A BROAD CLASS OF SHELLABLE LATTICES

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Abstract. We introduce a new class of lattices, the *modernistic* lattices, and their duals, the *comodernistic* lattices. We show that every modernistic or comodernistic lattice has shellable order complex. We go on to exhibit a large number of examples of (co)modernistic lattices. We show comodernism for two main families of lattices that were not previously known to be shellable: the order congruence lattices of finite posets, and a weighted generalization of the k-equal partition lattices.

We also exhibit many examples of (co)modernistic lattices that were already known to be shellable. To start with, the definition of modernistic is a common weakening of the definitions of semimodular and supersolvable. We thus obtain a unified proof that lattice in these classes are shellable.

Subgroup lattices of solvable groups form another family of comodernistic lattices that were already proved to be shellable. We show not only that subgroup lattices of solvable groups are comodernistic, but that solvability of a group is equivalent to the comodernistic property on its subgroup lattice. Indeed, the definition of comodernistic exactly requires on every interval a lattice-theoretic analogue of the composition series in a solvable group. Thus, the relation between comodernistic lattices and solvable groups resembles, in several respects, that between supersolvable lattices and supersolvable groups.

1. Introduction

Shellings are a main tool in topological combinatorics. An explicit shelling of a complex Δ simultaneously shows Δ to be sequentially Cohen-Macaulay, computes its homotopy type, and gives a cohomology basis. Frequently, a shelling also gives significant insight into the homeomorphy of Δ. The downside is that shellings are often difficult to find, and generally require a deep understanding of the complex.

In this paper, we describe a large class of lattices that admit shellings. The shellings are often straightforward to explicitly write down, and so give a large amount of information about the topology of the order complex. Included in our class of lattices are many examples which were not previously understood to be closely related.

The question that first motivated this research project involved shelling a particular family of lattices. The *order congruence lattice* $\mathcal{O}(P)$ of a finite poset P is the subposet of the partition lattice consisting of all equivalence classes arising as the level sets of an order-preserving function. Order congruence lattices interpolate between Boolean lattices and partition lattices, as we will make precise later. Such lattices were already considered by Sturm in [36]. More recently, Körtesi, Radeleczki and Szilágyi showed the order congruence lattice of any finite poset to be graded and relatively complemented [21], while Jenča and Sarkoci showed such lattices to be Cohen-Macaulay [20].

The Cohen-Macaulay result naturally suggested to us the question of whether every order congruence lattice is shellable. After proving the answer to this question to be "yes", we noticed that our techniques apply to a much broader class of lattices. Indeed, a large number of the lattices previously shown to be shellable lie in our class. Thus, our main result (Theorem 1.2 below) unifies numerous results on shellability of order complexes of lattices, in addition to proving shellability for new examples. It is our belief that our results will be useful to other researchers. Finding a shelling of a lattice can be a difficult problem. In many cases, showing that a lattice is in our class may be much simpler.

All lattices, posets, simplicial complexes, and groups considered in this paper will be finite.

1.1. **Modernistic and comodernistic lattices.** We now define the broad class of lattices described in the title and introduction. Our work relies heavily on the theory of modular elements in a lattice. Recall that an element m of a lattice L is *left-modular* if whenever $x < y$ are elements of L, then the expression $x \vee m \wedge y$ can be written without parentheses, that is, that $(x \vee m) \wedge y = x \vee (m \wedge y)$. An equivalent definition is that m is left-modular if m is not the nontrivial element in the short chain of any pentagonal sublattice of L; see Lemma 2.10 for a precise statement.

Our key object of study is the following class of lattices.

Definition 1.1. We say that a lattice L is *modernistic* if every interval in L has an atom that is left-modular. We say that L is *comodernistic* if the dual of L is modernistic, that is, if every interval has a left-modular coatom.

Our main theorem is as follows. (We will recall the definition of a CL-labeling in Section 2.4 below.)

Theorem 1.2. *If L is a comodernistic lattice, then L has a CL-labeling.*

Corollary 1.3. *If L is either comodernistic or modernistic, then the order complex of L is shellable.*

The CL-labeling is explicit from the left-modular coatoms, so Theorem 1.2 also gives a method for computing the Möbius function of L. See Lemma 3.6 for details.

We find it somewhat surprising that Theorem 1.2 was not proved before now. We speculate that the reason may be the focus of previous authors on atoms and CL-labelings, whereas Theorem 1.2 requires dualizing exactly one of the two.

Remark 1.4. The name "modernistic" comes from contracting "atomically modular" to "atomically mod". Since atomic was a common superlative from the the late 1940's, and since the mod (or modernistic) subculture was also active at about the same time, we find the name to be somewhat appropriate, as well as short and perhaps memorable.

1.2. **Examples and applications.** Theorem 1.2 has a large number of applications, which we briefly survey now. First, we can now solve the problem that motivated the project.

Theorem 1.5. *If P is any poset, then the order congruence lattice $\mathcal{O}(P)$ is comodernistic, hence CL-shellable.*

We also recover as examples many lattices already known to be shellable. The following theorem lists some of these, together with references to papers where they are shown to be shellable.

Proposition 1.6. *The following lattices are comodernistic, hence CL-shellable:*

(1) Supersolvable and left-modular lattices, and their order duals [2, 22, 25].
(2) Order duals of semimodular lattices [2]. (I.e., semimodular lattices are modernistic.)
(3) k-equal partition lattices [7], and their type B analogues [5].
(4) Subgroup lattices of solvable groups [31, 40].

We comment that many of these lattices are shown in the provided references to have EL-labelings. Theorem 1.2 provides only a CL-labeling. Since the CL-labeling constructed is explicit from the left-modular elements, Theorem 1.2 provides many of the benefits given by an EL-labeling. We do not know if every comodernistic lattice has an EL-labeling, and leave this interesting question open.

Experts in lattice theory will immediately recognize items (1) and (2) from Proposition 1.6 as being comodernistic. Theorem 1.2 thus unifies the theory of these well-understood lattices with the more difficult lattices on the list. The CL-labeling that we construct in the proof of Theorem 1.2 can moreover be seen as a generalization of the EL-labeling for a supersolvable lattice, further connecting these classes of lattices.

We will prove that k-equal partition lattices and their type B analogues are comodernistic in Section 6. In Section 6.3 we will show the same for a new generalization of k-equal partition lattices. The proofs show, broadly speaking, that coatoms of (intervals in) these subposets of the partition lattice inherit left-modularity from that in the partition lattice.

Although modernism and comodernism give a simple and unified framework for showing shellability of many lattices, not every shellable lattice is (co)modernistic. For an easy example, the face lattice of an n-gon has no left-modular elements when $n > 3$, so is neither modernistic nor comodernistic.

1.3. **Further remarks on subgroup lattices.** We can expand on the connection with group theory suggested by item (4) of Proposition 1.6. For a group G, the *subgroup lattice* referred to in this item consists of all the subgroups of G, ordered by inclusion; and is denoted by $L(G)$.

Theorem 1.7. *If G is a group, then G is solvable if and only if $L(G)$ is comodernistic.*

Stanley defined supersolvable lattices in [32] to abstract the interesting combinatorics of the subgroup lattices of supersolvable groups to general lattices. Theorem 1.7 says that comodernism is one possibility for a similar abstraction for solvable groups. A result of a similar flavor was earlier proved by Schmidt [28]; our innovation with comodernism is to require a lattice-theoretic analogue of a composition series in every interval of the lattice. We further discuss possible notions of solvability for lattices in Section 5.

Shareshian in [31] showed that a group G is solvable if and only if $L(G)$ is shellable. Theorems 1.2 and 1.7 give a new proof of the "only if" direction of this result. For the "if" direction, Shareshian needed a hard classification theorem from finite group theory. Our proof of Theorem 1.7 does not rely on hard classification theorems. On the other hand, it follows directly from Shareshian's proof that G is solvable if $L(G)$ is sequentially Cohen-Macaulay. Thus, Shareshian's Theorem gives a topological characterization of solvable groups. The characterization given in Theorem 1.7 is lattice-theoretic, rather than topological. It is an interesting open problem to give a classification-free proof that if $L(G)$ is sequentially Cohen-Macaulay, then G is solvable.

1.4. **Organization.** This paper is organized as follows. In Section 2 we recall some of the necessary background material. We prove Theorem 1.2 (our main theorem) in Section 3.

In the remainder of the paper we show how to apply comodernism and Theorem 1.2 to various classes of lattices. The techniques may be illustrative for those who wish to prove additional classes of lattices to be comodernistic. In Section 4, we examine order congruence lattices, and prove Theorem 1.5. In Section 5 we prove Theorem 1.7, and argue for comodernism as a notion of solvable for lattices. We close in Section 6 by showing that k-equal partition lattices (and variations thereof) are comodernistic.

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2. Preliminaries

We begin by recalling some necessary background and terminology. Many readers will be able to skip or skim this section, and refer back to it as necessary.

2.1. **Posets, lattices, and order complexes.** A poset P is *bounded* if P has a unique least element $\hat{0}$ and greatest element $\hat{1}$.

Associated to a bounded poset P is the *order complex*, denoted ΔP, a simplicial complex whose faces consist of all *chains* (totally ordered subsets) in $P \setminus \{\hat{0}, \hat{1}\}$. In particular, the vertices of ΔP are the chains of length 0 in $P \setminus \{\hat{0}, \hat{1}\}$, that is, the elements of $P \setminus \{\hat{0}, \hat{1}\}$. The importance of the order complex in poset theory arises since the Möbius function $\mu(P)$ (important for inclusion-exclusion) is given by the reduced Euler characteristic $\tilde{\chi}(\Delta P)$.

We often say that a bounded poset P possesses a property from simplicial topology (such as "shellability"), by which we mean that ΔP has the same property.

We say that a poset P is *Hasse-connected* if the Hasse diagram of P is connected as a graph. That is, P is Hasse-connected if and only for any $x, y \in P$, there is a sequence $x = x_0, x_1, \ldots, x_k = y$ such that x_i is comparable to x_{i+1} for each i.

A poset L is a *lattice* if every two elements $x, y \in L$ have a unique greatest lower bound (the *meet* $x \wedge y$) and unique least upper bound (the *join* $x \vee y$). It is obvious that every lattice is bounded, hence has an order complex ΔL.

A poset is *graded* if all its maximal chains have the same length, where the *length* of a chain is one less than its cardinality. The *height* of a bounded poset P is the length of the longest chain in P, and the *height* of an element x is the height of the interval $[\hat{0}, x]$. An *atom* of a bounded poset P is an element of height 1.

The *order dual* of a poset P is the poset P^* with reversed order relation, so that $x <^* y$ in P^* exactly when $x > y$ in P. Poset definitions may be applied to the dual by prepending a "co": for example, an element x is a *coatom* if x is an atom in P^*.

For more background on poset and lattice theory from a general perspective, we refer to [34]. For more on order complexes and poset topology, we refer to [39].

2.2. **Order congruence lattices.** If P and Q are posets, then a map $\varphi : P \to Q$ is *order-preserving* if whenever $x \leq y$, it also holds that $\varphi(x) \leq \varphi(y)$. The *level set partition* of a map $\varphi : P \to Q$ is the partition with blocks of the form $\varphi^{-1}(q)$. If π is the level set partition of an order preserving map $\varphi : P \to Q$, then π is an *order partition* of P. Since every poset has a linear extension, it is easy to see that it would be equivalent to restrict the definition of order partition to the case where $Q = \mathbb{Z}$.

As previously defined, the order congruence lattice $\mathcal{O}(P)$ is the subposet of the partition lattice Π_P consisting of all order partitions of P. The cover relations in $\mathcal{O}(P)$ correspond to merging blocks in an order partition, subject to a certain compatibility condition.

Example 2.1. Consider $P = [3]$ with the usual order. Then the function mapping $1, 2$ to 1 and 3 to 2 is order-preserving, so $12 \,|\, 3 \in \mathcal{O}([3])$. Similarly, the partition $1 \,|\, 23 \in \mathcal{O}([3])$. It is not difficult to see, however, that there is no order-preserving map with level set partition $13 \,|\, 2$. Thus, the lattice $\mathcal{O}([3])$ is isomorphic to the Boolean lattice on 2 elements.

More generally, a standard and elementary argument shows that the order congruence lattice of a chain on n elements is isomorphic to a Boolean lattice on $n - 1$ elements. It is obvious that the order congruence lattice of an antichain is the usual partition lattice. Thus, order congruence lattices interpolate between Boolean lattices and partition lattices.

It is easy to confuse $\mathcal{O}(P)$ with another closely related lattice defined on a poset P. We say that a subset $S \subseteq P$ is *order convex* if whenever $a \leq b \leq c$ with $a, c \in S$, then also $b \in S$. The *order convexity partition lattice* of P, denoted $\mathcal{O}^{\text{conv}}(P)$, consists of all partitions where every block is order convex.

Although we find it to be a natural object, the order convexity partition lattice does not seem to have been studied previously. There is some literature on the related lattice of all order convex subsets of a poset, going back to [1].

Example 2.2. Consider the *bowtie poset* B, with elements a_1, a_2, b_1, b_2 and relations $a_i < b_j$ (for $i, j \in \{1, 2\}$). As B has height 1, all subsets are order convex, so that $\mathcal{O}^{\text{conv}}(B) \cong \Pi_4$. However, the partitions $a_1b_1 \mid a_2b_2$ and $a_1b_2 \mid a_2b_1$ are not order congruence partitions, so are not in $\mathcal{O}(B)$.

In another point of view, it is straightforward to show that order-preserving partitions are in bijective correspondence with certain quotient objects of P. Thus, the order congruence lattice assigns a lattice structure to quotients of P. See [20, Section 3] for more on the quotient view of $\mathcal{O}(P)$.

For our purposes, it will be enough to understand intervals above atoms in $\mathcal{O}(P)$. Say that elements x, y of poset P are *compatible* if either $x \lessdot y$, $y \lessdot x$, or x, y are incomparable. If x, y are compatible in P, then $P_{x\sim y}$ is the poset obtained by identifying the two elements. That is, $P_{x\sim y}$ is obtained from P by replacing x, y with w, subject to the relations $z < w$ whenever $z < x$ or $z < y$, and $z > w$ whenever $z > x$ or $z > y$. We remark in passing that this identification is an easy special case of the quotienting viewpoint discussed above.

Lemma 2.3. *Let P be a poset. An order partition π is an atom of $\mathcal{O}(P)$ if and only if π has exactly one non-singleton block consisting of compatible elements $\{x, y\}$. In this situation, we have the lattice isomorphism $[\pi, \hat{1}]_{\mathcal{O}(P)} \cong P_{x\sim y}$.*

Repeated application of Lemma 2.3 allows us to understand any interval of the form $[\pi, \hat{1}]$ in $\mathcal{O}(P)$.

Although we will not use need this, intervals of the form $[\hat{0}, \pi]$ are also not difficult to understand. Let π have blocks $B_1, B_2, \ldots, B_k$. It is well-known (see e.g. [34, Example 3.10.4]) that an interval of this form in the full partition lattice is isomorphic to the product of smaller partition lattices $\Pi_{B_1} \times \cdots \times \Pi_{B_k}$. It is straightforward to see via the order-preserving mapping $P \to \mathbb{Z}$ definition that a similar result holds in the order congruence lattice. That is, $[\hat{0}, \pi]$ in $\mathcal{O}(P)$ is lattice-isomorphic to $\mathcal{O}(B_1) \times \cdots \times \mathcal{O}(B_k)$, where B_i refers to the induced subposet on $B_i \subseteq P$. Combining this observation with Lemma 2.3 allows us to write any interval in $\mathcal{O}(P)$ as a product of order congruence lattices of quotients of subposets. We find it simpler to give more direct arguments, but readers familiar with poset products may appreciate this connection.

2.3. **Simplicial complexes and shellings.** We assume basic familiarity with homology and cohomology, as exposited in e.g. [17, 26].

A *shelling* of a simplicial complex Δ is an ordering of the facets of Δ that obeys certain conditions, the precise details of which will not be important to us. Not every simplicial complex has a shelling; those that do are called *shellable*.

We remark that in the early history of the subject, shellings were defined only for balls and spheres [27]. Later, shellings were considered only for *pure* complexes, that is, complexes all of whose facets have the same dimension. Nowadays, shellings are studied on arbitrary simplicial complexes [7].

Shellable complexes are useful for showing a complex to satisfy the *Cohen-Macaulay* (in the pure case) or *sequentially Cohen-Macaulay* property (more generally). These properties are important in commutative algebra as well as combinatorics.

We refer to [33] for more on shellable and Cohen-Macaulay complexes.

2.4. **CL-labelings and EL-labelings.** The definition of a shelling is often somewhat unwieldy to work with directly, and it is desirable to find tools through which to work. One such tool is given by a CL-labeling, which we will now define.

If x and y are elements in a poset P, we say that y *covers* x when $x < y$ but there is no $z \in P$ so that $x < z < y$. In this situation, we write $x \lessdot y$, and may also say that $x \lessdot y$ is a *cover relation.* Thus, a cover relation is an edge in the Hasse diagram of P. A *rooted cover relation* is a cover relation $x \lessdot y$ together with a maximal chain from $\hat{0}$ to x (called the *root*).

A *rooted interval* is an interval $[x, y]$ together with a maximal chain $\mathbf{r}$ from $\hat{0}$ to x. In this situation, we use the notation $[x, y]_{\mathbf{r}}$. Notice that every atomic cover relation of $[x, y]_{\mathbf{r}}$ can be rooted by $\mathbf{r}$.

A *chain-edge labeling* of a bounded poset P is a function λ that assigns an element of an ordered set (which will always for us be $\mathbb{Z}$) to each rooted cover relation of P. Then λ assigns a word over $\mathbb{Z}$ to each maximal chain on any rooted interval by reading the cover relation labels in order, so e.g. the word associated with $\hat{0} \lessdot x_1 \lessdot x_2 \lessdot x_3 \lessdot \dots$ is $\lambda(\hat{0} \lessdot x_1, \hat{0})\lambda(x_1 \lessdot x_2, \hat{0} \lessdot x_1)\lambda(x_2 \lessdot x_3, \hat{0} \lessdot x_1 \lessdot x_2) \cdots$.

Remark 2.4. Since many researchers may be less familiar with CL-labelings and the machinery behind them, it may be helpful to think of a chain-edge labeling via the following dynamical process. Begin at $\hat{0}$, and walk up the maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot x_2 \lessdot \cdots \lessdot \hat{1}$. At each step i, assign a label to the label $x_{i-1} \lessdot x_i$. In assigning the label, you are allowed to look backwards at where you have been, but are <u>not</u> allowed to look forwards at where you may go. At each step, you add the assigned label to the end of a word associated with the maximal chain.

We say that a maximal chain $\mathbf{c}$ is *increasing* if the word associated with $\mathbf{c}$ is strictly increasing, and *decreasing* if the word is weakly decreasing. We order maximal chains by the lexicographic order on the associated words.

Definition 2.5. A *CL-labeling* is a chain-edge labeling that satisfies the following two conditions on each rooted interval $[x, y]_{\mathbf{r}}$:

(1) There is a unique increasing maximal chain $\mathbf{m}$ on $[x, y]_{\mathbf{r}}$, and
(2) the increasing chain $\mathbf{m}$ is strictly earlier in the lexicographic order than any other maximal chain on $[x, y]_{\mathbf{r}}$.

If a CL-labeling λ assigns the same value to $x \lessdot y$ irrespective of the choice of root, then we say λ is an *EL-labeling.*

Björner [2] and Björner and Wachs [6, 7] introduced CL-labelings, and proved the following theorem.

Theorem 2.6. *[8, Theorem 5.8] If λ is a CL-labeling of the bounded poset P, then the lexicographic order on the maximal chains of P is a shelling order of ΔP. In this case, a cohomology basis for ΔP is given by the decreasing maximal chains of P, and ΔP is homotopy equivalent to a bouquet of spheres in bijective correspondence with the decreasing maximal chains.*

For this reason, bounded posets with a CL- or EL-labeling are often called *CL- or EL-shellable.* Since the order complex of P and that of the order dual of P coincide, either a CL-labeling or a dual CL-labeling implies shellability of ΔP.

From the cohomology basis, it is straightforward to compute Euler characteristic, hence also Möbius number.

Corollary 2.7. *[8, Proposition 5.7] If λ is a CL-labeling of the bounded poset P, then the Möbius number of P is given by*

$$\begin{aligned}\mu(P) = \tilde{\chi}(\Delta P) = &\#\textit{even length decreasing maximal chains in } P \\ &- \#\textit{odd length decreasing maximal chains in } P.\end{aligned}$$

2.5. **Supersolvable and semimodular lattices.** We previously defined an element m of a lattice L to be left-modular if $(x \vee m) \wedge y = x \vee (m \wedge y)$ for all pairs $x < y$.

A lattice is *modular* if every element is left-modular. A lattice L is usually defined to be *semimodular* if whenever $a \wedge b \lessdot a$ in L, then $b \lessdot a \vee b$. We prefer the following equivalent definition, which highlights the close connection between semimodularity and comodernism:

Lemma 2.8 (see e.g. [35, essentially Theorem 1.7.2])**.** *A lattice L is semimodular if and only for every interval $[x, y]$ of L, every atom of $[x, y]$ is left-modular (as an element of $[x, y]$).*

Thus, the definition of a modernistic lattice is obtained from that of a semimodular lattice by weakening a single universal quantifier to an existential quantifier.

An *M-chain* in a lattice is a maximal chain consisting of left-modular elements. A lattice is *left-modular* if it has an M-chain, and *supersolvable* if it is graded and left-modular.

Supersolvable lattices were originally defined by Stanley [32], in a somewhat different form. The theory of left-modular lattices was developed in a series of papers [10, 22, 23, 25], and it was only in [25] that it was noticed that Stanley's original definition of supersolvable is equivalent to graded and left-modular.

There is an explicit cohomology basis for a supersolvable lattice, which does not seem to be as well-known as is deserved. A *chain of complements* to an M-chain $\mathbf{m} = \{\hat{0} = m_0 \lessdot m_1 \lessdot \cdots \lessdot m_n = \hat{1}\}$ is a chain of elements $\mathbf{c} = \{\hat{0} = c_n \lessdot c_{n-1} \lessdot \cdots \lessdot c_0 = \hat{1}\}$ so that each c_i is a *complement* to m_i, that is, so that $c_i \vee m_i = \hat{1}$ and $c_i \wedge m_i = \hat{0}$. A less explicit form of the following appears in [2, 32], and a special case in [38].

Theorem 2.9. *If L is a supersolvable lattice with a fixed M-chain* $\mathbf{m}$*, then a cohomology basis for ΔL is given by the chains of complements to* $\mathbf{m}$*. In particular, the Möbius number of L is (up to sign) the number of such chains.*

A (strong form of a) homology basis for supersolvable lattices appears in [30].

2.6. **Left-modularity.** We now recall some additional basic properties of left-modular elements. First, we state more carefully the equivalent "no pentagon" condition mentioned in the Introduction.

Lemma 2.10. *[23, Proposition 1.5] An element m of the lattice L is left-modular if and only if for every $a < c$ in L, we have $a \wedge m \neq c \wedge m$ or $a \vee m \neq c \vee m$.*

The pentagon lattice (usually notated as N_5) consists of elements $\hat{0}, \hat{1}, a, b, c$ with the only nontrivial relation being $a < c$. Lemma 2.10 says exactly that m is left-modular if and only if m never plays the role of b in a sublattice of L isomorphic to N_5. Thus, Lemma 2.10 is a pleasant generalization of the characterization of modular lattices as those with no pentagon sublattices.

Another useful fact is:

Lemma 2.11. *[22, Proposition 2.1.5] If m is a left-modular element of the lattice L, and $x < y$ in L, then $x \vee m \wedge y$ is a left-modular element of the interval $[x, y]$.*

Finally, we give an alternate characterization of left-modularity of coatoms, in the flavor of Lemma 2.8. This characterization will be useful for us in the proof of Theorems 1.2 and 3.4, and is also often easy to check.

Lemma 2.12 (Left-modular Coatom Criterion)**.** *Let m be a coatom of the lattice L. Then m is left-modular in L if and only if for every y such that $y \not\leq m$ we have $m \wedge y \lessdot y$.*

Proof. If $m \wedge y < z < y$, then $z < y$ violate the condition of Lemma 2.10 with m, hence m is not left-modular. Conversely, if $z < y$ violate the condition of Lemma 2.10, then $y \not\leq m$, and $m \wedge y = m \wedge z < z < y$. ☐

Corollary 2.13. *If L is a lattice and m a left-modular coatom of L, then for any $x < y$ in L either $x \vee m \wedge y = y$ or else $x \vee m \wedge y \lessdot y$.*

Proof. Apply Lemma 2.11 to m and $[x, \hat{1}]$, then Lemma 2.12 to $x \vee m$ and y. ☐

2.7. **Group theory.** We recall that a group G is said to be *solvable* if either of the following equivalent conditions is met:

(1) There is a chain $1 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_k = G$ of subgroups in G, so that each N_i is normal in G, and so that each factor N_i/N_{i-1} is abelian.
(2) There is a chain $1 = H_0 \subset\!\!\cdot\; H_1 \subset\!\!\cdot\; H_2 \subset\!\!\cdot\; \ldots \subset\!\!\cdot\; H_n = G$ of subgroups in G, so that each H_i is normal in H_{i+1} (but is not necessarily normal in G). Note that it follows in this case that each factor H_i/H_{i-1} is cyclic of prime order.

Since every subgroup of a solvable group is solvable, an alternative form of the latter is:

(2') For every subgroup $H \subseteq G$, there is a subgroup $K \subset\!\!\cdot\; H$ such that $K \triangleleft H$.

A subgroup H of G is said to be *subnormal* if there is a chain $H \triangleleft L_1 \triangleleft L_2 \triangleleft \ldots \triangleleft G$. Thus, Condition (2) says a group is solvable if and only if G has a maximal chain consisting of subnormal subgroups.

A group is *supersolvable* if there is a maximal chain in $L(G)$ consisting of subgroups normal in G. Thus, a group is supersolvable if there is a chain which simultaneously meets the conditions in (1) and (2). One important fact about the subgroup lattice of supersolvable groups is:

Theorem 2.14. *[19] For a group G, the subgroup lattice $L(G)$ is graded if and only if G is supersolvable.*

Subgroup lattices were one of the motivations for early lattice theorists in making the definition of (left-)modularity. It follows easily from the Dedekind identity (see Lemma 5.3) that if $N \lhd G$, then N is left-modular in $L(G)$. In particular, if G is a supersolvable group, then $L(G)$ is a supersolvable lattice.

Moreover, a normal subgroup N satisfies a stronger condition. An element m of a lattice is said to be *modular* (or *two-sided-modular*) if it neither plays the role of b nor of a in any pentagon sublattice, where a, b are as in the discussion following Lemma 2.10. A second application of the Dedekind identity shows any normal subgroup to be modular in $L(G)$.

The following lemma, whose proof is immediate from the definitions, says that left-modularity and two-sided-modularity are essentially the same for the purpose of comodernism arguments.

Lemma 2.15. *If L is a lattice, and m is a maximal left-modular element, then m is modular.*

We refer to e.g. [11, Chapter A] for further general background on group theory, and to [29] for the reader interested in further background on lattices of subgroups.

3. Proof of Theorem 1.2

3.1. **sub-M-chains.** As discussed in Section 2.5, a lattice is left-modular if it has an M-chain, that is, a maximal chain consisting of left-modular elements. The reader may be reminded of the maximal chain consisting of normal elements in the definition of a supersolvable group.

We extend the notion of M-chain to comodernistic lattices. A maximal chain $\hat{0} = m_0 \lessdot m_1 \lessdot \cdots \lessdot m_n = \hat{1}$ in L is a *sub-M-chain* if for every i, the element m_i is left-modular in the interval $[\hat{0}, m_{i+1}]$. The reader may be reminded of the maximal subnormal chain in a solvable group. It is straightforward to show that a lattice is comodernistic if and only if every interval has a sub-M-chain.

Stanley [32] and Björner [2] showed that any supersolvable lattice has an EL-labeling, and Liu [22] extended this to any left-modular lattice. If $\mathbf{m}^{(ss)} = \left\{\hat{0} = m_0 \lessdot m_1 \lessdot \cdots \lessdot m_n = \hat{1}\right\}$ is an M-chain, then the EL-labeling is defined as follows:

$$\begin{aligned} \lambda_{ss}(x \lessdot y) &= \max\{i \,:\, x \vee m_{i-1}^{(ss)} \wedge y = x\} \\ &= \min\{i \,:\, x \vee m_i^{(ss)} \wedge y = y\}. \end{aligned} \tag{3.1}$$

The essential observation involved in proving Theorem 1.2 is that, if we replace the M-chain used for λ_{ss} with a sub-M-chain, then we can still label the atomic cover relations of

L in the same manner as in λ_{ss}. More precisely, if $\mathbf{m}$ is a sub-M-chain in a lattice L, then let

$$\lambda(\hat{0} \lessdot a) = 1 + \max\{i : m_i \wedge a = \hat{0}\}. \tag{3.2}$$

Adding 1 is not essential, and we do so only so that the labels will be in the range 1 through n, rather than 0 through $n-1$.

3.2. **The CL-labeling.** We construct the full CL-labeling recursively from (3.2).

We say that a chain $\mathbf{c}$ is *indexed* by a subset $S = \{i_1 < \cdots < i_k\}$ of the integers if $\mathbf{c} = \{c_{i_1} < \cdots < c_{i_k}\}$. That is, we associate an index or label with each element of the chain. Notice that we require the indices to (strictly) respect order.

We will need the following somewhat-technical lemma to handle non-graded lattices.

Lemma 3.1. *Let L be a lattice with a sub-M-chain $\mathbf{m}$ of length n. Then no chain of L has length greater than n.*

Proof. We proceed by induction on n. The base case is trivial. Suppose that $\hat{0} = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_\ell = \hat{1}$ is some chain in L. Let $m_{n-1} \lessdot 1$ be the unique coatom in $\mathbf{m}$, and i be the greatest index such that $c_i \leq m_{n-1}$. Then $c_j \vee m_{n-1} = \hat{1}$ for any $j > i$, so by the left-modular property

$$c_{i+1} \wedge m_{n-1} < c_{i+2} \wedge m_{n-1} < \cdots < c_\ell \wedge m_{n-1} = m_{n-1}.$$

Thus,

$$\hat{0} = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_i = c_{i+1} \wedge m_{n-1} < c_{i+2} \wedge m_{n-1} < \cdots < c_\ell \wedge m_{n-1} = m_{n-1}$$

is a chain of length $\ell - 1$ on $[\hat{0}, m_{n-1}]$, and by induction $\ell - 1 \leq n - 1$. □

Definition 3.2. Let L be a comodernistic lattice of height n. Take a fixed sub-M-chain $\mathbf{m}$ given as $\hat{0} = m_0 \lessdot m_1 \lessdot \cdots \lessdot m_n = \hat{1}$ as the starting point for a recursive construction.

Let $x \lessdot a, \mathbf{r}$ be a rooted cover relation. Assume by recursion that we are given a sub-M-chain $\mathbf{m}^{(\mathbf{r})}$ on $[x, \hat{1}]$. Further assume that the elements of $\mathbf{m}^{(\mathbf{r})}$ are indexed by a subset $S \subseteq [n]$, and that $\hat{1} = m_n^{(\mathbf{r})}$. Label $x \lessdot a$ as in (3.2)that is, as

$$\lambda(x \lessdot a) = 1 + \max\{i : m_i^{(\mathbf{r})} \wedge a = x\}. \tag{3.3}$$

To continue the recursion, it remains to construct an indexed sub-M-chain $\mathbf{m}^{(\mathbf{r} \cup a)}$ on $[a, \hat{1}]$.

Suppose that $\lambda(x \lessdot a) = 1 + i$. It is clear that $m_i^{(\mathbf{r})}$ is the greatest element of $\mathbf{m}^{(\mathbf{r})}$ such that $a \not\leq m_i^{(\mathbf{r})}$. By abuse of notation, let $\mathbf{m}_{>i}^{(\mathbf{r})}$ be the portion of $\mathbf{m}$ that is greater than $m_i^{(\mathbf{r})}$, and let $S_{>i}$ be the indices greater than i on $\mathbf{m}^{(\mathbf{r})}$. Thus, the labels of $\mathbf{m}_{>i}^{(\mathbf{r})}$ are exactly $S_{>i}$. Let $S_{<i} = S \setminus (S_{>i} \cup i)$ similarly be the indices less than i on $\mathbf{m}^{(\mathbf{r})}$.

Now by construction, all elements of $\mathbf{m}_{>i}^{(\mathbf{r})}$ are greater than a. By the comodernistic property, the submodular chain $\mathbf{m}_{>i}^{(\mathbf{r})}$ may be completed to a sub-M-chain $\mathbf{m}^{(\mathbf{r} \cup a)}$ for $[a, \hat{1}]$. Preserve the indices on $\mathbf{m}_{>i}^{(\mathbf{r})}$, and index the elements of $\mathbf{m}^{(\mathbf{r} \cup a)} \setminus \mathbf{m}^{(\mathbf{r})}$ by elements of $S_{<i}$. It

follows by applying Lemma 3.1 on $[\hat{0}, m_{i+1}]$ that there are enough indices available in $S_{<i}$ to perform such indexing.

The recursion can now continue, which completes the definition of the CL-labeling.

Notation 3.3. Throughout the remainder of Section 3, we fix L to be a comodernistic lattice of height n, with a sub-M-chain $\mathbf{m} = \{\hat{0} = m_0 \lessdot m_1 \lessdot \cdots \lessdot m_n = \hat{1}\}$. Indeed, we select a sub-M-chain on every interval, which uniquely determines a chain-edge labeling λ as in Definition 3.2.

We are now ready to prove the following refinement of Theorem 1.2.

Theorem 3.4. *The labeling λ of Notation 3.3 is a CL-labeling.*

Proof. It is clear from construction that λ is a chain-edge labeling. By the recursive construction, it suffices to show that an interval of the form $[\hat{0}, y]$ has a unique increasing maximal chain, and that every lexicographically first maximal chain on $[\hat{0}, y]$ is increasing.

Let $\mathbf{m} = \{\hat{0} = m_0 \lessdot m_1 \lessdot \cdots \lessdot m_n = \hat{1}\}$ be the sub-M-chain used to define the labeling. Let $\ell = 1 + \max\{i : m_i \wedge y < y\}$. It is clear from the construction that every atom cover relation on $[\hat{0}, y]$ receives a label that is at most ℓ. Since the elements greater than $m_{\ell-1}$ are preserved until the corresponding labels are used, no chain on $[\hat{0}, y]$ receives any label greater than ℓ.

Similarly, the $m_{\ell-1}$ element in the sub-M-chain is by construction preserved until the ℓ label is used, and a chain receives an ℓ label when it leaves the interval $[\hat{0}, m_{\ell-1}]$. Since $y \notin [\hat{0}, m_{\ell-1}]$, we see that every maximal chain on $[\hat{0}, y]$ receives an ℓ label. Thus, an increasing chain must have the ℓ label on its last cover relation.

But Corollary 2.13 gives $m_{\ell-1} \wedge y$ to be a coatom of $[\hat{0}, y]$. It follows by the definition of the labeling that every increasing chain on $[\hat{0}, y]$ must end with $m_{\ell-1} \wedge y \lessdot y$. An easy induction now yields the only increasing chain to be $\hat{0} = m_0 \wedge y \leq m_1 \wedge y \leq \cdots \leq m_{\ell-1} \wedge y$, the "projection" of the sub-M-chain to $[\hat{0}, y]$. As $m_{\ell-1} \wedge y \lessdot y$, the projection chain is in particular maximal.

We now show that this chain is the unique lexicographically first chain. In the construction, the least label of an atomic cover relation on $[\hat{0}, y]$ corresponds with the least m_{i+1} such that $m_{i+1} \wedge y > \hat{0}$. But this is the (unique) first non-$\hat{0}$ element of the increasing chain. The desired follows. ☐

3.3. **More details about the CL-labeling.** Any chain-edge labeling assigns a word to each maximal chain of L. Since when we label a cover relation with i according to λ, we remove i from the index set (used for available labels), we obtain a result extending one direction of [24, Theorem 1].

Lemma 3.5. *The chain-edge labeling λ assigns a word with no repeated labels to each maximal chain in L.*

Thus, if L is graded of height n, then λ assigns a permutation in S_n to each maximal chain.

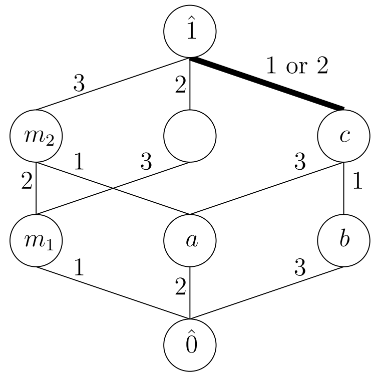

FIGURE 3.1. A comodernistic labeling of a lattice

The decreasing chains are also easy to (recursively) understand. Recall that if x and y are lattice elements, then x is a *complement* to y if $x \vee y = \hat{1}$ and $x \wedge y = \hat{0}$. The following is an extension of Theorem 2.9 for comodernistic lattices.

Lemma 3.6. *If $\hat{0} \lessdot c_1 \lessdot \cdots \lessdot \hat{1}$ be a decreasing chain of L with respect to λ, then c_1 is a complement to m_{n-1}.*

Proof. As in the proof of Theorem 3.4, every maximal chain on $[\hat{0}, \hat{1}]$ contains an n label. Thus $\lambda(\hat{0} \lessdot c_1) = n$, so $m_{n-1} \wedge c_1 = \hat{0}$. The result follows. □

We close this section by working out a small example in detail.

Example 3.7. We consider the lattice C in Figure 3.1, which we obtained by removing a single cover relation from the Boolean lattice on 3 elements. It is easy to check that m_2 is modular in C, but that m_1 is not modular in C. (Indeed, m_1 together with $b < c$ generate a pentagon sublattice.) Since any lattice of height at most 2 is modular, the chain $\hat{0} \lessdot m_1 \lessdot m_2 \lessdot \hat{1}$ is a sub-M-chain.

With the exception of $c \lessdot \hat{1}$, the label of every cover relation in C is independent of the choice of root. We have indicated these labels in the diagram. But we notice that the interval $[a, \hat{1}]$ inherits the sub-M-chain $a \lessdot m_2 \lessdot \hat{1}$, while the interval $[b, \hat{1}]$ has unique maximal (sub-M-)chain $b \lessdot c \lessdot 1$. Thus, the edge $c \lessdot \hat{1}$ receives a label of 1 with respect to root $\hat{0} \lessdot a \lessdot c$, but a label of 2 with respect to root $\hat{0} \lessdot b \lessdot c$.

The reader may have noticed that the atom a is indeed left-modular. Thus, although we have shown the comodernistic labeling determined by the given sub-M-chain, there is also a supersolvable EL-labeling of C. We will see in Example 4.8 and Figure 4.1 a lattice that is neither geometric nor supersolvable, but that is comodernistic.

4. Order congruence and order convexity lattices

In this section, we examine the order congruence lattices of posets, as considered in the introduction and in Section 2.2. We prove Theorem 1.5, and apply Lemma 3.6 to calculate the Möbius number of $\mathcal{O}(P)$.

4.1. **Order congruence lattices are comodernistic.** A useful tool for showing certain lattices to be comodernistic is given by the following lemma.

Lemma 4.1. *Let L be a meet subsemilattice of a lattice L_+. If $m \in L_+$ is a left-modular coatom in L_+, and $m \in L$, then m is also left-modular in L.*

Proof. Since m is a coatom in both L_+ and therefore in L, the join of x and m is either m (if $x \leq m$) or $\hat{1}$ (otherwise) in both lattices. In particular, the join operations in L and L_+ agree on m, and we already know the meet operations agree by the subsemilattice condition. The result now follows by Lemma 2.10. □

The following theorem follows immediately.

Theorem 4.2. *If L is a meet subsemilattice of the partition lattice Π_S, and $m \in L$ is a partition of the form $x \,|\, (S \setminus x)$, then m is a left-modular coatom of L.*

We now show:

Lemma 4.3. *If P is any poset, then the order congruence lattice $\mathcal{O}(P)$ is a meet subsemilattice of Π_P.*

Proof. It is clear from definition that $\mathcal{O}(P)$ is a subposet of Π_P. It suffices to show that if π_1, π_2 are in $\mathcal{O}(P)$, then their meet $\pi_1 \wedge \pi_2$ also is in $\mathcal{O}(P)$. Let $f_1, f_2 : P \to \mathbb{Z}$ be such that π_1 and π_2 are the level sets of f_1 and f_2. But then the product map $f_1 \times f_2 : P \to \mathbb{Z} \times \mathbb{Z}$ (where $\mathbb{Z} \times \mathbb{Z}$ is taken with the product order) has the desired level set partition. □

That order congruence lattices are comodernistic now follows easily.

Proof (of Theorem 1.5). Let P be a poset, and let x be a maximal element of P. Assume by induction that the result holds for all smaller posets. It is straightforward to see $m = x \,|\, (P \setminus x)$ is the level set partition of an order preserving map. By Theorem 4.2 and Lemma 4.3, the element m is a left-modular coatom on the interval $[\hat{0}, \hat{1}]$. Since $[\hat{0}, m]$ is lattice-isomorphic to $\mathcal{O}(P \setminus x)$, we get by induction that $[\hat{0}, \pi]$ is comodernistic when $\pi < m$. If π is incomparable to m, then $\pi \wedge m$ is a left-modular coatom of $[\hat{0}, \pi]$ by Corollary 2.13. Finally, repeated application of Lemma 2.3 and induction gives that intervals of the form $[\pi', \hat{1}]$ are comodernistic. The result follows for general intervals $[\pi', \pi]$. □

We do not know whether the order convexity partition lattice of every poset is shellable or comodernistic. It is straightforward to show that (as $\mathcal{O}^{\text{conv}}(P)$ is clearly a meet subsemilattice of $\mathcal{O}(P)$) every interval of the form $[\hat{0}, \pi]$ has a left-modular coatom, but intervals of the form $[\pi', \hat{1}]$ are somewhat more difficult to understand.

4.2. **The Möbius number of an order congruence lattice.** We now use the comodernism of the order congruence lattice $\mathcal{O}(P)$ to recover the Möbius number calculation due to Jenča and Sarkoci. Denote by $\mathrm{Compat}(x)$ the set of all $y \in P$ that are compatible with x. That is, $\mathrm{Compat}(x)$ consists of all y such that either $y \lessdot x$, $x \lessdot y$, or y is incomparable to x.

Our proof is short and simple.

Theorem 4.4. *[20, Theorem 3.8] For any poset P with maximal element x, the Möbius function of the order congruence lattices satisfies the recurrence*

$$\mu(\mathcal{O}(P)) = -\sum_{y \in \mathrm{Compat}(x)} \mu(\mathcal{O}(P_{x\sim y}).$$

Proof. By Lemma 3.6 together with the proof of Theorem 1.5, every decreasing chain of $\mathcal{O}(P)$ begins with a complement to the (left-modular) order partition $x \mid P \setminus x$. Such complements are easily seen to be atoms a whose non-singleton block is $\{x, y\}$, where $y \in P$ is compatible with x. The result now follows by observing that the interval $[a, \hat{1}]$ in $\mathcal{O}(P)$ is isomorphic to $\mathcal{O}(P_{x\sim y})$. ☐

Jenča and Sarkoci also show in [20] that if P is a Hasse-connected poset, then the number of linear extensions of P satisfies the same recurrence as $\mu(\mathcal{O}(P))$. We give a short bijective proof of the same, which has the same flavor as the proof of the main result in [12]. Let $\mathcal{LE}(P)$ denote the set of linear extensions of P.

Lemma 4.5. *Let P be a poset and x a maximal element of P. If x is not also minimal, then there is a bijection*

$$\mathcal{LE}(P) \to \bigcup_{y \in \mathrm{Compat}(x)} \mathcal{LE}(P_{x\sim y}).$$

Proof. Since x is not minimal, it cannot be the first element in any linear extension L of P. If y is the element immediately preceding x in L, then it is clear that x and y are compatible. Then $L_{x\sim y}$ is a linear extension of $P_{x\sim y}$.

To show this map is a bijection, we notice that the process is reversible. If L is a linear extension of $P_{x\sim y}$, replace the element corresponding to the identification of x and y with x followed by y to get a linear extension of P. ☐

Corollary 4.6. *If P is a Hasse-connected poset, then the decreasing chains of $\mathcal{O}(P)$ are in bijective correspondence with linear extensions of P.*

In particular, $|\mu(\mathcal{O}(P)|$ is the number of linear extensions of P, and $\Delta\mathcal{O}(P)$ is homotopy equivalent to a bouquet of this number of $(|P|-3)$-dimensional spheres.

Proof. Since P is Hasse-connected, a maximal element x cannot also be minimal. Moreover, if P is Hasse-connected then $P_{x\sim y}$ is also Hasse-connected. The result now follows immediately by Theorem 4.4 and Lemma 4.5. ☐

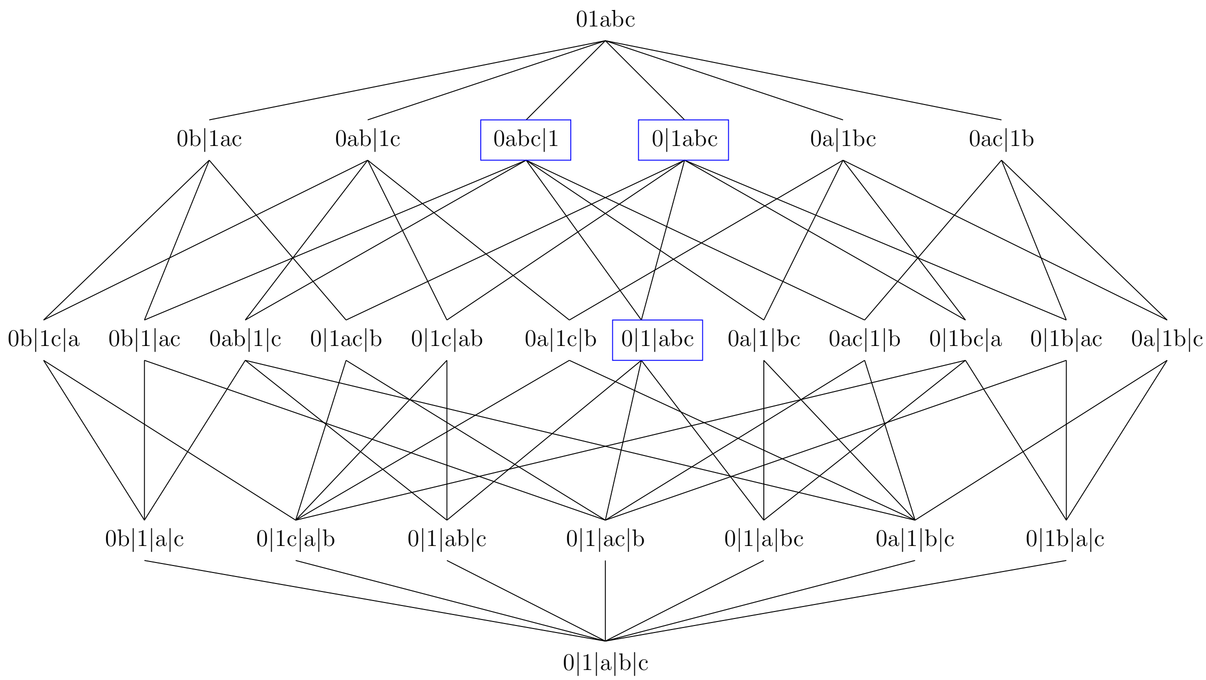

FIGURE 4.1. The order congruence lattice $\mathcal{O}(N_5)$ of the pentagon lattice N_5. Nontrivial left-modular elements are shown with rectangles.

In the case where P is not Hasse-connected, we can still say something. The proof of the following is exactly similar to that of Lemma 4.5.

Lemma 4.7. *Let P be a poset, and x an element that is incomparable with every other element. Then there is a bijection*

$$\mathcal{LE}^*(P) \to \bigcup_{y \in \mathrm{Compat}(x)} \mathcal{LE}(P_{x\sim y}),$$

where $\mathcal{LE}^*(P)$ denotes the linear extensions of P that do not have x as first element.

It follows that there is a bijection between decreasing chains of $\mathcal{O}(P)$ and a certain subset of linear extensions, described recursively by Lemmas 4.5 and 4.7. We omit a non-recursive description. Another approach to this problem was worked out in detail by Jenča and Sarkoci in [20, Theorem 4.5].

4.3. **An order congruence lattice that is neither geometric nor supersolvable.**

Example 4.8. Consider the pentagon lattice N_5, obtained by attaching a bottom and top element $\hat{0}$ and $\hat{1}$ to the poset with elements a, b, c and relation $a < c$. In this case, $\mathcal{O}(N_5)$ and $\mathcal{O}^{\mathrm{conv}}(N_5)$ coincide, and are pictured in Figure 4.1.

Group class	Lattice class for $L(G)$	Characterizes group class?	Self-dual?
cyclic	distributive	Yes	Yes
abelian, Hamiltonian	modular	No	Yes
nilpotent	lower semimodular	No	No
supersolvable	supersolvable	Yes	Yes
solvable	???		

Table 1. Classes of groups and related classes of lattices.

The reader can verify by inspection that no atom of $\mathcal{O}(N_5)$ is left-modular, thus, the lattice $\mathcal{O}(N_5)$ is comodernistic but neither geometric nor supersolvable. We remark that order congruence lattices that fail to be geometric were examined earlier in [21].

5. Solvable subgroup lattices

In this section, we discuss applications to and connections with the subgroup lattice of a group.

5.1. **Known lattice-theoretic analogues of classes of groups.** Since the early days of the subject, a main motivating object for lattice theory has been the subgroup lattice of a finite group. Indeed, a (left-)modular element may be viewed as a purely lattice-theoretic analogue or extension of a normal subgroup. Focusing on the normal subgroups characterizing a class of groups then typically gives in a straightforward way an analogous class of lattices with interesting properties. For example, every subgroup of an abelian (or more generally Hamiltonian) group is normal, so a corresponding class of lattices is that of the modular lattices.

We summarize some of these analogies in Table 1.

We remark that, although every normal subgroup is modular in the subgroup lattice, not every modular subgroup is normal. Similarly, although every nilpotent group has lower semimodular subgroup lattice, group that are not nilpotent may also have lower semimodular subgroup lattice. For example, the subgroup lattice $L(S_3)$ has height 2, hence is modular (despite being neither abelian nor nilpotent). As $L(S_3)$ is lattice isomorphic to $L(\mathbb{Z}_3^2)$, a subgroup lattice characterization of these classes of groups is not possible.

It may then be surprising that a group G is supersolvable if and only if $L(G)$ is a supersolvable lattice. The reason for this is more superficial than one might hope: Iwasawa proved in [19] that a group is supersolvable if and only if its subgroup lattice is graded. However, the definition of supersolvable lattice seems to capture the pleasant combinatorial properties of supersolvable groups much better than the definition of graded lattice does.

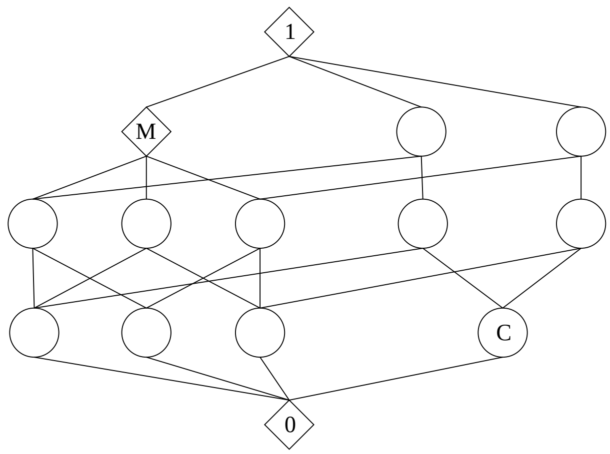

FIGURE 5.1. A non-shellable lattice satisfying Conditions (2) and (3) of Proposition 5.1.

Semimodular and supersolvable lattices have been of great importance in algebraic and topological combinatorics. In particular, both classes of lattices are EL-shellable, and the EL-labeling gives an efficient method of computing homotopy type, Möbius invariants, etc.

5.2. **Towards a definition of solvable lattice.** After the previous subsection, it may come as some surprise that there is no widely-accepted definition of solvable lattice. It is the purpose of this subsection to make the case for the definition of comodernistic lattices as one good candidate.

It was first proved by Suzuki in [37] that solvable groups are characterized by their subgroup lattices. Later Schmidt gave an explicit characterization:

Proposition 5.1 (Schmidt [28]; see also [29, Chapter 5.3])**.** *For a group G, the following are equivalent:*

(1) G is solvable.
(2) $L(G)$ has a chain of subgroups $1 = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_k = G$ such that each G_i is modular in $L(G)$, and such that each interval $[G_i, G_{i+1}]$ is a modular lattice.
(3) $L(G)$ has a chain of subgroups $1 = G_0 \lessdot G_1 \lessdot \cdots \lessdot G_n = G$ such that each G_i is modular in the interval $[1, G_{i+1}]$.

The reader will recognize the conditions in Proposition 5.1 as direct analogues of the conditions from Section 2.7. Despite this close correspondence, proving that Conditions (2) and (3) of the proposition imply solvability is not at all trivial.

Although Proposition 5.1 combinatorially characterizes solvable subgroup lattices, we find it somewhat unsatisfactory. We don't know how to use the implicit lattice conditions to calculate Möbius numbers. And, as the following example will show, lattices satisfying the implicit conditions need not be shellable.

Example 5.2. Consider the lattice whose Hasse diagram is pictured in Figure 5.1. The element M is easily verified to be modular in this lattice, and the interval $[\hat{0}, M]$ is a Boolean lattice, hence modular. But since the interval $[C, \hat{1}]$ is disconnected, the lattice is not shellable or Cohen-Macaulay.

It is our opinion that a good definition of "solvable lattice" should be equivalent to solvability on subgroup lattices, and obey as many of the useful properties possessed by supersolvable lattices as possible. Among these are: EL-shellability, efficient computation of homotopy type and/or homology bases and/or Möbius numbers, and self-duality of the property. Perhaps even more importantly, such a definition should have many combinatorial examples.

We believe comodernistic lattices to be an important step towards understanding a definition or definitions of "solvable lattice". As Theorem 1.7 states, comodernism is equivalent to solvability on subgroup lattices. While we do not know if a comodernistic lattice is always EL-shellable, we have shown such a lattice to be CL-shellable. The CL-labeling allows efficient calculation of homotopy type, and consequences thereof. Perhaps most importantly, there are many natural examples of comodernistic lattices.

Unfortunately, comodernism is not a self-dual property, as is made clear by Example 4.8.

5.3. **Proof of Theorem 1.7.** We begin the proof by reviewing a few elementary facts about groups and subgroup lattices, all of which can be found in [29], or easily verified by the reader. We say that H *permutes* with K if $HK = KH$.

Lemma 5.3. *Let H, K, and L be subgroups of a group G.*

(1) If H permutes with K, then $HK = KH$ is the join in $L(G)$ of H and K.
(2) If $N \lhd G$, then N permutes with every subgroup of G.
(3) (Dedekind Identity) If $H \subseteq K$, then $H(K \cap L) = K \cap HL$ and $(K \cap L)H = K \cap LH$.

Corollary 5.4. *If $K \supset H$ permutes with every subgroup on the interval $[H, G]$, then K is a modular element of this interval.*

By Lemma 2.15, the elements of a sub-M-chain of a comodernistic lattice satisfy the modularity condition as in part (3) of Proposition 5.1. It follows immediately by the same proposition that G is solvable if $L(G)$ is comodernistic.

For the other direction, every subgroup of a solvable group is solvable. Thus, it suffices to find a modular coatom in the interval $[H, G]$ over any subgroup H. Let $1 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_k = G$ be a chief series. It follows by that each HN_i is a subgroup for each i. Let ℓ be the maximal index such that $HN_\ell < G$, and let K be any coatom of the interval $[HN_\ell, G]$.

We will show that K permutes with every subgroup L on $[H, G]$, hence is modular on the same interval. For any such L, we have

$$H(L \cap N_{\ell+1}) = L \cap HN_{\ell+1} = L \cap G = L, \text{ and similarly}$$
$$(L \cap N_{\ell+1})H = L \cap N_{\ell+1}H = L \cap G = L,$$

so that H permutes with $L \cap N_{\ell+1}$. Moreover, $N_\ell \subseteq K \cap N_{\ell+1} \subseteq N_{\ell+1}$, and since $N_{\ell+1}/N_\ell$ is abelian, the Correspondence Theorem gives that $K \cap N_{\ell+1} \lhd N_{\ell+1}$. Thus, $K \cap N_{\ell+1}$ permutes with $L \cap N_{\ell+1}$. Now

$$KL = H(K \cap N_{\ell+1})H(L \cap N_{\ell+1}) = H(L \cap N_{\ell+1})H(K \cap N_{\ell+1}) = LK,$$

as desired.

5.4. **Homotopy type of the subgroup lattice of a solvable group.** It is immediate from Lemma 5.3 that if $N \triangleleft G$ then HN permutes with all subgroups on the interval $[H, G]$. Thus, a chief series $\{N_i\}$ lifts to a chain of left-modular elements $\{HN_i\}$ on the interval $[H, G]$. In a solvable group we can (by the proof of Theorem 1.7) complete the chain $\{HN_i\}$ to a sub-M-chain. Let λ be constructed according to this choice of sub-M-chain in all applicable intervals, and consider the decreasing chains of λ.

It is immediate by basic facts about left-modular elements that a decreasing maximal chain on $L(G)$ contains as a subset a chain of complements to the chief series $\{N_i\}$. Since a chain of complements to $\{N_i\}$ in a solvable group is a maximal chain [11, Lemma 9.10], such chains are exactly the decreasing maximal chains.

The order complex of a CL-shellable poset is a bouquet of spheres, where the spheres are in bijective correspondence with the decreasing chains of the poset. Thus, we recover the homotopy-type calculation of [38] (see also [40, 41]).

6. k-equal partition and related lattices

In this section, we will show that the k-equal partition lattices are comodernistic. We'll also show two related families of lattices to be comodernistic.

6.1. **k-equal partition lattices.** Recall that the *k-equal partition lattice* $\Pi_{n,k}$ is the subposet of the partition lattice Π_n consisting of all partitions whose non-singleton blocks have size at least k.

Theorem 6.1. *For any $1 \leq k \leq n$, the k-equal partition lattice $\Pi_{n,k}$ is comodernistic.*

Proof. Consider an interval $[\pi', \pi]$ in $\Pi_{n,k}$. Let π be $C_1 \mid C_2 \mid \dots \mid C_m$, and assume without loss of generality that C_1 is not a block in π'. Then C_1 is formed by merging blocks $B_1, \dots, B_\ell$ of π'. Suppose that the B_i's are ordered by increasing size, so that $|B_1| \leq |B_2| \leq \cdots$. Consider the element

$$m = B_1 \mid B_2 \cup \cdots \cup B_\ell \mid C_2 \dots \mid C_m.$$

Suppose that σ is some partition on $[\pi', \pi]$, and D is the block of σ containing B_1. If $D = B_1$, then $\sigma \wedge m = \sigma$. Otherwise, there are two cases:

Case 1: $|B_1| > 1$. Then $\sigma \wedge m$ is formed by splitting D into blocks B_1, $D \setminus B_1$. Notice that $|D \setminus B_1| \geq |B_1| \geq k$ by the ordering of the B_i's.

Case 2: $|B_1| = 1$. Then if $|D| > k$, then $\sigma \wedge m$ is formed by splitting D into smaller blocks $B_1, D \setminus B_1$. Otherwise, we have $|D| = k$, and $\sigma \wedge m$ is formed by splitting D into k singletons.

In either situation, we have $\sigma \wedge m \lessdot \sigma$, so Lemma 2.12 gives m to be left-modular on the desired interval. $\square$

We recover from Theorem 6.1 a weaker form of the result [7, Theorem 6.1] that k-equal partition lattices are EL-shellable. Repeated application of Lemma 3.6 recovers the same set of decreasing chains for the comodernistic labeling as in [7, Corollary 6.2]. See also [9].

6.2. **k, h-equal partition lattices in type B.** By a *sign pattern* of a set S, we refer to an assignment of $+$ or $-$ to each element of S, considered up to reversing the signs of every element of S. Thus, if S has an order, an equivalent notion is to assign a $+$ to the first element of S and an arbitrary sign to each remaining element.

A *signed partition* of $\{0, 1, \ldots, n\}$ then consists of a partition of the set, together with a sign pattern assignment for each block not containing 0. The block containing 0 is called the *zero block*, and other blocks are called *signed blocks*. The *signed partition lattice* Π_n^B consists of all signed partitions of $\{0, 1, \ldots, n\}$. The cover relations in Π_n^B are of two types: merging two signed blocks, and selecting one of the two possible patterns of the merged set; or merging a signed block with the zero block (thereby 'forgetting' the sign pattern on the signed block).

The signed partition lattice is well-known to be supersolvable. Indeed, if π is a signed partition where every signed block is a singleton, then π is left-modular in Π_n^B.

Björner and Sagan [5] considered the *signed k, h-equal partition lattice*, where $1 \leq h < k \leq n$. This is the subposet $\Pi_{n,k,h}^B$ consisting of all signed partitions whose signed blocks have size at least k, and whose zero block has size at least $h + 1$.

Theorem 6.2. *For any $1 \leq h < k \leq n$, the signed k, h-equal partition lattice $\Pi_{n,k,h}^B$ is comodernistic.*

Proof. We proceed similarly to the proof of Theorem 6.1. Let $[\pi', \pi]$ be an interval in $\Pi_{n,k,h}^B$, and π be $C_1 \,|\, C_2 \,|\, \ldots \,|\, C_m$. Assume without loss of generality that C_1 is not a block in π'. Then C_1 is formed by merging blocks $B_1, \ldots, B_\ell$ of π'. Let B_1 be the smallest signed block in this list, and consider the element

$$m = B_1 \,|\, B_2 \cup \cdots \cup B_\ell \,|\, C_2 \,\ldots\, |\, C_m.$$

Now let σ be some partition in the interval $[\pi', \pi]$, and D be the block of σ containing B_1. If $D = B_1$, then $\sigma \wedge m = \sigma$. Otherwise, there are two cases:

Case 1: $|B_1| > 1$. Since σ is on $[\pi', \pi]$, we see that $D \subseteq C_1$. If $0 \in D \setminus B_1$, then $|D \setminus B_1| \geq h + 1$, as otherwise there would be a (signed) singleton block inside C_1 on $[\pi', \pi]$. If $0 \notin D \setminus B_1$, then $|D \setminus B_1| \geq |B_1| \geq k$ by the ordering of the blocks. In either case, it follows that $\sigma \wedge m$ is formed by splitting D into blocks B_1, $D \setminus B_1$.

Case 2: $|B_1| = 1$. If $|D| > k$, then $\sigma \wedge m$ is formed by splitting D into smaller blocks $B_1, D \setminus B_1$. Similarly if $0 \in D$ and $|D| > h + 1$. Otherwise, we have $|D| = k$ or $|D| = h$ (depending on whether $0 \in D$), and $\sigma \wedge m$ is formed by splitting D into singletons.

In either situation, we have $\sigma \wedge m \lessdot \sigma$, hence that m is left-modular on the desired interval by Lemma 2.12. ☐

Björner and Sagan [5, Theorem 4.4] showed $\Pi^B_{n,k,h}$ to be EL-shellable. We recover from Theorem 6.2 the weaker result of CL-shellability. However, we remark that our proof is significantly simpler, and still allows easy computation of a cohomology basis, etc.

There is also a "type D analogue" of $\Pi_{n,k}$ and $\Pi^B_{n,k,h}$. Björner and Sagan considered this lattice in [5], but left the question of shellability open. Feichtner and Kozlov gave a partial answer to the type D shellability question in [15].

Our basic technique in the proofs of Theorems 6.1 and 6.2 is to show that left-modularity of coatoms in Π_n and Π^B_n is sometimes inherited in the join subsemilattices $\Pi_{n,k}$ and $\Pi^B_{n,k,h}$. It is easy to verify from the (here omitted) definition that the type D analogue of Π_n and Π^B_n has no left-modular coatoms, see also [18]. For this reason, the straightforward translation of our techniques will not work in type D. We leave open the question of under what circumstances the type D analogue of $\Pi_{n,k}$ has left-modular coatoms, or is comodernistic.

6.3. **Partition lattices with restricted element-block size incidences.** The k-equal partition lattices admit generalizations in several directions. One such generalization, examined in [7], is that of the subposet of partitions where the size of every block is in some set T. Further generalizations in similar directions are studied in [13, 14].

We consider here a different direction. Motivated by the signed k, h-equal partition lattices, Gottlieb [16] examined a related sublattice of the (unsigned) partition lattice. In Gottlieb's lattice, the size of a block is restricted to be at least k or at least h, depending on whether or not the block contains a distinguished element.

We further generalize to allow each element to have a different block-size restriction associated to it. More formally, we consider a map $\operatorname{aff} : [n] \to [n]$, which we consider as providing an *affinity* to each element x of $[n]$. We now consider two subposets of the partition lattice Π_n:

$$\Pi^{\forall}_{\mathrm{aff}} \triangleq \{\pi \in \Pi_n \,:\, \text{every nonsingleton block } B \text{ has } |B| \geq \operatorname{aff}(x) \text{ for every } x \in B\}\,, \text{ and}$$

$$\Pi^{\exists}_{\mathrm{aff}} \triangleq \{\pi \in \Pi_n \,:\, \text{every nonsingleton block } B \text{ contains some } x \text{ such that } |B| \geq \operatorname{aff}(x)\}\,.$$

It is clear that both subposets are join subsemilattices of Π_n, hence lattices.

Theorem 6.3. *For any selection of affinity map* aff*, the lattices* $\Pi^{\exists}_{\mathrm{aff}}$ *and* $\Pi^{\forall}_{\mathrm{aff}}$ *are comodernistic.*

Proof. As in the proof of Theorem 6.1, consider an interval $[\pi', \pi]$. Let some block of π split nontrivially into blocks $B_1, B_2, \dots, B_\ell$ of π'. As in Theorem 6.1, assume that the blocks are sorted by increasing size, and in particular that $|B_1| \leq |B_i|$ for all i.

Now, if there are multiple singleton blocks

$$B_1 = \{x_1\}, B_2 = \{x_2\}, \dots B_j = \{x_j\},$$

then sort these by affinity. In the case of $\Pi^{\exists}_{\mathrm{aff}}$, let $\operatorname{aff}(x_1) \geq \operatorname{aff}(x_2) \geq \dots$; while for $\Pi^{\forall}_{\mathrm{aff}}$ reverse to require $\operatorname{aff}(x_1) \leq \operatorname{aff}(x_2) \leq \dots$.

The remainder of the proof now goes through entirely similarly to that of Theorems 6.1 and 6.2. ∎

Theorem 6.3 has applications to lower bounds of the complexity of a certain computational problem, directly analogous to the work in [3, 4] with $\Pi_{n,k}$.

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