CATALAN NUMBERS, BINARY TREES, AND POINTED PSEUDOTRIANGULATIONS

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Abstract. We study connections among structures in commutative algebra, combinatorics, and discrete geometry, introducing an array of numbers, called Borel's triangle, that arises in counting objects in each area. By defining natural combinatorial bijections between the sets, we prove that Borel's triangle counts the Betti numbers of certain Borel-fixed ideals, the number of binary trees on a fixed number of vertices with a fixed number of "marked" leaves or branching nodes, and the number of pointed pseudotriangulations of a certain class of planar point configurations.

1. Introduction

The Catalan numbers C_n, defined for $n \geq 0$ by $C_n = \frac{1}{n+1}\binom{2n}{n}$, appear in countless places throughout enumerative combinatorics. For example, in a well-known exercise from [St2], the reader is asked to show that the Catalan numbers count the elements of each of 66 sets. A related construction is Catalan's triangle, a triangular array whose right boundary gives the classical Catalan numbers.

Let $S = K[x_1, x_2, \ldots, x_n]$. One of the many sets counted by the Catalan numbers is the set of minimal monomial generators of the smallest Borel ideal containing the monomial $x_1 x_2 x_3 \cdots x_n$. As a result, the Betti numbers of these ideals can be obtained by applying an invertible transformation to the rows of Catalan's triangle, which is an array closely linked to the Catalan numbers. This results in a new triangular array, which we call *Borel's triangle.*

In [FMS], we show a curious correspondence: The Betti numbers of these ideals also count certain pointed pseudotriangulations, which are planar configurations often studied in discrete geometry and rigidity theory. Here, our interest is in illustrating this correspondence by producing bijections between these two seemingly unrelated (yet equinumerous) sets. Our bijections involve binary trees and a new labeling of their leaves, which we believe to be of independent interest (for instance, these so-called *marked* binary trees are also enumerated by the entries of Borel's triangle).

With its close connection to the Catalan numbers, we believe Borel's triangle will prove useful in other enumerative applications as well.

Let $f_{n,k}$ to denote the k^{th} element of the n^{th} row of Borel's triangle (where we begin counting rows and elements at 0). We summarize our main results in Theorem 1.1. For relevant definitions, see Sections 2 and 4.

Theorem 1.1. *Let $n \geq 1$ and $k \leq n-1$. Then the coefficient $f_{n-1,k}$ counts each of the following sets, and there exist natural bijections between them:*

1. *The number of branch-marked binary trees with n unmarked vertices and k marked branching vertices.*
2. *The number of leaf-marked binary trees with n unmarked vertices and k marked leaves.*
3. *The number of EK-symbols (m, α) of I_n in which $\deg \alpha = k$.*
4. *The number of pointed pseudotriangulations of the single chain of length n in which k interior vertices are not connected to the tip.*

We begin the paper by reviewing preliminaries on Catalan numbers and Borel ideals in Section 2. We also introduce some language in this section to help us discuss binary trees. In Section 3, we construct a labeling of the vertices of binary trees which provides a bijection between leaf-marked trees and Eliahou-Kervaire symbols (that is, between items (2) and (3) above). We begin Section 4 by reviewing the background on pointed pseudotriangulations, and we go on to produce a bijection between pointed pseudotriangulations of the single chain and branch-marked binary trees (that is, between items (1) and (4) above). (The bijection between items (1) and (2) is trivial.)

We conclude the paper by mentioning further directions for our study and by giving a class of binary parenthesizations that are enumerated by Borel's triangle.

2. Preliminaries

Definition 2.1. *Catalan's triangle* is the array $\{C_{n,k}\}$, where $0 \leq k \leq n$, defined by $C_{n,0} = 1$ for all n, $C_{n,k} = C_{n-1,k} + C_{n,k-1}$ for $0 < k < n$, and $C_{n,n} = C_{n,n-1}$.

Below we show rows zero through four of Catalan's triangle.

$$\begin{array}{lllll}
1 \\
1 & 1 \\
1 & 2 & 2 \\
1 & 3 & 5 & 5 \\
1 & 4 & 9 & 14 & 14
\end{array}$$

Definition 2.2. We define a corresponding array $\{f_{n,k}\}$ by setting $f_{n,k} = \sum_{s=0}^{n} \binom{s}{k} C_{n,s}$. We call this array *Borel's triangle.* Equivalently,

$$\sum_{k=0}^{n} C_{n,k}(t+1)^k = \sum_{k=0}^{n} f_{n,k} t^k.$$

Rows zero through four of this array are below.

1				
2	1			
5	6	2		
14	28	20	5	
42	120	135	70	14

Remark 2.3. The classical Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ occur on the diagonal of both Catalan's and Borel's triangles as well as in the first column of Borel's triangle. That is, $f_{n,0} = C_{n+1} = f_{n+1,n+1}$.

Remark 2.4. Borel's triangle appears in slightly different forms in the Online Encyclopedia of Integer Sequences [OEIS]. The sequence A062991 is the triangle with alternating signs, and P. Barry gives an explicit form for Borel's triangle in the comments, noting that

$$f_{n,k} = \frac{1}{n+1}\binom{2n+2}{n-k}\binom{n+k}{k}.$$

The entry A094385, written in triangular form, has an additional column on the left of the triangle and the entries of the row reversed from Borel's triangle. This sequence arises in Barry's recent work on generalized Pascal matrices defined by Riordan arrays; see, in particular, [B, Example 18].

Let T be a binary tree. As we work only with binary trees, we often drop the word "binary." Recall that T has a unique *root* vertex, which we call r_T. If v and w are two vertices of T, recall that w is a *descendant* of v if the unique path from r_T to w goes through v. In this case, we also say that v is an *ancestor* of w. If in addition v and w are the endpoints of an edge of T, then w is called an *child* of v and v is called the *parent* of w.

If e is an edge of T with positive (resp., negative) slope, we say e is a *left* (resp., *right*) edge. If w is a child of v and the two are connected by a left edge, we say that w is a *left* child of v, and we define the notion of a *right* child analogously. If w is a descendant of a left child of v, we say that w is a *left descendant* of v and define a right descendant similarly.

A vertex of T with no children is a *leaf*, and we call a vertex *branching* if it has two children.

Definition 2.5. Let T be a binary tree. The *rightmost* leaf of T is the leaf obtained by starting at the root and descending, take a right edge whenever possible. Let X be a set of leaves of T not containing T's rightmost leaf. Then the pair (T, X) is called a *leaf-marked tree.* We call the vertices in X *marked*, and the other vertices *unmarked.*

If B is a set of branching vertices of T, we call the pair (T, B) a *branch-marked tree.*

We often use T to refer both to the tree and its set of vertices. For instance, we may write $v \in T \smallsetminus X$ if v is a vertex of T not contained in X.

2.1. **Borel ideals.** All our ideals are in the ring $S = K[x_1, x_2, \ldots]$ for a field K. If m is a monomial and x_j divides m, replacing m with $\frac{x_i}{x_j}m$ for some $i < j$ is called a *Borel move.*

Definition 2.6. A *Borel ideal* $I \subseteq S$ is a monomial ideal closed under Borel moves. We write I_n to denote the smallest Borel ideal containing the monomial $x_1 x_2 x_2 \cdots x_n$.

We will be particularly interested in the minimal monomial generating set of I_n, for which we write $\mathrm{Gens}(I_n)$.

Fact 2.7. *The number of monomials in* $\mathrm{Gens}(I_n)$ *is* $C_n = \frac{1}{n+1}\binom{2n}{n}$, *the* n^{th} *Catalan number.*

For example, there are $C_3 = \frac{1}{4}\binom{6}{3} = 5$ generators of I_3: $x_1^3, x_1^2 x_2, x_1^2 x_3, x_1 x_2^2, x_1 x_2 x_3$. If m is a monomial, we let $\max(m)$ denote the greatest index of a variable dividing m.

Definition 2.8. An *Eliahou-Kervaire symbol* (or *EK-symbol*) of I_n is a pair (m, α) where $m \in \mathrm{Gens}(I_n)$ and α is a squarefree monomial with $\max(m) > \max(\alpha)$. The (*homological*) *degree* of an EK-symbol (m, α) is the degree of α.

The EK-symbols of a Borel ideal give a basis for its resolution. See [EK] for a proof and [PS] for a mapping cone approach to resolving a Borel ideal minimally. Thus we can calculate the Betti numbers of I_n by counting EK-symbols:

Theorem 2.9 (See, for instance, [FMS])**.** *The* i^{th} *Betti number* b_i *of* I_n *is the number of EK-symbols of* I_n *of homological degree* i.

3. Leaf-marked trees and EK-symbols

In this section, we construct a bijection between (2) and (3) in Theorem 1.1. Let T be a tree. Throughout, we refer to an ordering on the vertices of T, defined as follows. Given vertices v and w, let u be their unique common ancestor of greatest depth. (If w is an ancestor of v, let $u = w$.) If v is a left descendant of u (so that w must be a right descendant) we set $v < w$ and say that v is *on the left* of w.

This defines a total order on the vertices of T. Moreover, this order is the transitive closure of the order defined by $v < w$ whenever w is a right child of v or v is a left child of w.

Construction 3.1. Let (T, X) be a leaf-marked tree on $n + k$ vertices with $|X| = k$. We define a labeling $\phi : T \to [n]$ as follows. For a vertex $v \in T$, define $\phi(v)$ by:

$$\phi(v) = 1 + |\{w \in T \smallsetminus X : w < v \text{ and } w \text{ is not a descendant of } v\}|.$$

Using this labeling, we define a pair of monomials $\mathrm{EK}(T,X) = (m,\alpha)$ by

$$m = \prod_{v \in T \smallsetminus X} x_{\phi(v)} \text{ and } \alpha = \prod_{v \in X} x_{\phi(v)}.$$

If $X = \varnothing$, we set $\alpha = 1$.

An alternate recursive definition of ϕ is as follows. Let (T,X) be a leaf-marked tree, and set $\phi(r_T) = 1$, where r_T is the root of T. Now let w be a child of v.

- If w is a left child of v, set $\phi(w) = \phi(v)$.
- If w is a right child of v, set $\phi(w) = \phi(v) + k + 1$, where k is the number of left descendants of v not contained in X.

See Figure 1 for an example of the labeling and also Figure 8, which appears later in the paper. (For readability, we set $x_1 = a$, $x_2 = b$, etc.)

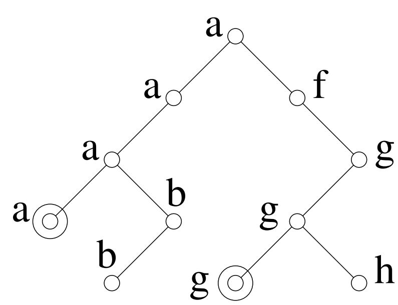


FIGURE 1. The labeling of a leaf-marked tree

Lemma 3.2. *Let μ : {trees with n vertices} $\to$ {monomials} be the map sending a tree T to the first entry of* $\mathrm{EK}(T,\varnothing)$. *Then μ is a bijection between trees with n vertices and generators of I_n.*

Proof. Write $\mu(T) = x_{i_1}x_{i_2}\cdots x_{i_n}$, where $i_1 \le i_2 \le \cdots \le i_n$, and let $v_1 < v_2 < \cdots < v_n$ be the vertices of T. By construction, $\phi(v_j) \le j$ for all j. Thus, for all j, $i_j = \phi(v_j) \le j$, meaning $\mu(T)$ is a minimal monomial generator of I_n.

A straightforward induction shows that $\mu(T) \ne \mu(T')$ for non-isomorphic trees T and T'. As C_n counts both the number of generators of I_n and the number of n-vertex binary trees [St2], the result follows. □

Lemma 3.3. *Let (T,X) be a leaf-marked tree and let v and w be two leaves of T. Then $\phi(v) \ne \phi(w)$.*

Proof. Since v and w are both leaves, neither has any descendants. Without loss, let $v < w$. Let u be the unique ancestor of both v and w of greatest depth. Then $v < u < w$, so $\phi(v) < \phi(w)$. □

Let (T, X) be a leaf-marked tree with $\mathrm{EK}(T, X) = (m, \alpha)$. By Lemma 3.2, m is a generator of I_n. By Lemma 3.3, α is squarefree. As X cannot contain the last leaf of T, $\max(m) > \max(\alpha)$. Thus, $\mathrm{EK}(T, X)$ is an EK-symbol for I_n. We now show that this map is a bijection.

Theorem 3.4. *The map $(T, X) \to \mathrm{EK}(T, X)$ is a bijection between leaf-marked trees on $n + k$ vertices with $|X| = k$ and EK-symbols of I_n of homological degree k.*

Proof. Let T' be an arbitrary unmarked tree, with rightmost leaf z, and let $s = \phi(z)$. We will show that, for $i \le n$, there exists a unique way to affix a new marked leaf w_i to T' such that $\phi(w_i) = i$, and that z is the rightmost leaf of the resulting tree if and only if $i < s$.

Let the vertices of T' be $v_1, \ldots, v_n$, written in ascending order. Observe that if v_{i-1} has a right child w, then $\phi(w) = i$. (If u is left of v_{i-1}, then it cannot be a descendant of w. If u is right of v_{i-1}, then it is right of w as well.) If such a child does not exist, we may affix w_i on the right of v_{i-1}. If such a child does exist, then, if it has a left child w', we have $\phi(w') = i$ as well. If no such w' exists, we may affix w_i on the left of w. If w' does exist, we may look to its left as well. Because T' is finite, we eventually reach a vertex without a left child, and affix w_i there.

Thus, for all $i \le n + 1$, there is at least one way to affix a marked leaf with label x_i. If there were two such ways, we could affix both leaves, and they would have the same label, violating Lemma 3.3.

Observe that w_i is left of v_i and $z = v_s$. If $i < s$, it follows that z is still the rightmost leaf after affixing w_i. If $i > s$, then w_i becomes the rightmost leaf. If $i = s$, then $\phi(w_i) = \phi(z)$, so by Lemma 3.3 z and w_i cannot both be leaves. Thus z is no longer a leaf and in particular is not the rightmost leaf. (In fact, one can show that w_i is the left child of z.)

Now let (m, α) be an EK-symbol with $\max(m) = s$. To construct a marked tree (T, X) satisfying $\mathrm{EK}(T, X) = (m, \alpha)$, let T' be the (unmarked) tree on n vertices guaranteed by Lemma 3.2 whose associated monomial is m. Now for each x_i dividing α, append the marked leaf w_i found above. Let X be the set of added leaves and $T = T' \cup X$ be the resulting tree. Since each new leaf is marked, it does not affect the label of any other vertex. Thus $EK(T, X) = (m, \alpha)$.

We have shown above that the leaf-marked trees (T, X) are in bijection with the pairs (T', α) consisting of unmarked trees T' with n vertices and squarefree monomials α such that $\max(\alpha) < \phi(z)$. By Lemma 3.2, the set of such pairs is in bijection with the set of EK-symbols. The map $(T, X) \to \mathrm{EK}(T, X)$ is the composition of these bijections. $\square$

Remark 3.5. Note that the bijection between branch-marked trees and leaf-marked trees, items (1) and (2) in Theorem 1.1, is immediate: Suppose T is a branch-marked tree with k branching vertices. Then T necessarily has $k+1$ leaves. Writing branching vertices $b_1, b_2, \ldots, b_k$ in ascending order and the leaves $w_1, w_2, \ldots, w_{k+1}$ in ascending order, the map sending b_i to w_i yields a leaf-marked tree, and this map is easily seen to be a bijection. (Note that under this map the rightmost leaf is never marked, which is a requirement of leaf-marked trees.)

4. Branch-marked trees and ppts

Pseudotriangulations and pointed pseudotriangulations are the subject of much investigation in combinatorial geometry, robotics, and rigidity theory. The authors of [AOSS], concerned with enumerative properties of these objects, examined pointed pseudotriangulations (which we will often call *ppts*) of the *single chain*. We first define these notions and show some simple properties of this configuration. Our purpose is to construct a bijection between items (1) and (4) of Theorem 1.1.

Definition 4.1. A *pseudotriangle* is a polygon with exactly three convex interior angles.

Example 4.2. The hexagon in Figure 2 is a pseudotriangle because only three of the six interior angles are convex.

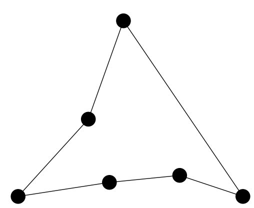


Figure 2. A pseudotriangle

Definition 4.3. Let V be a collection of points in the plane, and suppose that no three points in V are collinear. Then a *pseudotriangulation* of V is a connected whiskerless planar graph with vertex set V, with the properties that all bounded regions are pseudotriangles, and that the union of the bounded regions is the convex hull of V.

A vertex $v \in V$ is called *pointed* if it is incident to a concave angle (on a bounded or unbounded region), and the pseudotriangulation is called *pointed* if every vertex is pointed.

Example 4.4. Figure 3 is a pseudotriangulation, but it is not a pointed pseudotriangulation because the unshaded vertex is not pointed. Every angle to which the unshaded vertex is incident is convex.

Definition 4.5. The *single chain* of length n is a point configuration consisting of $n+1$ points $p_0, \ldots, p_n$ arranged in order on an open semicircular arc, together with another point z, called the *tip*, at the intersection of the tangent lines from p_0 and

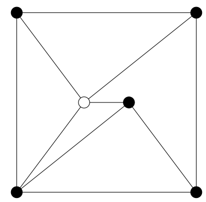


FIGURE 3. A pseudotriangulation

p_n. Observe that the convex hull of the single chain is a triangle with vertices p_0, p_n, and z, and that no line segment from the tip to any p_i intersects any chord.

Remark 4.6. Our notation differs from that of [AOSS]. In that paper, the authors refer to a single chain with $\ell = r$ to mean that there are r interior points plus the tip and the first and last vertices arranged on the semicircular arc.

Example 4.7. Figure 4 shows all pointed pseudotriangulations of the single chain of length three. There are five in which both interior vertices are connected to the tip, six in which exactly one interior vertex is connected to the tip, and two in which neither interior vertex is connected to the tip. Note that the second row $(f_{2,0}, f_{2,1}, f_{2,2})$ of Borel's triangle is $(5, 6, 2)$, which is the sequence of Betti numbers of I_3, the smallest Borel ideal containing $x_1x_2x_3$.

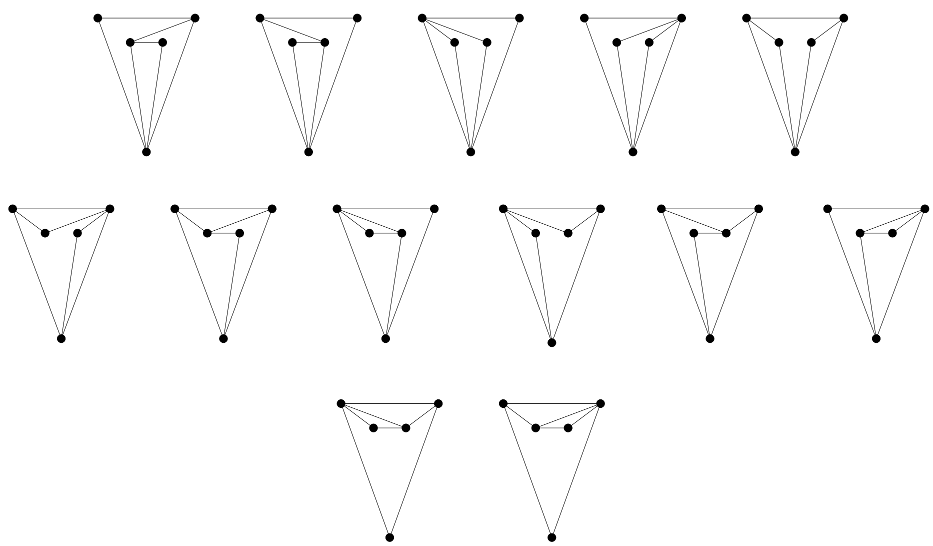


FIGURE 4. The pointed pseudotriangulations of the single chain of length three

The authors of [AOSS] study the pointed pseudotriangulations of the single chain and several other point configurations in considerable detail. In order to handle this context, they introduce notation that is unwieldy for our narrower purposes. We now recover a few basic results using our notation.

Proposition 4.8. *Let V be any point configuration with $n+2$ points, and suppose that $\mathcal{P}$ is a pointed pseudotriangulation of V. Then $\mathcal{P}$ contains $2n+1$ edges.*

Proof. Let V, E, and F be the sets of vertices, edges, and regions of $\mathcal{P}$, respectively. Then $\mathcal{P}$ has $|V|$ concave angles, and $2|E|-|V|$ convex angles. Consequently $|F| = 1 + \frac{2|E|-|V|}{3}$. The lemma follows from Euler's formula $|V|-|E|+|F| = 2$. □

Proposition 4.9. *Let $\mathcal{P}$ be a pointed pseudotriangulation of the single chain, and let p_j be a vertex on the semicircle. Then exactly two of the following three statements hold:*

(1) *p_j is connected to the tip.*
(2) *p_j is connected to some vertex p_i with $i < j$.*
(3) *p_j is connected to some vertex p_k with $j < k$.*

Proof. If all three statements hold, then the vertex p_j is not pointed. If all three fail, then $\mathcal{P}$ is not connected.

If only statement (i) holds, then the vertex p_j has degree one.

Now suppose that only statement (iii) holds. Let R be the region containing both p_j and the tip in its boundary. Then R has convex angles at the tip and at its leftmost point p_i; observe that $i < j$. Now consider the path along ∂R connecting p_j to p_i that doesn't pass through the tip, and the path connecting p_j to the tip that doesn't pass through p_i. Observe that R has a convex angle at the rightmost point of each path. Thus R is not a pseudotriangle. □

Lemma 4.10. *Let $\mathcal{P}$ be a pointed pseudotriangulation of the single chain of length at least four. Then the triangles of $\mathcal{P}$ are in bijection with the p_j that are not connected to the tip. In particular, $\mathcal{P}$ contains a triangle with vertices p_i, p_j, and p_k satisfying $i < j < k$ if and only if p_j is not connected to the tip.*

Proof. The first statement follows from the second since two such triangles cannot share middle vertex p_j. We prove the second statement.

If such a triangle exists, then p_j cannot be connected to the tip by Proposition 4.9. Conversely, if p_j is not connected to the tip, let I be minimal and K maximal such that p_j is connected to both p_I and p_K. Let R be the region above p_j, which contains both these edges in its boundary. If R contains the tip it has convex angles at p_j and the tip, as well as at its leftmost and rightmost points, and so cannot be a pseudotriangle. If R does not contain the tip it is convex and consequently is the triangle with vertices p_I, p_j, and p_K. □

Construction 4.11. Given a branch-marked tree (T, B), we construct a pointed pseudotriangulation of the single chain by induction on a depth-first traversal of T.

We begin at the root of T by constructing two points p_0 and p_1, and create an edge between them, labelled by the root vertex. Then at every vertex v of T, starting with the root, we find the edge (p_i, p_ℓ) labelled by v and do the following.

If v is a leaf, we leave our construction unmodified.

If v has a child on the left but not on the right, we choose a real number j with $i < j < \ell$, construct a new point p_j, and construct an edge from p_i to p_j labelled by the child vertex.

If v has a child on the right but not on the left, we choose a real number j with $i < j < \ell$, construct a new point p_j, and construct an edge from p_ℓ to p_j labelled by the child vertex.

If v is a marked branching vertex, we choose a real number j with $i < j < \ell$, construct a new point p_j, and construct edges from p_i to p_j and from p_ℓ to p_j labelled by the children on the left and right, respectively.

If v is an unmarked branching vertex, we choose real numbers j and k with $i < j < k < \ell$, construct new points p_j and p_k, and construct edges from p_i to p_j and from p_ℓ to p_k, labelled by the children on the left and right, respectively.

After traversing T, we delete the edge labels, reindex the points with consecutive integers, and finally add edges from z to each point p_j that is not connected to points p_i and p_k with $i < j < k$.

The result is a pointed pseudotriangulation of the single chain of length n, which we call $P(T, B)$.

Example 4.12. Given the branch-marked tree in Figure 5, we begin by labeling the vertices as in a depth-first search. We form an edge labeled 1 corresponding to the root of the tree, and we create edges labeled 2 and 6 that correspond to the left and right children of 1 respectively. The edge 2 intersects 1 in its left vertex because 2 is a left child of 1, and similarly for 6. Additionally, because 1 is marked in the original tree, the edges labeled 2 and 6 meet at a common vertex. We construct the edges 3 and 5 in the same way, noting that 2 is also marked. Finally, we create an edge labeled 7 that meets 6 at its left endpoint because 7 is a left child of 6, and similarly we create an edge 4 that shares its right endpoint with edge 3.

Proposition 4.13. *If (T, B) has n unmarked vertices and k marked vertices, then $P(T, B)$ is a pointed pseudotriangulation of the single chain of length n in which k of the points are not connected to the tip.*

Proof. We begin the construction with two points, then add two more points for each unmarked branching vertex, and one more point for each marked branching vertex and for each non-branch, non-leaf vertex, for a total of

It remains to show that we have in fact constructed a pointed pseudotriangulation. Clearly our configuration is pointed: the tip has a concave angle on the exterior, and every other point has a convex angle either on the left (if it it is not connected to any

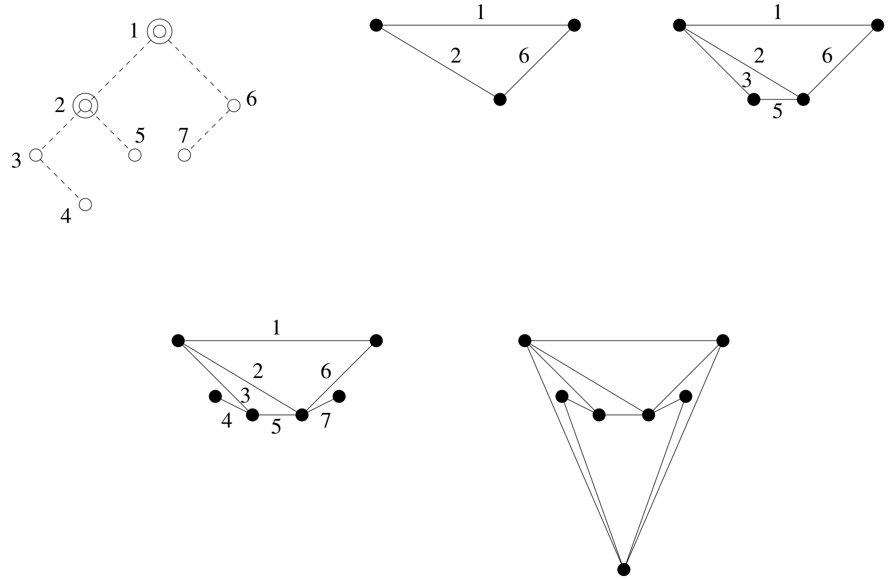


FIGURE 5. The construction of $P(T)$

other points on its left), on the right (if it is not connected to any other points on its right), or in the direction of the tip (if it is connected on both its left and its right).

To see that our configuration is a pseudotriangulation, first observe that any vertex p_j which is connected to any p_i and p_ℓ with $i < j < \ell$ must have arisen in our construction from a marked branching vertex. In this case, let I and L be minimal and maximal indices, respectively, such that p_I and p_L are connected to p_j; then there was a marked vertex labelling an edge from p_I to p_L, and its children labelled edges from p_I to p_j and from p_L to p_j.

Now suppose some region R within our configuration is not a pseudotriangle; that is, it has at least four convex vertices. Then in particular, R has at least three convex vertices other than the tip z; choose three and write them p_i, p_j, and p_ℓ, in order. Let p_I and p_L be as in the discussion above; then we claim that $p_I = p_i$ and $p_L = p_\ell$. Indeed, if not the lines from p_I to p_j would cut through R. It follows that R is in fact the triangle with vertices p_i, p_j, and p_ℓ, contradicting the assumption that it is not a pseudotriangle. □

Finally, given a pointed pseudotriangulation $\mathcal{P}$ of the single chain of length n, we construct a branch-marked tree $T(\mathcal{P})$ by imposing a tree order on the edges of $\mathcal{P}$.

Construction 4.14. Given an edge $e_{i,\ell}$ from p_i to p_ℓ, let $J < \ell$ be maximal such that there is an edge $e_{i,J}$ from p_i to p_J, and let K be minimal such that there is an edge $e_{K,\ell}$ from p_K to p_ℓ. Then we set $e_{i,J}$ and $e_{K,\ell}$ to the left and right children of $e_{i,\ell}$ (provided that they exist). Finally, if $J = K$ (i.e., $\mathcal{P}$ contains a triangle on p_i, p_J, and p_ℓ), we mark $e_{i,\ell}$. The resulting branch-marked tree is $T(\mathcal{P})$.

Example 4.15. Let $\mathcal{P}$ be the pointed pseudotriangulation in Figure 6. In Figure 7 we construct the associated binary tree from the covering relations on its edges. We mark both branch vertices because they each come from the top edge of a triangle. The resulting branch-marked tree is the first tree in Figure 8, which also displays the associated leaf-marked tree and its vertex labels. Observe that this reverses the process illustrated in Example 4.12.

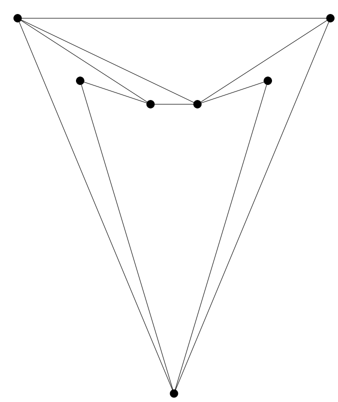


FIGURE 6. A pointed pseudotriangulation $\mathcal{P}$ of the single chain of length five

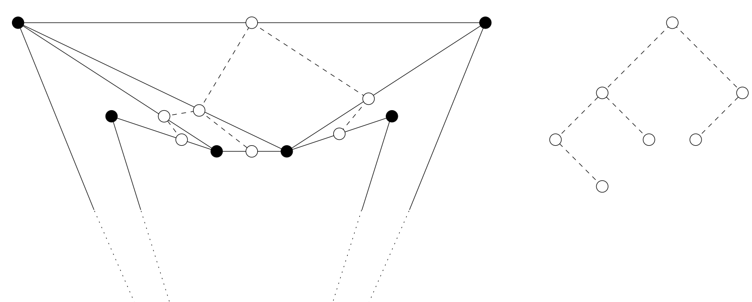


FIGURE 7. The associated binary tree $T(\mathcal{P})$

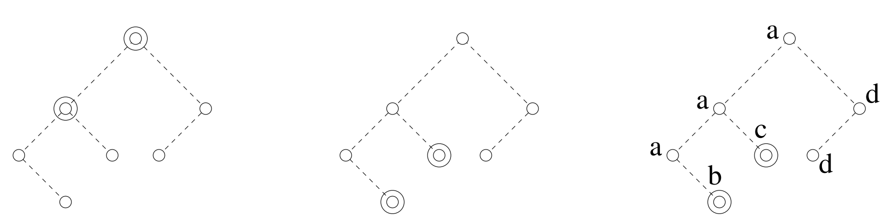


FIGURE 8. $T(\mathcal{P})$ and its associated leaf-marked tree with labeling

Proposition 4.16. *If $\mathcal{P}$ is a pointed pseudotriangulation of the single chain of length n in which k vertices are not connected to the tip, then $T(\mathcal{P})$ is a branch-marked tree with n unmarked vertices and k marked branching vertices.*

Proof. By Proposition 4.8 $\mathcal{P}$ contains $2n+1$ edges, of which $n+1-k$ are connected to the tip. Thus $T(\mathcal{P})$ contains $n+k$ vertices; the number of marked vertices is equal to the number of triangles in $\mathcal{P}$, which by Lemma 4.10 is equal to the number of points not connected to the tip, i.e., k.

Clearly, each marked vertex is a branching vertex.

It remains to show that $T(\mathcal{P})$ is a tree. We will show that every vertex of $T(\mathcal{P})$, except the root vertex $e_{0,n}$, is the child of a unique vertex. Indeed, let $e_{j,k}$ be an edge of $\mathcal{P}$. Let I be minimal such that $e_{I,k}$ is an edge, and let L be maximal such that $e_{j,L}$ is an edge. If $I = j$ and $L = k$, then the region outside the triangle connecting e_i, e_j, and the tip is not simply connected, so we must have $e_{j,k} = e_{0,n}$. Otherwise, suppose without loss of generality that $I \neq j$. Let i be maximal such that $I \leq i < j$; then $e_{j,k}$ is the left child of $e_{i,k}$ but not of any other edge. In this case we claim that $L = k$ (so that $e_{j,k}$ is not the right child of any edge). Indeed, if not, then $e_{I,k}$ and $e_{j,L}$ would intersect. $\square$

The following proposition is straightforward.

Proposition 4.17. *T and P are inverse bijections.*

5. Further Research

Theorem 1.1 gives several classes of objects counted by Borel's triangle. However, we believe the entries of Borel's triangle will prove useful in many other enumerative contexts. For example, Stanley's book [St2] lists many sets counted by the n^{th} Catalan number, including binary trees on n vertices, generators of Borel ideals I_n (implicit in the ballot-path bijection), and triangulations of n points in convex position. As these classes of objects (with some extra structure) correspond to marked trees, minimal syzygies of Borel ideals, and pointed pseudotriangulations, we anticipate that most objects on Stanley's list similarly correspond to a new class of objects counted by Borel's triangle.

Open-ended Problem 5.1. Find classes of objects counted by Borel's triangle. In particular, for each class of objects counted by the Catalan numbers, identify extra structure that defines a new class of objects counted by Borel's triangle.

We begin addressing this problem by studying binary parenthesizations, which are counted by the Catalan numbers [St2]. For example, there are $C_3 = 5$ binary parenthesizations of 4 symbols:

$$(x_1x_2)(x_3x_4) \qquad x_1(x_2(x_3x_4)) \qquad (x_1(x_2x_3))x_4 \qquad x_1((x_2x_3)x_4) \qquad ((x_1x_2)x_3)x_4$$

Now let $X \subseteq \{2, 3, \ldots, n-1\}$. We let the *$X$-word on $[n]$* be the word obtained from $x_1 x_2 \cdots x_n$ by doubling x_i whenever $i \in X$. For example, if $X = \{2, 4\}$, then the X-word on $[5]$ is $x_1x_2x_2x_3x_4x_4x_5$. Similarly, an *X-parenthesization on $[n]$* is a binary parenthesization of the X-word on $[n]$ satisfying the following property: For each doubled letter x_i, both types of parentheses appear between the two copies of the letter. For example, if $X = \{2\}$, then the X-word on $[4]$ is $x_1x_2x_2x_3x_4$, and there are three X-parenthesizations on $[4]$:

$$(x_1x_2)((x_2x_3)x_4) \qquad (x_1x_2)(x_2(x_3x_4)) \qquad ((x_1x_2)(x_2x_3))x_4$$

Note that if $X \subseteq [n]$ is empty, the number of X-parenthesizations on $[n]$ is C_{n-1}, as this is simply the number of binary parenthesizations on n symbols.

We define a k-parenthesization of $[n]$ to be an X-parenthesization of $[n]$ for some X with cardinality k.

Theorem 5.2. *Recall that we write $f_{n,k}$ for the k^{th} entry in the n^{th} row of Borel's triangle. Then $f_{n,k}$ is the number of k-parenthesizations of $[n+2]$.*

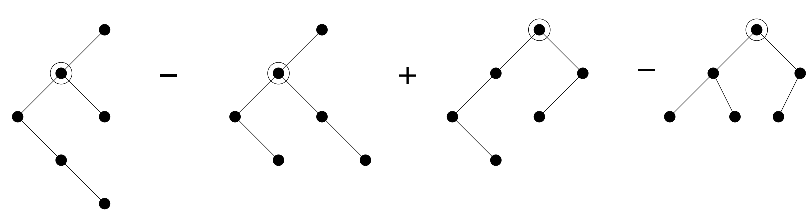
For example, a symmetry argument coupled with the example above shows that the number of X-parenthesizations on $[4]$ for which $|X| = 1$ is 6, which is $f_{2,1}$.

Sketch of proof of Theorem 5.2. The bijection from pointed pseudotriangulations is as follows: For a pointed pseudotriangulation $\mathcal{P}$ of the single chain of length n, let $X = \{i : p_i \text{ is not connected to the tip}\}$. We associate to $\mathcal{P}$ the X-parenthesization obtained by attaching a left parenthesis to x_i for each edge of $\mathcal{P}$ having p_i as its left vertex, and attaching a right parenthesis to x_i for each edge of $\mathcal{P}$ having p_i as its right vertex. For example, let $\mathcal{P}$ be the pointed pseudotriangulation in Example 4.15. Then $X = \{3, 4\}$ and the associated X-parenthesization is $(((x_1(x_2x_3))(x_3x_4))((x_4x_5)x_6))$. $\square$

As marked trees count the minimal syzygies of I_n, one can ask whether the combinatorics yield an interesting differential. We can of course translate the EK differential along our bijections, but the resulting maps do not seem at all natural on sets of marked trees.

Example 5.3. Let T be the tree from Figure 5. Tracing the bijections, T corresponds to the Eliahou-Kervaire symbol $[a^3d^2, bc]$. The differential of this Eliahou-Kervaire symbol is $[a^3bd, c] - [a^3cd, b] + [a^3d^2, b] - [a^3d^2, c]$, which corresponds to the linear combination below:

D(T) =



As this figure illustrates, there is no obvious combinatorial rule for defining the differential on a marked tree.

While the Eliahou-Kervaire differential does not appear to make sense on the space of marked trees, it is possible that there is a natural combinatorial differential. Such a structure would be of considerable interest to algebraists; for example, in [NR, Si], other bases for the resolution of a Borel ideal are constructed. Unfortunately, thus far we have been unable to find a reasonable differential on either trees or the other structures discussed in this paper. For example, on the space of X-parenthesizations, the following map looks tantalizingly like a differential, but it is not.

Example 5.4. For an X-parenthesization w, and $j \in X$, define $L_j(w)$ to be the $(X \setminus j)$-parenthesization obtained by deleting the first copy of x_j from w and simplifying (or, if this is not an $(X \setminus j)$-parenthesization, set $L_j(w)$ equal to zero). Similarly define $R_j(w)$ by deleting the second copy of x_j. Finally, if $X = \{x_{j_1}, \dots, x_{j_k}\}$, define $D(w) = \sum(-1)^i(L_{j_i}(w) - R_{j_i}(w))$. For example, if $w = ((x_1(x_2x_3))(x_3x_4))(x_4x_5)$, then

$$D(w) = ((x_1x_2)(x_3x_4))(x_4x_5) - ((x_1(x_2x_3))x_4)(x_4x_5) - 0 + ((x_1(x_2x_3))(x_3x_4))x_5.$$

The definition of D looks like a boundary operator, and appears to mimic the Eliahou-Kervaire differential, but it is not a differential. In fact, with w as above, $D^2(w) \neq 0$.

It seems natural to ask the following question:

Question 5.5. Among the classes of objects counted by Borel's triangle, is there one that admits a natural differential structure?

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