$\mathbb{Q}(\sqrt{-7})$, t^2-t+2, $-\frac{5}{16}$, $\left\{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{1}{2}g+\frac{1}{4}, \frac{1}{2}g-\frac{3}{4}, \frac{1}{2}g-\frac{1}{4}\right\}$

12(3): $\mathbb{Q}(\sqrt{73})$, t^2-t-18, $-\frac{301}{144}$, $\left\{\frac{1}{6}g-\frac{1}{12}, \frac{5}{12}, \frac{19}{12}, \frac{1}{3}g+\frac{1}{12}, \frac{1}{3}g-\frac{5}{12}, \frac{23}{12}\right\}$

12(4): $\mathbb{Q}(\sqrt{105})$, t^2-t-26, $-\frac{95}{48}$,

$$\left\{\tfrac{1}{6}g-\tfrac{13}{12}, \tfrac{1}{6}g+\tfrac{11}{12}, \tfrac{1}{3}g-\tfrac{5}{12}, \tfrac{1}{6}g+\tfrac{5}{12}, \tfrac{1}{6}g-\tfrac{7}{12}, \tfrac{1}{3}g+\tfrac{1}{12}\right\}$$

12(4,2): $\mathbb{Q}(\sqrt{-15})$, t^2-t+4, $-\frac{31}{48}$,

$$\left\{\tfrac{1}{3}g+\tfrac{1}{12}, \tfrac{1}{6}g-\tfrac{13}{12}, \tfrac{1}{3}g-\tfrac{5}{12}, \tfrac{1}{6}g+\tfrac{5}{12}, \tfrac{1}{6}g-\tfrac{7}{12}, \tfrac{1}{6}g+\tfrac{11}{12}\right\}$$

12(6): $\mathbb{Q}(\sqrt{33})$, $t^2 - t - 8$, $-\frac{71}{48}$,

$$\left\{\tfrac{1}{6}g - \tfrac{13}{12}, \tfrac{1}{6}g - \tfrac{7}{12}, \tfrac{1}{3}g - \tfrac{5}{12}, \tfrac{1}{6}g + \tfrac{5}{12}, \tfrac{1}{3}g + \tfrac{1}{12}, \tfrac{1}{6}g + \tfrac{11}{12}\right\}$$

14(2,1,1): $\mathbb{Q}(\sqrt{17})$, $t^2 - t - 4$, $-\frac{21}{16}$, $\left\{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{1}{2}g - \frac{1}{4}, \frac{1}{2}g - \frac{3}{4}, \frac{1}{2}g + \frac{1}{4}\right\}$

14(3,1,1): $\mathbb{Q}(\sqrt{33})$, $t^2 - t - 8$, $-\frac{29}{16}$, $\left\{\frac{1}{4}, \frac{5}{4}, \frac{3}{4}, \frac{1}{2}g - \frac{3}{4}, \frac{1}{2}g + \frac{1}{4}, \frac{1}{2}g - \frac{1}{4}, \frac{7}{4}\right\}$

14(3,2): $\mathbb{Q}(\sqrt{17})$, $t^2 - t - 4$, $-\frac{29}{16}$, $\left\{\frac{1}{4}, \frac{5}{4}, \frac{3}{4}, \frac{1}{2}g - \frac{1}{4}, \frac{1}{2}g - \frac{3}{4}, \frac{1}{2}g + \frac{1}{4}, \frac{7}{4}\right\}$

Bibliography

[1] Johan Bosman et al. "Ranks of Elliptic Curves with Prescribed Torsion over Number Fields". *International Mathematics Research Notices* (2013).

[2] Thierry Bousch. "Sur quelques problèmes de dynamique holomorphe". PhD thesis. Université de Paris-Sud, Centre d'Orsay, 1992.

[3] Lennart Carleson and Theodore W. Gamelin. *Complex dynamics*. Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993, pp. x+175.

[4] J. W. S. Cassels and E. V. Flynn. *Prolegomena to a middlebrow arithmetic of curves of genus* 2. Vol. 230. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 1996, pp. xiv+219.

[5] Claude Chabauty. "Sur les points rationnels des courbes algébriques de genre supérieur à l'unité". *C. R. Acad. Sci. Paris* 212 (1941), pp. 882–885.

[6] Robert F. Coleman. "Effective Chabauty". *Duke Math. J.* 52(3) (1985), pp. 765–770.

[7] J. E. Cremona. *Algorithms for modular elliptic curves*. Second Ed. Cambridge: Cambridge University Press, 1997.

[8] John R. Doyle, Xander Faber, and David Krumm. "Preperiodic points for quadratic polynomials over quadratic fields". Preprint.

[9] E. V. Flynn. "A flexible method for applying Chabauty's theorem". *Compositio Math.* 105(1) (1997), pp. 79–94.

[10] E. V. Flynn, Bjorn Poonen, and Edward F. Schaefer. "Cycles of quadratic polynomials and rational points on a genus-2 curve". *Duke Math. J.* 90(3) (1997), pp. 435–463.

[11] Benjamin Hutz. "Determination of all rational preperiodic points for morphisms of $\mathbb{P}^N$". *Math. Comp.* To appear.

[12] Benjamin Hutz and Patrick Ingram. "On Poonen's conjecture concerning rational preperiodic points of quadratic maps". *Rocky Mountain J. Math.* 43(1) (2013), pp. 193–204.

[13] S. Kamienny. "Torsion points on elliptic curves and q-coefficients of modular forms". *Invent. Math.* 109(2) (1992), pp. 221–229.

[14] M. A. Kenku and F. Momose. "Torsion points on elliptic curves defined over quadratic fields". *Nagoya Math. J.* 109 (1988), pp. 125–149.

[15] David Krumm. "Quadratic points on modular curves". PhD thesis. University of Georgia, 2013.

[16] Dino Lorenzini and Thomas J. Tucker. "Thue equations and the method of Chabauty-Coleman". *Invent. Math.* 148(1) (2002), pp. 47–77.

[17] B. Mazur. "Modular curves and the Eisenstein ideal". *Inst. Hautes Études Sci. Publ. Math.* (47) (1977), 33–186 (1978).

[18] William McCallum and Bjorn Poonen. "The method of Chabauty and Coleman". *Explicit methods in number theory: Rational points and Diophantine equations.* Société Math. de France, 2013, pp. 99–117.

[19] Loïc Merel. "Bornes pour la torsion des courbes elliptiques sur les corps de nombres". *Invent. Math.* 124(1-3) (1996), pp. 437–449.

[20] J. S. Milne. "Jacobian varieties". *Arithmetic geometry (Storrs, Conn., 1984).* New York: Springer, 1986, pp. 167–212.

[21] Patrick Morton. "Arithmetic properties of periodic points of quadratic maps". *Acta Arith.* 62 (1992), pp. 343–372.

[22] Patrick Morton. "Arithmetic properties of periodic points of quadratic maps, II". *Acta Arith.* 87 (1998), pp. 89–102.

[23] Patrick Morton. "On certain algebraic curves related to polynomial maps". *Compositio Math.* 103(3) (1996), pp. 319–350.

[24] Patrick Morton and Pratiksha Patel. "The Galois theory of periodic points of polynomial maps". *Proc. London Math. Soc. (3)* 68(2) (1994), pp. 225–263.

[25] Patrick Morton and Joseph H. Silverman. "Rational periodic points of rational functions". *Int. Math. Res. Not.* 1994(2) (1994), pp. 97–110.

[26] Patrick Morton and Franco Vivaldi. "Bifurcations and discriminants for polynomial maps". *Nonlinearity* 8(4) (1995), pp. 571–584.

[27] Filip Najman. "Complete classification of torsion of elliptic curves over quadratic cyclotomic fields". *J. Number Theory* 130(9) (2010), pp. 1964–1968.

[28] Filip Najman. "Torsion of elliptic curves over quadratic cyclotomic fields". *Math. J. Okayama Univ.* 53 (2011), pp. 75–82.

[29] André Néron. "Problèmes arithmétiques et géométriques rattachés à la notion de rang d'une courbe algébrique dans un corps". *Bull. Soc. Math. France* 80 (1952), pp. 101–166.

[30] D. G. Northcott. "Periodic points on an algebraic variety". *Ann. of Math.* 51(1) (1950), pp. 167–177.

[31] Chatchawan Panraksa. "Arithmetic dynamics of quadratic polynomials and dynamical units". PhD thesis. University of Maryland, College Park, 2011.

[32] Bjorn Poonen. “The classification of rational preperiodic points of quadratic polynomials over $\mathbb{Q}$: a refined conjecture”. *Math. Z.* 228(1) (1998), pp. 11–29.

[33] F. Patrick Rabarison. “Structure de torsion des courbes elliptiques sur les corps quadratiques”. *Acta Arith.* 144(1) (2010), pp. 17–52.

[34] Joseph H. Silverman. *The Arithmetic of Dynamical Systems*. Graduate Texts in Mathematics. New York: Springer, 2007.

[35] Joseph H. Silverman. *The Arithmetic of Elliptic Curves*. Second Ed. Vol. 106. Graduate Texts in Mathematics. Dordrecht: Springer, 2009.

[36] Michael Stoll. “Independence of rational points on twists of a given curve”. *Compos. Math.* 142(5) (2006), pp. 1201–1214.

[37] R. Walde and P. Russo. “Rational periodic points of the quadratic function $Q_c(x) = x^2 + c$”. *Amer. Math. Monthly* 101 (1994), pp. 318–331.

[38] Lawrence C. Washington. “A family of cyclic quartic fields arising from modular curves”. *Math. Comp.* 57(196) (1991), pp. 763–775.