Simplicial complexes with many facets are vertex decomposable

Anton Dochtermann[a] Ritika Nair[b] Jay Schweig[b] Adam Van Tuyl[c]
Russ Woodroofe[d]

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Abstract

Suppose Δ is a pure simplicial complex on n vertices having dimension d and let $c = n-d-1$ be its codimension in the simplex. Terai and Yoshida proved that if the number of facets of Δ is at least $\binom{n}{c}-2c+1$, then Δ is Cohen-Macaulay. We improve this result by showing that these hypotheses imply the stronger condition that Δ is vertex decomposable. We give examples to show that this bound is optimal, and that the conclusion cannot be strengthened to the class of matroids or shifted complexes. We explore an application to Simon's Conjecture and discuss connections to other results from the literature.

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1 Introduction

Suppose Δ is a simplicial complex on vertex set $[n] = \{1, \dots, n\}$. Recall that Δ is *pure* if all the facets of Δ (the maximal elements of Δ under inclusion) have the same cardinality. As first shown by Terai and Yoshida [14, Theorem 3.1], if the number of facets of Δ is "large enough," then Δ is Cohen-Macaulay. In other words, the simplicial complex Δ enjoys additional topological properties that can be detected by simply counting the number of facets.

In this short note we strengthen Terai and Yoshida's results. As in their paper, we express our theorem in terms of the codimension of Δ relative to the simplex on the same vertex set. Recall that the *dimension* of Δ is given by $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$,

[a]Department of Mathematics, Texas State University, San Marcos, TX 78666, USA
dochtermann@txstate.edu
[b]Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA
ritika.nair@okstate.edu, jay.schweig@okstate.edu
[c] Department of Mathematics and Statistics, McMaster University, Hamilton, ON, L8S 4L8, Canada
vantuyla@mcmaster.ca
[d] FAMNIT, University of Primorska, 6000 Koper, Slovenia russ.woodroofe@famnit.upr.si

where $\dim F = |F| - 1$. If Δ has n vertices, then the *codimension* of Δ, denoted $\operatorname{codim} \Delta$, is given by $c = n - \dim \Delta - 1$ (the number of vertices in the complement of a facet). Our main result can now be stated as follows.

Theorem 1. *Let Δ be a pure simplicial complex on vertex set $[n]$ of codimension c. If Δ has at least $\binom{n}{c} - 2c + 1$ facets, then Δ is vertex decomposable.*

The class of vertex decomposable simplicial complexes was introduced by Provan and Billera in [12], and have a recursive definition (see Section 2 for the formal statement). Terai and Yoshida [14, Theorem 3.1] showed that, under the same hypotheses as Theorem 1, the simplicial complex Δ is Cohen-Macaulay. Their result was stated in terms of the multiplicity of the Stanley-Reisner ring, which agrees with the number of facets of Δ when Δ is pure. Since being vertex decomposable implies being Cohen-Macaulay, our main theorem strengthens their result. To prove [14, Theorem 3.1], Terai and Yoshida used an algebraic approach, employing the Stanley-Reisner correspondence between simplicial complexes and square-free monomials. Our proof, in contrast, requires only simple tools from topological combinatorics.

The bound in Theorem 1 is tight. In Example 13 we describe a family of simplicial complexes on n vertices of codimension c with $\binom{n}{c} - 2c$ facets, each of which fails to be vertex decomposable. In Remark 15 we also show that Theorem 1 cannot be strengthened to conclude that Δ is a matroid complex or shifted complex (both of which are classes known to be vertex decomposable), even if we require $\binom{n}{c} - 2$ facets.

It is not hard to see that a simplicial complex Δ on vertex set $[n]$ with dimension d and codimension c can have more than $\binom{n}{c} - 2c$ faces of dimension d, but fail to be pure. For instance, consider the 1-dimensional complex consisting of a triangle and an isolated vertex. However, we observe in Lemma 16 that if we require at least $\binom{n}{c} - c$ facets of this top dimension, then purity is guaranteed. This leads to Corollary 17 for arbitrary simplicial complexes, which strengthens [14, Theorem 2.1].

We also interpret our results in terms of square-free monomial ideals (the language used by Terai and Yoshida). In Corollary 18 we describe a sufficient condition for a square-free monomial ideal to have a linear resolution, which may be useful for researchers working in combinatorial commutative algebra.

Our main result also has connections to other work involving decompositions of simplicial complexes. Simon's Conjecture is a well-known open question regarding the structure of *shellable* complexes (see Conjecture 21). Combining our results with those from [3], we see that any counterexample to Simon's Conjecture cannot have "too many" facets. On the other hand, a result of Lasoń from [9] provides a sufficient condition for a complex Δ to be vertex decomposable in terms of extremal properties of the f-vector of Δ. In Example 23 we discuss how Theorem 1 relates to the results of Lasoń.

Our paper is organized as follows. In Section 2 we provide the necessary definitions and background, and in particular recall relevant properties of vertex decomposable simplicial complexes. In Section 3 we prove Theorem 1. In Section 4, we address the optimality of Theorem 1, interpret our result in the language of square-free monomial ideals, and discuss the connection to Simon's Conjecture and Lasoń's result.

2 Background

In this section, we recall additional relevant definitions and results needed to prove our main theorem. We refer the reader to [6] or [15] for additional background on combinatorial commutative algebra, and to [1] for topological combinatorics. We continue to use the notation from the introduction.

Let Δ be a simplicial complex on vertex set V, which we typically take as $V = [n] := \{1, 2, \ldots, n\}$. An element $F \in \Delta$ is called a *face*, and a face not properly contained in any other face is called a *facet*. For each $i \in [n]$ we always assume that $\{i\}$ is a face of Δ, which we call a vertex and denote simply as i. We will use the notation $\Delta = \langle F_1, \ldots, F_s \rangle$ to denote the complete list of distinct facets. After fixing a vertex $x \in [n]$, we can define two other simplicial complexes from Δ. Namely, the *link* of $\{x\}$ is the simplicial complex

$$\mathrm{link}_\Delta(\{x\}) = \{G \in \Delta \mid x \notin G, \quad G \cup \{x\} \in \Delta\},$$

and the *deletion* of $\{x\}$ is the complex

$$\mathrm{del}_\Delta(\{x\}) = \{G \in \Delta \mid x \notin G\}.$$

We will abuse notation and write $\mathrm{link}_\Delta(x)$ (respectively, $\mathrm{del}_\Delta(x)$) instead of $\mathrm{link}_\Delta(\{x\})$ (respectively, $\mathrm{del}_\Delta(\{x\})$). A vertex x is a *shedding vertex* if for any facet $F \in \Delta$ with $x \in F$, there exists a vertex $y \notin F$ such that $(F \backslash \{x\}) \cup \{y\}$ is also a face of Δ. Equivalently, x is a shedding vertex if every facet of $\mathrm{del}_\Delta(x)$ is a facet of Δ.

Vertex decomposable simplicial complexes are then defined recursively as follows.

Definition 2. A simplicial complex Δ is *vertex decomposable* if either

1. $\Delta = \{\emptyset\}$, or Δ is a simplex (i.e., $\Delta = \langle F \rangle$ for some $F \subseteq [n]$), or
2. there exists a shedding vertex $x \in [n]$ such that $\mathrm{link}_\Delta(x)$ and $\mathrm{del}_\Delta(x)$ are vertex decomposable.

Remark 3. We take this definition from Björner and Wachs [2]. All the complexes we consider will be pure, and in this case our definition recovers the notion of vertex decomposability introduced by Provan and Billera in [12].

We next collect some results regarding vertex decomposable complexes that will be useful for our study. Recall that a simplicial complex Δ is a *cone* if there exists a vertex $v \in V$ such that $v \in F$ for every facet $F \in \Delta$. This implies that there exists a simplicial complex Γ on vertex set $V \backslash \{v\}$ such that every face of Δ is either a face of Γ, or else of the form $F \cup \{v\}$ for some face $F \in \Gamma$. In this case we write $\Delta = \Gamma * v$. In the statement below, the *k-skeleton* of a pure simplicial complex Δ is (for $k \leqslant \dim \Delta$) the simplicial complex whose facets are the faces of Δ having dimension k.

Lemma 4. *The following properties regarding vertex decomposability hold.*

1. *[12, Proposition 3.1.1] All* 0*-dimensional complexes are vertex decomposable.*
2. *[12, Proposition 3.1.2] A pure* 1*-dimensional complex* Δ *is vertex decomposable if and only if* Δ *is connected.*
3. *[16, Lemma 3.10] Any* k*-skeleton of a vertex decomposable complex* Δ *is vertex decomposable.*
4. *[12, Proposition 2.4] A cone* $\Delta = \Gamma * v$ *is vertex decomposable if and only if* Γ *is vertex decomposable.*

We also record the following observation.

Lemma 5. *Suppose* Δ *is a pure simplicial complex on* $[n]$ *with* $n \geqslant 3$ *obtained by removing a single facet from the boundary of a simplex. Then* Δ *is vertex decomposable.*

Proof. Suppose Δ has vertex set $[n]$, and without loss of generality suppose the facet removed to obtain Δ is $\{1, 2, \ldots, n-1\}$. Then the facets of Δ consist of all subsets of $[n]$ of the form $\{n\} \cup F$, where F is any $(n-2)$-subset of $[n-1]$. Hence $\Delta = \partial\Delta_{n-1} * n$. Here $\partial\Delta_{n-1}$ denotes the boundary of the simplex on $[n-1]$, which can be recovered as the $(n-3)$-skeleton of Δ_{n-1}. From Lemma 4 we conclude that Δ is vertex decomposable. □

The following lemma is immediate from the recursive definition of vertex decomposability, and we will use it without comment.

Lemma 6. *If* $\mathcal{F}$ *is a family of simplicial complexes, so that every* $\Delta \in \mathcal{F}$ *is either a simplex or has a shedding vertex* v *with the property that both* $\operatorname{link}_\Delta(v)$ *and* $\operatorname{del}_\Delta(v)$ *are in* $\mathcal{F}$*, then every complex in* $\mathcal{F}$ *is vertex decomposable.*

3 Proof of Main Result

In this section we provide the proof of Theorem 1. In what follows, we let Δ be a pure simplicial complex on n vertices with codimension $c = n - \dim\Delta - 1$. If F is a set of $\dim\Delta + 1$ vertices that is not a face of Δ, we say F is an *antifacet*. We write $e(\Delta)$ for the number of facets of Δ and $\bar{e}(\Delta)$ for the number of antifacets of Δ. Note that $\bar{e}(\Delta) = \binom{n}{c} - e(\Delta)$.

We start with a technical lemma.

Lemma 7. *If* $\dim\Delta \geqslant 1$*,* $c \geqslant 2$ *and* $\bar{e}(\Delta) \leqslant 2c-1$*, then* Δ *has at least two vertices that each avoids at most* $2c-3$ *antifacets.*

Proof. We proceed by double counting on the set

$$\mathcal{P} = \{(v, F) : v \text{ a vertex of } \Delta, F \text{ an antifacet of } \Delta, v \notin F\}.$$

Counting by antifacets, we see that

$$|\mathcal{P}| \leqslant c \cdot (2c-1) = 2c^2 - c. \tag{1}$$

Suppose for a contradiction that all but at most one vertex avoids more than $2c-3$ antifacets. Then

$$|\mathcal{P}| \geqslant (n-1) \cdot (2c-2) \geqslant (c+1) \cdot (2c-2) = 2c^2 - 2. \tag{2}$$

As $2c^2 - c \geqslant 2c^2 - 2$ holds only if $c \leqslant 2$, we have reduced to the case $c = 2$. But in this case, the inequalities (1) and (2) give us that $2 \cdot 3 \geqslant (n-1) \cdot 2$, so we are left with the case $n = 4$.

Thus, the proof reduces to the case where Δ is a one-dimensional complex on 4 vertices. In this case, $\bar{e}(\Delta) \leqslant 3$ and $c = 2$, and so we need to show there must be at least two vertices that each avoids at most $2c - 3 = 1$ antifacets. Viewing the antifacets as edges in a graph on four vertices, the antifacets give a graph on four vertices where every vertex has degree at most two (since Δ cannot have an isolated vertex). We finish the proof with the following easily-verified observation: For any graph on four vertices with three or fewer edges, where the degree of each vertex is $\leqslant 2$, there are at least two vertices that are each disjoint from at most one edge. □

Remark 8. Note that if Δ has the property that all vertices are shedding, then for any v we have that $\mathrm{del}_\Delta(v)$ is pure of the same dimension as Δ. Thus, in this situation Lemma 7 implies that Δ has at least two vertices satisfying $\bar{e}(\mathrm{del}_\Delta(v)) \leqslant 2c - 3$.

We make some other easy observations. Recall that a *minimal nonface* of Δ is a nonface of Δ that is minimal under inclusion with respect to this property. Note that if $\{v, w\}$ is a minimal nonface of Δ of cardinality two, then w is *not* a vertex of $\mathrm{link}_\Delta(v)$. The following lemma says that minimal nonfaces of cardinality two are rare when $\dim \Delta \neq 1$ and the number of antifacets is small enough (equivalently, if the number of facets is large enough relative to n).

Lemma 9. *If $\bar{e}(\Delta) \leqslant 2c$, then no nonface of Δ has fewer than $\dim \Delta$ vertices, and at most one (minimal) nonface has $\dim \Delta$ vertices.*

Proof. A nonface of size $\dim \Delta - 1$ is contained in $\binom{c+2}{2}$ antifacets. But $(c+2)(c+1)/2$ is greater than $2c$ for all c, so no such nonface can exist.

Each nonface with $\dim \Delta$ vertices is contained in $c + 1$ antifacets. If there existed a pair of such nonfaces, they would be contained in at most one common antifacet. But $2(c+1) - 1$ is greater than $2c$, contradicting our assumptions. □

Corollary 10. *Suppose v is a vertex of Δ. If $\bar{e}(\Delta) \leqslant 2c - 1$ and b is the codimension of $\mathrm{link}_\Delta(v)$, then $\bar{e}(\mathrm{link}_\Delta(v)) \leqslant 2b - 1$.*

Proof. Note that the statement is trivial if $\dim \Delta \leqslant 1$ since in this case $\bar{e}(\mathrm{link}_\Delta(v)) = 0$. Now since Δ is pure, we have that $\dim \mathrm{link}_\Delta(v) = \dim \Delta - 1$. If $\dim \Delta > 2$, then by Lemma 9 we have that Δ has no nonface of size two, and hence $\mathrm{link}_\Delta(v)$ has $n-1$ vertices by the comment after Remark 8. From this we get that $b = (n-1) - (\dim \Delta - 1) - 1 = c$. The result then follows by noting that if F is an antifacet of $\mathrm{link}_\Delta(v)$, then $F \cup \{v\}$ is an antifacet of Δ.

If $\dim \Delta = 2$, then there are two cases. If Δ has no nonface of size two, then the argument follows as above. Otherwise, Δ has a nonface of size two, and then again by Lemma 9 we have that $b = c - 1$ (and hence also $c \geqslant 1$), and there is a unique minimal nonface $\{v, w\}$ having two vertices. Now, as before there are $\bar{e}(\mathrm{link}_\Delta(v))$ antifacets of Δ containing v but not w, and an additional $n-2$ antifacets of Δ containing $\{v, w\}$. We obtain that $\bar{e}(\mathrm{link}_\Delta(v)) \leqslant \bar{e}(\Delta) - (n-2) \leqslant (2c-1) - (c+1) = 2b - c$, which yields the desired result. □

We restate the definition of a shedding vertex in the language of antifacets.

Lemma 11. *A vertex w of Δ is a shedding vertex unless w is contained in some facet F with the property that, for each vertex $u \notin F$, the set $(F \setminus \{w\}) \cup \{u\}$ is an antifacet.*

With this we have the following observation.

Lemma 12. *If $\bar{e}(\Delta) \leqslant 2c - 1$, then:*

1. *Any vertex that is contained in at least c antifacets is a shedding vertex.*
2. *If w is not a shedding vertex, and F is a facet as in Lemma 11, then any vertex $v \in F \setminus \{w\}$ is contained in c antifacets, and in particular is a shedding vertex.*

Proof. If w is not a shedding vertex, then w is contained in some facet F so that $(F \setminus \{w\}) \cup \{u\}$ is an antifacet for each $u \notin F$. In particular, there are c antifacets that do not contain w, hence at most $c - 1$ that do contain w.

Moreover, since the set $F \setminus \{w\}$ is contained in c antifacets, the same holds for every $v \in F \setminus \{w\}$. □

We are now ready to prove our main result.

Proof of Theorem 1. Recall that Δ is assumed to be a pure simplicial complex on vertex set $[n]$ with codimension c, with at least $\binom{n}{c} - 2c + 1$ facets, so that $\bar{e}(\Delta) \leqslant 2c - 1$. The statement can be checked directly for all simplicial complexes on $n \leqslant 3$ vertices, so we suppose that $n > 3$. It is convenient to handle the cases when $c = 0, 1, n-2, n-1$ separately. If $c = 0$ or 1, then Δ is a simplex, or a simplex boundary with at most one face removed, and the result follows by Lemma 5. On the other hand, if $c = n - 1$, then $\dim \Delta = 0$, and the result is immediate.

Moving forward, we can assume that $c \leqslant n - 2$ (or equivalently, that $\dim \Delta \geqslant 1$) and that $c \geqslant 2$. To prove the theorem we wish to find a shedding vertex v so that $\mathrm{link}_\Delta(v)$ and $\mathrm{del}_\Delta(v)$ also satisfy the hypotheses of the theorem. Note that we are using Lemma 6. First observe that for any vertex v, the complex $\mathrm{link}_\Delta(v)$ is pure of dimension $\dim \Delta - 1$, and also satisfies the hypotheses in Corollary 10.

We next turn to the deletion. For this note that if v is a shedding vertex, then by definition $\dim \mathrm{del}_\Delta(v) = \dim \Delta$. Since the number of vertices of $\mathrm{del}_\Delta(v)$ is $n - 1$, we have that the codimension of $\mathrm{del}_\Delta(v)$ is $c-1$. Hence to complete the proof we need to find a shedding vertex v satisfying $\bar{e}(\mathrm{del}_\Delta(v)) \leqslant 2c-3$.

If every (or even all but one) vertex of Δ is a shedding vertex, then we are done by Lemma 7. Otherwise, Lemma 12 yields a shedding vertex v that is contained in at least c antifacets, so that $\bar{e}(\mathrm{del}_\Delta(v)) \leqslant 2c-1-c$. In either case, we have produced the desired shedding vertex and the result follows. □

4 Consequences and Concluding Remarks

We end with some additional comments and examples related to our main result. We first give an example to show that the bound in Theorem 1 is optimal.

Example 13. For any $2 \leqslant c \leqslant n-2$, we construct a pure simplicial complex Δ on n vertices that has codimension c and $\binom{n}{c}-2c$ facets, and that is not Cohen–Macaulay. Since any vertex decomposable complex is also Cohen–Macaulay, this shows that the bound in Theorem 1 is optimal.

We start with the complete $(n-c-1)$-skeleton of the simplex on n vertices, and obtain Δ by removing the facets containing the following vertices: all of $\{1,\dots,n-c-2\}$ (this set is $\{1\}$ if $n-c-2=1$ and is empty if $n-c-2=0$), exactly one of $\{n-c-1, n-c\}$, and exactly one of $\{n-c+1,\dots,n\}$. Then we have removed exactly $2c$ facets, and $\mathrm{link}_\Delta(\{1,\dots,n-c-2\})$ is $1-$dimensional and disconnected, meaning Δ is not Cohen–Macaulay, by Reisner's Criterion. Since every vertex decomposable complex is Cohen-Macaulay, Δ is not vertex decomposable.

Remark 14. In the situation where $\dim \Delta = 1$, so that $c = n-2$, we may view Δ as a graph on vertex set $[n]$ and our main result has a graph theoretic interpretation. Since Δ is assumed to be pure, we consider graphs with no isolated vertex. In this context, Example 13 yields a complex Δ that is the union of a complete graph with an isolated edge, and the link being considered is $\mathrm{link}_\Delta(\emptyset) = \Delta$. Indeed, from Lemma 4 we know that a pure 1-dimensional complex is vertex decomposable if and only if it is connected. Hence in this case Theorem 1 is equivalent to the following graph theoretic statement: an edge-cut of the complete graph K_n into nontrivial parts has size at least $2c = 2n-4$. We refer to [4] for definitions from graph theory.

Remark 15. One can also ask whether the conclusion of Theorem 1 can be strengthened, that is whether there is some property *stronger* than vertex decomposable that is implied by Δ having "enough facets". Two well-studied classes of vertex decomposable simplicial complexes are independence complexes of matroids and shifted complexes.

Recall that a pure simplicial complex Δ is the independence complex of a *matroid* if it satisfies the following exchange property: if F and G are facets of Δ such that $v \in F\backslash G$, then there exists a $w \in G\backslash F$ such that $(F\backslash\{v\}) \cup \{w\}$ is again a facet of Δ. A pure simplicial complex Δ is *shifted* if there exists a (re)labeling of the vertex set $V = [n] = \{1,\dots,n\}$ such that whenever $\{v_1, v_2, \dots, v_k\}$ is a face of Δ, replacing any v_i by a vertex with a smaller label again results in a face of Δ.

We show that neither of these properties are implied by the assumptions in Theorem 1. As an example, consider the complete graph K_4 as a 1-dimensional simplex on $n = 4$ vertices (so that $c = 2$). Theorem 1 implies that if we remove 3 or fewer facets (edges) and are left with no isolated vertices, we end up with a vertex decomposable complex. It turns out that we can remove just two facets from K_4 to destroy these stronger properties.

On the one hand, we can delete the facets $\{1, 2\}$ and $\{1, 3\}$, resulting in a complex that is easily seen not to be a matroid complex. On the other hand, if we start with K_4 and delete the facets $\{1, 2\}$ and $\{3, 4\}$ we are left with a 4-cycle, which is not shifted. Indeed the class of shifted 1-dimensional complexes coincide with the class of *threshold* graphs [8], which are known to be chordal [10].

We can also recover a result similar to [14, Theorem 2.1]. In particular, if we assume Δ has "even more" facets, then Δ is automatically pure, so it must be vertex decomposable.

Lemma 16. *Let Δ be a simplicial complex on vertex set $[n]$ of codimension c. Suppose Δ has at least $\binom{n}{c} - c$ facets of dimension $\dim \Delta = n - c - 1$. Then Δ is pure.*

Proof. If Δ has a facet F of dimension smaller than $\dim \Delta$, then F can be extended to a set S containing $\dim \Delta$ vertices. Namely, if G is a facet of dimension $\dim \Delta$, then $F \not\subset G$, giving $|G \backslash F| \geqslant \dim \Delta - \dim F + 1$. We obtain the set S by adding $\dim \Delta - \dim F$ vertices from $G \backslash F$ to F. Now, S is contained in $c + 1$ sets of size $n - c$, none of which are facets of Δ (as S properly contains the facet F). This is a contradiction to the assumption that there are at least $\binom{n}{c} - c$ facets of dimension $\dim \Delta = n - c - 1$. □

We now derive a strengthening of [14, Theorem 2.1]:

Corollary 17. *Let Δ be a simplicial complex on vertex set $[n]$ of codimension c. If Δ has at least $\binom{n}{c} - c$ facets of dimension $\dim \Delta = n - c - 1$, then Δ is vertex decomposable.*

Proof. If $c = 0$, then Δ is a simplex and the result follows immediately. So, suppose that $c \geqslant 1$. By Lemma 16, Δ is a pure simplicial complex of codimension c with at least $\binom{n}{c} - c \geqslant \binom{n}{c} - 2c + 1$ facets. Now apply Theorem 1. □

Terai and Yoshida showed that Stanley-Reisner rings having large multiplicities are Cohen-Macaulay [14, Theorem 3.1], by proving that the dual ideal of the Stanley-Reisner ideal of Δ has a linear resolution [14, Theorem 3.3]. We can also recover this dual version as a direct consequence of our main result by employing Stanley-Reisner theory and properties of Alexander duality, and in particular Eagon and Reiner's classification of square-free monomial ideals with a linear resolution. The following corollary may be of interest for those researchers working with monomial ideals. We assume that the reader is familiar with the Stanley-Reisner correspondence; for undefined terminology, see [11]. The following result originally appeared as [14, Theorem 3.3].

Corollary 18. *Let I be a square-free monomial ideal of $R = \mathbb{K}[x_1, \ldots, x_n]$ with $\mathbb{K}$ an arbitary field, and suppose that every generator has degree c. If I has at least $\binom{n}{c} - 2c + 1$ generators, then I has a linear resolution.*

Proof. Given a square-free monomial $m = x_{i_1} \cdots x_{i_c}$ of degree c, set $F_m = [n] \setminus \{i_1, \ldots, i_c\}$. So, if $I = \langle m_1, \ldots, m_t \rangle$ is a square-free monomial ideal where each generator has degree c, then $\Delta = \langle F_{m_1}, \ldots, F_{m_t} \rangle$ is a pure simplicial complex of codimension c with at least $\binom{n}{c} - 2c + 1$ facets. By Theorem 1, Δ is a vertex decomposable simplicial complex, and consequently, Δ is Cohen-Macaulay.

The ideal I is the Alexander dual of I_Δ, the Stanley-Reisner ideal of Δ. Since Δ is Cohen-Macaulay, the Eagon-Reiner Theorem [5] implies that I has a linear resolution. □

Our results also have applications to the study of *Simon's Conjecture* [13], a statement involving the structure of shellable complexes. To explain this connection we first review some relevant concepts.

Definition 19. A pure d-dimensional simplicial complex Δ is *shellable* if there exists an ordering of its facets $F_1, F_2, \ldots, F_s$ such that for all $k = 2, 3, \ldots, s$ the simplicial complex

$$\left(\bigcup_{i=1}^{k-1} \langle F_i \rangle \right) \cap \langle F_k \rangle$$

is pure of dimension $d - 1$.

By convention the void complex $\emptyset$ and the empty complex $\{\emptyset\}$ are both shellable. Note that any pure d-dimensional complex Δ on vertex set $[n]$ is a subcomplex of $\Delta_n^{(d)}$, the d-dimensional skeleton of the simplex on $[n]$. It is known that $\Delta_n^{(d)}$ is shellable, and one can ask if it is possible to produce a shelling of $\Delta_n^{(d)}$ that begins with a shelling of Δ.

Definition 20. A pure d-dimensional simplicial complex Δ on vertex set $[n]$ is *shelling completable* if there exists a shelling of Δ that is the initial sequence of a shelling of $\Delta_n^{(d)}$.

With these notions, we can formulate Simon's Conjecture as follows.

Conjecture 21. Every shellable complex is shelling completable.

In [3] it is shown that every pure vertex decomposable complex is shelling completable. Hence our Theorem 1 implies that any counterexample to Simon's Conjecture (a pure shellable complex that is not shelling completable) must have "not too many facets". In particular if Δ has codimension c, then any counterexample can have at most $\binom{n}{c} - 2c$ facets.

There are other ways to detect the vertex decomposability of a simplicial complex Δ in terms of combinatorial data. In [9] Lasoń provides a sufficient condition for vertex decomposability in terms of the f-vector of Δ (more specifically the number of facets and *ridges*). To describe these results, suppose Δ is a pure simplicial complex on vertex set $[n]$, with dimension $\dim \Delta = d$ and codimension $\operatorname{codim} \Delta = c = n - d - 1$. Recall that the *$f$-vector* of Δ is $f(\Delta) = (f_0, f_1, \ldots, f_d)$ where f_i is the number of faces of Δ of dimension i. Theorem 1 implies that if $f_d \geqslant \binom{n}{c} - 2c + 1$, then Δ is vertex decomposable.

We say that a simplicial complex Δ is *extremal* if Δ is pure of dimension t with r facets such that it has the minimum number of faces of dimension $(t - 1)$ among all simplicial complexes of dimension t- and r-faces. As shown by [9, Corollary 1], an extremal simplicial complex can be determined from its f-vector. The main result from [9] is the following.

Theorem 22. *[9, Theorem 5] An extremal simplicial complex is vertex decomposable.*

Example 23. To see that our result does not follow from Lasoń's work, let $n = 6$ and consider the pure two-dimensional simplicial complex with facets

$$\{1,2,3\},\ \{1,4,5\},\ \{1,2,6\},\ \{1,4,6\},\ \{2,4,5\},\ \{3,4,5\},\ \{3,5,6\}$$

and any other 9 facets of dimension 2, for a total of 16 facets of dimension 2. In this case, the codimension is $c = 3$, and $16 \geqslant \binom{6}{3} - 2 \cdot 3 + 1 = 15$, so, this simplicial complex is vertex decomposable by our Theorem 1. By our construction, this simplicial complex has f-vector $(6, 15, 16)$ (our initial choice of facets forces this simplicial complex to contain all the 1-dimensional faces). However, this f-vector does not satisfy [9, Corollary 1]. Indeed, there exists a pure simplicial complex with f-vector $(6, 14, 16)$, namely the simplicial complex which contains all the facets of dimension two *except* $\{1, 5, 6\}$, $\{2, 5, 6\}$, $\{3, 5, 6\}$, and $\{4, 5, 6\}$ (this simplicial complex does not contain the face $\{5, 6\}$). We see that our simplicial complex is not extremal, and hence one can not deduce that it is vertex decomposable from [9].

As a final consequence, we mention a connection to geometrically vertex decomposable ideals, as first defined by Klein and Rajchgot [7]. While we do not reproduce the formal definition here, a geometrically vertex decomposable ideal is a generalization of the properties of the square-free monomial ideals that corresponds to a vertex decomposable simplicial complex via the Stanley-Reisner correspondence. As shown in [7, Proposition 2.14], under a suitable lexicographical monomial order $<$, in some cases we can determine if an ideal I is a geometrically vertex decomposable ideal from its initial ideal $\mathrm{in}_<(I)$. In particular, if $\mathrm{in}_<(I)$ is the Stanley-Reisner ideal of a vertex decomposable simplicial complex, then I is geometrically vertex decomposable. By combining Klein and Rajchgot's result with our Theorem 1, we get a new technique to check if an ideal is geometrically vertex decomposable. Precisely, if there is an appropriate monomial order $<$ such that the initial ideal $\mathrm{in}_<(I)$ is the Stanley-Reisner ideal of a simplicial complex with "enough" facets, as determined by Theorem 1, then I is geometrically vertex decomposable.

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