Toric Ideals of Lattice Path Matroids and Polymatroids

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Abstract

White has conjectured that the toric ideal of a matroid is generated by quadric binomials corresponding to symmetric basis exchanges. We prove a stronger version of this conjecture for lattice path polymatroids by constructing a monomial order under which these sets of quadrics form Gröbner bases. We then introduce a larger class of polymatroids for which an analogous theorem holds. Finally, we obtain the same result for lattice path matroids as a corollary.

1 Introduction

If $B = \{i_1, i_2, \ldots, i_r\}$ is a basis of a matroid M, the toric map of M sends the base ring variable Y_B to the monomial $x_{i_1} x_{i_2} \cdots x_{i_r}$ and is naturally extended over all polynomials in variables indexed by bases of M. White's conjecture (see [13]) posits that the kernel of this map is generated by quadratic binomials corresponding to symmetric exchanges between pairs of bases of M. This conjecture has received much attention, but has only been proven for graphic matroids (shown by Blasiak in [2]) and matroids of rank 3 (shown by Kashiwabara in [9]).

Lattice path matroids, introduced by Bonin, de Mier and Noy in [3] and studied further in [4], constitute a nice class of matroids whose bases are in correspondence with certain planar lattice paths. Subclasses of these matroids appeared in [10] and [1]. In [12], the study of enumerative properties of such matroids gave rise to a related class of discrete polymatroids, in the sense of Herzog and Hibi [8], known as lattice path polymatroids.

As in [8], toric maps can be defined for discrete polymatroids as well, inspiring a generalization of White's conjecture. In [5], Conca shows that toric ideals of transversal polymatroids (a class of polymatroids containing lattice path polymatroids) are generated by binomials, although White's conjecture for these polymatroids remains open. In Theorem 3.1, we show that White's conjecture holds for lattice path polymatroids. We also provide a monomial order under which the

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generating set of symmetric exchange binomials forms a Gröbner basis for the toric ideal.

In Section 4 we introduce pruned lattice path polymatroids, a larger class of polymatroids for which an analogue of Theorem 3.1 holds. Finally, we show how any lattice path matroid may be recognized as a pruned lattice path polymatroid, proving an analogue of Theorem 3.1 for lattice path matroids.

2 Preliminaries

We assume the reader has a basic knowledge of matroid theory (see [11]). All our monomials are in the variables $\{x_0, x_1, x_2, \ldots\}$. When m is a monomial, we write $d_i(m)$ to mean the degree of x_i in m.

Definition 2.1. *Let Γ be a finite collection of monomials. Then Γ is a* discrete polymatroid *if it satisfies the following two properties.*

1: If $m \in \Gamma$ and m' divides m, then $m' \in \Gamma$, and

2: If $m, m' \in \Gamma$ and the degree of m is greater than that of m', there exists an index i such that $d_i(m) > d_i(m')$ and $x_i m' \in \Gamma$.

Thus, a matroid can be viewed as a squarefree discrete polymatroid. It is easily seen that the maximal monomials of a discrete polymatroid Γ are all of the same degree. In keeping with standard matroid terminology, we refer to these maximal monomials as *bases*, and we say their degree is the *rank* of Γ. Discrete polymatroids were introduced by Herzog and Hibi in [8], where the following polymatroidal analogue of the classical symmetric exchange property for matroids was proven.

Proposition 2.2. *Let m and m' be bases of a discrete polymatroid Γ, and choose i with $d_i(m) > d_i(m')$. Then there exists an index j with $d_j(m) < d_j(m')$, such that both $\frac{x_j}{x_i}m$ and $\frac{x_i}{x_j}m'$ are bases of Γ.*

In the case of the above proposition, we say that the bases $\frac{x_j}{x_i}m$ and $\frac{x_i}{x_j}m'$ are obtained from m and m' via a *symmetric exchange*.

2.1 *Lattice paths*

Fix two integers $n, r > 0$. For our purposes, a *lattice path* is a sequence of unit-length steps in the plane, each either due north or east, beginning at the origin and ending at the point (n, r). For a lattice path σ, define a set $N(\sigma) \subseteq [n + r]$ by the following rule: $i \in N(\sigma) \Leftrightarrow$ the i^{th} step of σ is north.

Let σ and τ be lattice paths. We say that σ is above τ if for all $i \leq n$ the i^{th} east step of σ lies on or above the i^{th} east step of τ. In this case, we write $\sigma \succeq \tau$.

Now fix two lattice paths α and ω with $\alpha \succeq \omega$.

Theorem 2.3 ([3]). *The collection $\{N(\sigma) : \alpha \succeq \sigma \succeq \omega\}$ is the set of bases of a matroid.*

We write $\mathcal{M}(\alpha, \omega)$ to denote the matroid determined by the paths α and ω. Matroids arising in this fashion are known as *lattice path matroids*. For any lattice path σ, define a monomial $m(\sigma)$ by the following rule: the degree of x_i in $m(\sigma)$ is the number of north steps taken by σ along the vertical line $x = i$.

Theorem 2.4 ([12]). *The collection $\{m(\sigma) : \alpha \succeq \sigma \succeq \omega\}$ is the set of bases of a discrete polymatroid.*

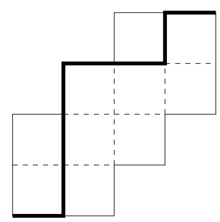

Figure 1: The lattice path matroid $\mathcal{M}(\alpha, \omega)$ where $N(\alpha) = \{1, 2, 4, 6\}$ and $N(\omega) = \{3, 5, 7, 8\}$. If σ is the bold path, $m(\sigma) = x_1^3 x_3$.

We call such discrete polymatroids *lattice path polymatroids*, and write $\Gamma(\alpha, \omega)$ to denote the polymatroid determined by α and ω. For a lattice path σ whose last step is east, let σ^+ be the path obtained from σ by removing its last east step, and adding an east step at the beginning. It is easily seen that the lattice path matroid $\mathcal{M}(\alpha, \omega)$ is coloop-free if and only if α and ω share no north steps (and thus the last step of α is east). That is, $\mathcal{M}(\alpha, \omega)$ is coloop-free if and only if $\alpha^+ \succeq \omega$. The following theorem motivated the introduction of lattice path polymatroids.

Theorem 2.5 ([12]). *Suppose the lattice path matroid $\mathcal{M}(\alpha, \omega)$ is coloop-free. Then its h-vector is the f-vector (or degree sequence) of $\Gamma(\alpha^+, \omega)$.*

Example 2.6. *If α is the path consisting of r north steps followed by n east steps and ω is any other path, then $\mathcal{M}(\alpha, \omega)$ is a* shifted *matroid. In [10], it is shown that every shifted matroid is of this form. In this case, bases of the polymatroid $\Gamma(\alpha, \omega)$ are generators of the smallest Borel-fixed ideal containing $m(\omega)$ (see [7]).*

In general, the bases of a lattice path polymatroid correspond to an interval in the Borel order of monomials of degree r.

2.2 *Toric ideals*

The *base ring* of a polymatroid Γ is the polynomial ring $\mathbb{C}[Y_m : m \text{ is a basis of } \Gamma]$. If n and n' are obtained from m and m' by a symmetric exchange (that is, $n = (x_i/x_j)m$ and $n' = (x_j/x_i)m'$ for some i and j), we call $Y_m Y_{m'} - Y_n Y_{n'}$ a *symmetric exchange* *binomial.* The *toric ideal* of Γ is the kernel of the map $\phi : \mathbb{C}[Y_m : m \text{ is a basis of } \Gamma] \to \mathbb{C}[x_0, x_1, x_2, \ldots]$ defined by

$$\phi(Y_{m_1} Y_{m_2} \cdots Y_{m_t}) = m_1 m_2 \cdots m_t$$

and extended by linearity. Clearly, any symmetric exchange binomial is in the toric ideal of Γ.

Conjecture 2.7 (White's conjecture, adapted for polymatroids)**.** *The toric ideal of Γ is generated by symmetric exchange binomials.*

For a set $V = \{m_1, m_2, \ldots, m_t\}$ of bases of Γ, write M_V as short for the base ring monomial $Y_{m_1} Y_{m_2} \cdots Y_{m_t}$. Now for any monomial μ of degree $> r$, we define a simple graph $\mathcal{G}(\mu)$, known as a *symmetric exchange graph*, as follows. The vertices of $\mathcal{G}(\mu)$ are all sets $V = \{m_1, m_2, \ldots, m_t\}$ of bases of Γ with $\phi(M_V) = \mu$ (that is, $m_1 m_2 \cdots m_t = \mu$), and two vertices V and W are connected by an edge whenever $M_V - M_W = NS$ for some monomial N and symmetric exchange binomial S. Put another way, V and W are connected by an edge if W can be obtained from V by performing a symmetric exchange on two of its constituent bases. Although $\mathcal{G}(\mu)$ depends on Γ, we omit this from the notation whenever it will be clear from context.

The following was inspired by Blasiak's techniques in [2], where Conjecture 2.7 was proven for graphic matroids.

Theorem 2.8. *Suppose that $\mathcal{G}(\mu)$ is connected for any monomial μ of degree $> r$. Then Conjecture 2.7 holds for Γ.*

Proof. Any polynomial in the toric ideal of Γ can be written as a sum of binomials of the form $M_V - M_W$, where V and W are vertices of some $\mathcal{G}(\mu)$. Since $\mathcal{G}(\mu)$ is connected, there exists a path $V = V_0, V_1, V_2, \ldots, V_k = W$ where each V_i and V_{i+1} are connected by an edge. Now write

$$M_V - M_W = (M_V - M_{V_1}) + (M_{V_1} - M_{V_2}) + (M_{V_2} - M_{V_3}) + \cdots + (M_{V_{k-1}} - M_W).$$

Since each parenthesized term in this sum is the product of a monomial with a symmetric exchange binomial, the result follows. $\square$

2.3 *Gröbner bases*

Our treatment here is brief; the reader unfamiliar with the theory of Gröbner bases is referred to [6].

Let $>_\ell$ be a total order on monomials of the base ring of a polymatroid Γ with $M >_\ell 1$ for any monomial $M \neq 1$. The order $>_\ell$ is called a *monomial order* if $M >_\ell M'$ implies that $MN >_\ell M'N$ for any monomials M, M', and N.

If $>_\ell$ is a monomial order on the base ring and μ is a monomial, define a directed graph $\mathcal{G}^\ell(\mu)$ as follows: the vertices and edges are those of $\mathcal{G}(\mu)$. If V and W are vertices of $\mathcal{G}(\mu)$ joined by an edge, direct the corresponding edge of $\mathcal{G}^\ell(\mu)$ towards W if $M_V >_\ell M_W$ and towards V if $M_W >_\ell M_V$. Note that $\mathcal{G}^\ell(\mu)$ is acyclic, since $>_\ell$ is a total order.

The following lemma, whose straightforward proof we omit, is an elementary result from graph theory.

Lemma 2.9. *Let G be a finite and acyclic directed graph, and suppose G has a unique sink v_0. Then for any vertex w of G, there exists a directed path from w to v_0.*

Theorem 2.10. *Let $>_\ell$ be a monomial order on the base ring of Γ, and suppose that $\mathcal{G}^\ell(\mu)$ has a unique sink anytime it is nonempty. Then Conjecture 2.7 holds for Γ and the symmetric exchange binomials, under the order $>_\ell$, form a Gröbner basis for its toric ideal.*

Proof. To see that Conjecture 2.7 holds for Γ, note that Lemma 2.9 implies that any two vertices of some $\mathcal{G}(\mu)$ are in the same connected component (since $\mathcal{G}(\mu)$ is just $\mathcal{G}^\ell(\mu)$ with the edge orientations removed). Therefore every $\mathcal{G}(\mu)$ is connected, and Theorem 2.8 gives us that the toric ideal of Γ is generated by symmetric exchange binomials.

To finish the proof, we apply Buchberger's algorithm (again, see [6]) to the set of symmetric exchange binomials. The S-pair of two symmetric exchange binomials can be represented as $M_V - M_W$, for two vertices V, W of some $\mathcal{G}^\ell(\mu)$. A step in the reduction of this binomial with respect to the set of symmetric exchange binomials consists either of replacing M_V with $M_{V'}$ where $V \to V'$ is a directed edge of $\mathcal{G}^\ell(\mu)$ or of replacing M_W with $M_{W'}$ where $W \to W'$ is a directed edge of $\mathcal{G}^\ell(\mu)$. Let V_0 be the unique sink of $\mathcal{G}^\ell(\mu)$. By Lemma 2.9, this binomial reduces to $M_{V_0} - M_{V_0} = 0$. □

3 Lattice path polymatroids

For the remainder of this paper, fix n and r and let α and ω be two lattice paths to (n, r) with $\alpha \succeq \omega$. To eliminate excess notation we often identify a path σ with the associated monomial $m(\sigma)$, writing, for example, Y_σ rather than $Y_{m(\sigma)}$ and $d_i(\sigma)$ rather than $d_i(m(\sigma))$. This section is devoted to proving the following theorem.

Theorem 3.1. *Let $\Gamma = \Gamma(\alpha, \omega)$ be a lattice path polymatroid. Then the toric ideal of Γ is generated by symmetric exchange binomials. That is, White's conjecture holds for Γ. Moreover, there exists a monomial order on the base ring of Γ under which the symmetric exchange binomials form a Gröbner basis for the toric ideal.*

First, we build a monomial order on the base ring of a lattice path polymatroid Γ so that we may apply Theorem 2.10.

For any lattice path σ, define $\ell(\sigma)$ to be the following nr-tuple:

$$(\ell_{0,r}, \ell_{0,r-1}, \ldots, \ell_{0,1}, \ell_{1,r}, \ell_{1,r-1}, \ldots, \ell_{1,1}, \ldots, \ell_{n-1,r}, \ell_{n-1,r-1}, \ldots, \ell_{n-1,1})$$

where $\ell_{i,j} = 1$ if the topmost north step of σ along the line $x = i$ goes from $(i, j-1)$ to (i, j), and $\ell_{i,j} = 0$ otherwise. For a base ring monomial $M = Y_{\sigma_1} Y_{\sigma_2} \cdots Y_{\sigma_t}$, let $\ell(M) = \sum_{1 \leq i \leq t} \ell(\sigma_i)$, where the sum of vectors is taken componentwise.

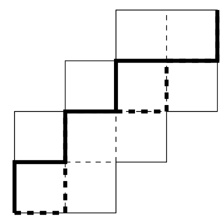

Figure 2: If $n = r = 4$ and σ_1 and σ_2 are the paths above, $\ell(Y_{\sigma_1} Y_{\sigma_2}) = (0, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 0, 1, 0, 0)$.

Now let M and M' be base ring monomials, and write $M >_\ell M'$ whenever $\ell(M)$ lexicographically precedes $\ell(M')$. Note that $>_\ell$ is not yet a *total* order, as clearly there may be monomials $M \neq M'$ with $\ell(M) = \ell(M')$.

Indeed, let $M = Y_{\sigma_1} Y_{\sigma_2} \cdots Y_{\sigma_t}$ and $M' = Y_{\tau_1} Y_{\tau_2} \cdots Y_{\tau_t}$ be two distinct base ring monomials with $\ell(M) = \ell(M')$, where the indexing paths of these monomials are ordered so that whenever $i < j$, $\ell(\sigma_i)$ lexicographically precedes $\ell(\sigma_j)$ and $\ell(\tau_i)$ precedes $\ell(\tau_j)$. Extend the definition of $>_\ell$ to say that $M >_\ell M'$ if $\ell(\tau_i)$ lexicographically precedes $\ell(\sigma_i)$ for the least i such that $\sigma_i \neq \tau_i$.

Since a path σ is clearly determined by the vector $\ell(\sigma)$, this completes $>_\ell$ to a total order on all monomials in the base ring (once we set $M >_\ell 1$ for any monomial M). Moreover, if $M >_\ell M'$, then $MN >_\ell M'N$, since $\ell(MN) = \ell(M) + \ell(N)$ and $\ell(M'N) = \ell(M') + \ell(N)$. Thus $>_\ell$ is a monomial order.

Definition 3.2. *Let $V = \{\sigma_1, \sigma_2, \ldots, \sigma_t\}$ be a vertex of $\mathcal{G}(\mu)$ for some μ, where again we have written σ_i to mean $m(\sigma_i)$. We call the vertex V* thin *if it has the following two properties:*

1: For any two paths $\sigma_i, \sigma_j \in V$, either $\sigma_i \succeq \sigma_j$ or $\sigma_j \succeq \sigma_i$.

2: For any i, the i^{th} east steps of any two paths in V are at most a unit length apart.

Thin vertices, as shown by Proposition 3.3 and Lemma 3.4, will be sinks in the directed graphs $\mathcal{G}^\ell(\mu)$.

Proposition 3.3. *Let V be a vertex of some $\mathcal{G}(\mu)$ that is not thin. Then there is a vertex V' of $\mathcal{G}(\mu)$ resulting from a symmetric exchange between two bases in V such that $M_V >_\ell M_{V'}$. In other words, $V \to V'$ is a directed edge of $\mathcal{G}^\ell(\mu)$.*

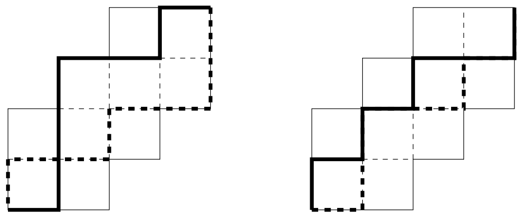

Figure 3: Two vertices of the graph $\mathcal{G}(x_0x_1^3x_2x_3x_4^2)$. The second is thin, while the first is not.

Proof. Let $V = \{\sigma_1, \sigma_2, \ldots, \sigma_t\}$. We handle two cases, each corresponding to a way in which a vertex may fail to be thin.

First suppose that the $(i+1)^{st}$ east step of some path in V is more than a unit length above the $(i+1)^{st}$ east step of another path in V, and let i be minimal with this property. Let σ_p be the path with the highest $(i+1)^{st}$ east step, and let σ_q be the path with the lowest. By the minimality of i, $d_i(\sigma_p) > d_i(\sigma_q)$. Since the two paths eventually meet, there must be some $j > i$ such that $d_j(\sigma_p) < d_j(\sigma_q)$. Let j be minimal with this property, let σ_p' be the path obtained from σ_p by adding a north step along $x = j$ and removing one along $x = i$, and let σ_q' be the path obtained from σ_q by adding a north step along $x = i$ and removing one along $x = j$. Note that σ_p' and σ_q' are the results of a symmetric exchange between σ_p and σ_q, although we still need to show that both σ_p' and σ_q' are paths in $\Gamma(\alpha, \omega)$. To see this, note that the minimality of j implies that every east step of σ_p between $x = i$ and $x = j$ is *strictly* above the corresponding east step of σ_q. Thus $\sigma_p \succeq \sigma_p' \succeq \sigma_q$ and $\sigma_p \succeq \sigma_q' \succeq \sigma_q$, meaning both σ_p' and σ_q' are between α and ω. Let V' be the vertex resulting from this symmetric exchange. Then V' is identical to V to the left of the line $x = i$. Since neither σ_p' nor σ_q' attains the same height on the line $x = i$ as σ_p, it follows that $M_V >_\ell M_{V'}$.

Now suppose that no two paths in V are ever more than a unit length apart, and let i be the least index so that V fails to be thin at the line $x = i$. Then there are paths σ_p and σ_q of V such that every east step of σ_p to the left of $x = i$ is on or above the corresponding east step of σ_q (though the two do not always coincide), but the $(i+1)^{st}$ step of σ_q is a unit length above that of σ_p. It is clear that $d_i(\sigma_p) < d_i(\sigma_q)$. Let j be the least index greater than i such that $d_j(\sigma_p) > d_j(\sigma_q)$, let σ_p' be the path obtained from σ_p by deleting a north step along $x = j$ and adding one along $x = i$, and let σ_q' be the path obtained from σ_q by deleting a north step along $x = i$ and adding one along $x = j$. The same argument from the first paragraph of this proof shows that both σ_p' and σ_q' are paths in $\Gamma(\alpha, \omega)$. Again, let V' be the vertex resulting from this symmetric exchange. Since every east step of σ_p in between $x = i$ and $x = j$ is exactly a unit length above the corresponding east step of σ_q, it follows that $\ell(M_V) = \ell(M_{V'})$. Writing $>_{lex}$ for lexicographic order, we have the following chain:

$$\ell(\sigma'_p) >_{lex} \ell(\sigma_p) >_{lex} \ell(\sigma_q) >_{lex} \ell(\sigma'_q).$$

Thus $M_V >_\ell M_{V'}$. $\square$

Lemma 3.4. *Let μ be a monomial so that $\mathcal{G}(\mu)$ is nonempty. Then $\mathcal{G}(\mu)$ has exactly one thin vertex.*

Proof. Existence follows from Proposition 3.3 and the easy fact that a finite acyclic directed graph has at least one sink.

To prove uniqueness, let $V = \{\sigma_1, \sigma_2, \dots, \sigma_t\}$ be a thin vertex, ordered so that $\sigma_1 \succeq \sigma_2 \succeq \cdots \succeq \sigma_t$, and suppose V is uniquely determined to the left of the line $x = i$ (where we allow $i = 0$). Since V is thin, there is an index k and a number p so that the i^{th} east steps of the paths $\sigma_1, \sigma_2, \dots, \sigma_k$ coincide and lie on the line $y = p$ and the i^{th} east steps of the paths $\sigma_{k+1}, \sigma_{k+2}, \dots, \sigma_t$ coincide and lie on the line $y = p - 1$. Now write $d_i(\mu) = qt + r$, with $r < t$.

If $r \leq t - k$, then the paths $\sigma_{k+1}, \sigma_{k+2}, \dots, \sigma_{k+r}$ must each have $q+1$ steps along the line $x = i$, while the rest must have q north steps along this line. If $r > t - k$, then each of the paths $\sigma_1, \sigma_2, \dots, \sigma_{r-t+k}$ and $\sigma_{k+1}, \sigma_{k+2}, \dots, \sigma_t$ must have $q+1$ steps along $x = i$, and the rest must have q steps.

Thus, V is uniquely determined to the left of the line $x = i + 1$, and the result follows. $\square$

Proof of Theorem 3.1. Proposition 3.3 and Lemma 3.4 imply that any $\mathcal{G}^\ell(\mu)$ has a unique sink (namely its thin vertex), so Theorem 2.10 finishes the proof. $\square$

4 Pruned polymatroids and lattice path matroids

Definition 4.1. *Let Γ be a rank-r polymatroid in the variables $\{x_0, x_1, x_2, \dots, x_m\}$, and let $\boldsymbol{v} = (v_0, v_1, v_2, \dots, v_m)$ be a vector of nonnegative integers. Let $B(\Gamma)$ denote the set of bases of Γ. The* pruned polymatroid $\Gamma_{\boldsymbol{v}}$ *is the polymatroid with bases*

$$\{x_0^{a_0} x_1^{a_1} \cdots x_m^{a_m} \in B(\Gamma) : a_i \leq v_i \text{ for all } i\}.$$

It is straightforward to verify that $\Gamma_{\mathbf{v}}$ is a discrete polymatroid.

Observation 4.2. *The class of lattice path polymatroids is not closed under pruning. To see this, let Γ be the lattice path polymatroid whose bases are x_0x_1, x_0x_2, x_1^2, and x_1x_2. Setting $\mathbf{v} = (1,1,1)$, we see $\Gamma_{\mathbf{v}}$ has bases x_0x_1, x_0x_2, and x_1x_2. It is easy to show that $\Gamma_{\mathbf{v}}$ is not a lattice path polymatroid.*

By Observation 4.2, the following corollary expands upon Theorem 3.1.

Corollary 4.3. *Let Γ be a lattice path polymatroid in $\{x_0, x_1, \dots, x_m\}$, and let $\boldsymbol{v} = (v_0, v_1, \dots, v_m)$ be a vector of nonnegative integers. Then the toric ideal of $\Gamma_{\boldsymbol{v}}$ is generated by symmetric exchange binomials, and there exists a monomial order under which these binomials form a Gröbner basis for the toric ideal.*

Proof. Consider a monomial μ. Using the same monomial order ℓ as in Theorem 3.1, we see that $\mathcal{G}^{\ell}_{\Gamma_{\mathbf{v}}}(\mu)$ is a directed subgraph of $\mathcal{G}^{\ell}_{\Gamma}(\mu)$. Because a symmetric exchange in Γ between two bases of $\Gamma_{\mathbf{v}}$ results in two bases of $\Gamma_{\mathbf{v}}$ (this is easy to see), Proposition 3.3 and Lemma 3.4 together imply that $\mathcal{G}^{\ell}_{\Gamma_{\mathbf{v}}}(\mu)$ has a unique sink. Applying Theorem 2.10 completes the proof. $\square$

Corollary 4.4. *Let $\mathcal{M} = \mathcal{M}(\alpha, \omega)$ be a lattice path matroid. Then the toric ideal of $\mathcal{M}$ is generated by symmetric exchange binomials, and there exists a monomial order under which these binomials form a Gröbner basis for the toric ideal.*

Proof. Let σ be a lattice path to the point (n, r) with $N(\sigma) = \{a_1, a_2, \dots, a_r\}$, where $a_1 < a_2 < \cdots < a_r$. Define a lattice path $\overline{\sigma}$ to the point $(n + r, r)$ by $N(\overline{\sigma}) = \{a_1 + 1, a_2 + 2, \dots, a_r + r\}$, and note that $m(\overline{\sigma}) = x_{a_1}x_{a_2}\cdots x_{a_r}$ (see Figure 4).

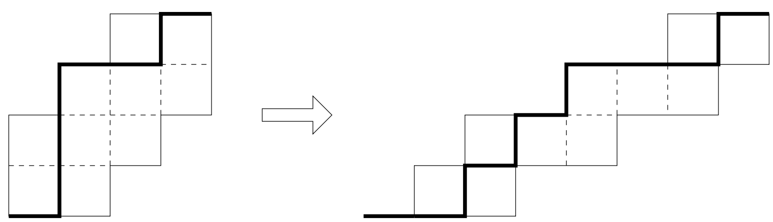

Figure 4: A lattice path matroid $\mathcal{M}(\alpha, \omega)$ and the associated polymatroid $\Gamma(\overline{\alpha}, \overline{\omega})$. If σ is the bold path, note that $N(\sigma) = \{2, 3, 4, 7\}$ and $m(\overline{\sigma}) = x_2x_3x_4x_7$.

Define a function from $\mathcal{M} = \mathcal{M}(\alpha, \omega)$ to $\Gamma = \Gamma(\overline{\alpha}, \overline{\omega})$ by $\sigma \to \overline{\sigma}$, and note that a lattice path σ in between $\overline{\alpha}$ and $\overline{\omega}$ is in the image of this map if and only if it has no more than one north step along every line $x = i$, which is equivalent to $m(\sigma)$ being squarefree. Letting $\mathbf{v} = (1, 1, \dots, 1)$, we may thus identify $\mathcal{M}$ with the pruned polymatroid $\Gamma_{\mathbf{v}}$ and apply Corollary 4.3. $\square$

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