POSET CONVEX-EAR DECOMPOSITIONS AND APPLICATIONS TO THE FLAG H-VECTOR

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Possibly the most fundamental combinatorial invariant associated to a finite simplicial complex is its f-vector, the integral sequence expressing the number of faces of the complex in each dimension. The h-vector of a complex is obtained by applying a simple invertible transformation to its f-vector, and thus the two contain the same information. Because some properties of the f-vector are easier expressed after applying this transformation, the h-vector has been the subject of much study in geometric and algebraic combinatorics. A convex-ear decomposition, first introduced by Chari in [7], is a way of writing a simplicial complex as a union of subcomplexes of simplicial polytope boundaries. When a $(d-1)$-dimensional complex admits such a decomposition, its h-vector satisfies, for $i < d/2$, $h_i \leq h_{i+1}$ and $h_i \leq h_{d-i}$. Furthermore, its g-vector is an M-vector.

We give convex-ear decompositions for the order complexes of rank-selected subposets of supersolvable lattices with nowhere-zero Möbius functions, rank-selected subposets of geometric lattices, and rank-selected face posets of shellable complexes (when the rank-selection does not include the maximal rank). Using these decompositions, we are able to show inequalities for the flag h-vectors of supersolvable lattices and face posets of Cohen-Macaulay complexes.

Finally, we turn our attention to the h-vectors of lattice path matroids. A lattice path matroid is a certain type of transversal matroid whose bases correspond to planar lattice paths. We verify a conjecture of Stanley in the special case of lattice path matroids and, in doing so, introduce an interesting new class of monomial order ideals.

BIOGRAPHICAL SKETCH

Jay Schweig was born September 7th, 1978 in Arlington, Massachusetts. After much relocation, he and his family finally settled in the Washington, D.C. suburb of Herndon, Virginia. Jay attended Oakton High School, and later enrolled in nearby George Mason University as a philosophy major. Two years and many Aristotle treatises later, Jay switched to a mathematics concentration. After obtaining a bachelor's degree, Jay worked several odd jobs in the Northern Virginia area. He eventually entered the mathematics Ph.D. program at Cornell University in Ithaca, New York, initially as a topologist. Six short years later, his advisor suggested that he graduate. After graduation, Jay will be a Robert D. Adams visiting assistant professor at the University of Kansas.

To Mom & Sarah

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Preliminaries and Background

1.1 The h-Vector of a Finite Simplicial Complex

Let Δ be a $(d-1)$-dimensional finite simplicial complex (in fact, all simplicial complexes considered herein will be finite). One of the most basic invariants associated to Δ is its *f-vector*, the integral sequence that expresses that number of faces of Δ in each dimension.

Formally, the f-vector of Δ, written $f(\Delta)$, is the $(d+1)$-tuple $(f_0, f_1, f_2, \ldots, f_d)$, where f_i is the number of $(i-1)$-dimensional faces of Δ. By convention we set $f_0 = 1$ whenever Δ is a non-empty complex. (Some authors use f_i to denote the number of i-dimensional faces, causing the indices of their f-vectors to differ from ours.) A simplicial complex whose f-vector is $(1, 4, 4, 1)$ is shown in Figure 1.1.

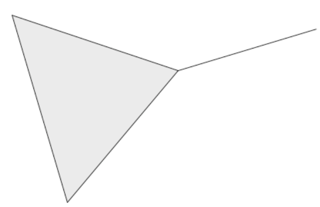

Figure 1.1: A simplicial complex with f-vector $(1, 4, 4, 1)$.

The *Kruskal-Katona-Schützenberger Theorem*, whose proof and statement we omit here, provides a classification of all possible simplicial complex f-vectors. The interested reader can refer to [23].

The *f-polynomial* of Δ, $f_\Delta(t)$, is $\sum_{i=0}^{d} f_i t^{d-i}$. Many of the results that follow concern the *h-vector* of a finite simplicial complex, defined as follows:

Definition 1.1.1 *The h-vector of Δ, $h(\Delta)$, is the $(d+1)$-tuple $(h_0, h_1, h_2, \ldots, h_d)$, where*

$f_\Delta(t-1) = h_0 t^d + h_1 t^{d-1} + h_2 t^{d-2} + \ldots + h_{d-1} t + h_d.$

The polynomial $f_\Delta(t-1)$ is called the *h-polynomial* of Δ, and is denoted $h_\Delta(t)$. A few facts about the h-vector are immediate: First, by substituting the value $t = 1$ into the h-polynomial, $h_0 + h_1 + \ldots + h_d = h_\Delta(1) = f_\Delta(0) = f_d$, and so $\sum h_i = f_d$, the number of full-dimensional faces of Δ. Next, substituting the value $t = 0$ into the h-polynomial yields $h_d = h_\Delta(0) = f_\Delta(-1) = (-1)^{d-1}(-f_0 + f_1 - f_2 + \ldots + (-1)^{d-1} f_d) = (-1)^{d-1}\tilde{\chi}(\Delta)$, where $\tilde{\chi}$ denotes the *reduced* Euler characteristic of Δ: $\tilde{\chi}(\Delta) = \chi(\Delta) - 1$. For example, if Δ is the complex in Figure 1.1, then $f_\Delta(t) = t^3 + 4t^2 + 4t + 1$, so $h_\Delta(t) = f_\Delta(t-1) = (t-1)^3 + 4(t-1)^2 + 4(t-1) + 1 = t^3 + t^2 - t$ and $h(\Delta) = (1, 1, -1, 0)$. Thus the h-vector of a complex is not necessarily nonnegative.

For a second example, take Δ to be the boundary of the octahedron. Then $f(\Delta) = (1, 6, 12, 8)$ and $h(\Delta) = (1, 3, 3, 1)$.

Upon first glance, one might wonder what purpose the h-vector could serve. After all, it holds the same information as the f-vector. However, it turns out that certain properties of a complex's f-vector are sometimes much better expressed through the associated h-vector. A shining example of this phenomenon are the Dehn-Sommerville relations (see, for instance, [28]):

Theorem 1.1.2 *Suppose Δ is the boundary complex of a simplicial d-polytope. Then the h-vector of Δ satisfies $h_i = h_{d-i}$ for $0 \le i \le d$.*

In fact, all possible h-vectors (and thus all possible f-vectors) of simplicial polytope boundaries have been characterized. To state the result, a few definitions are required.

Definition 1.1.3 *Let Γ be a finite set of monomials. Γ is an order ideal if $\alpha \in \Gamma$ whenever $\alpha | \alpha'$ for some $\alpha' \in \Gamma$.*

Any simplicial complex can be viewed as a squarefree monomial order ideal by mapping the face $\{v_{i_1}, v_{i_2}, \ldots, v_{i_n}\}$ to the monomial $x_{i_1}x_{i_2}\ldots x_{i_n}$. Thus the class of monomial order ideals properly contains all finite simplicial complexes.

Definition 1.1.4 *A finite sequence $(m_0, m_1, \ldots, m_d)$ is an M-vector (also called an O-sequence by some authors) if there exists a monomial order ideal Γ such that, for all i, m_i is the number of monomials of Γ of degree i.*

Definition 1.1.5 *Let Δ be a $(d-1)$-dimensional simplicial complex. The g-vector of Δ is the sequence $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$.*

The theorem classifying all simplicial polytope boundary h-vectors, known as the *g-Theorem*, can now be stated. The "only if" direction was proven by Stanley in [21], while the "if" direction was proven in [1] by Billera and Lee.

Theorem 1.1.6 *An integral sequence $(h_0, h_1, h_2, \ldots, h_d)$ is the h-vector of a simplicial d-polytope boundary if and only if:*

(i) $h_i = h_{d-i}$ *whenever* $0 \leq i \leq d$*, and*

(ii) *the associated g-vector is an M-vector.*

One may be tempted to ask if other simplicial complexes satisfy all or part of the above theorem. In fact, Swartz has shown that a large class of complexes have g-vectors that are M-vectors, namely all complexes admitting convex-ear decompositions, a concept introduced in the next section.

1.2 Shellability and Convex-Ear Decompositions

Let G be a finite connected graph without loops, and let $T \subseteq E(G)$ be a spanning tree of G. It is a basic result in topology that G is homotopy equivalent to a wedge of $|E(G) \setminus T|$-many circles. (Simply contract the spanning tree to see this: Every edge not in the spanning tree has its endpoints identified.) When a simplicial complex is *shellable*, a similar phenomenon takes place. Recall that a simplicial complex is called *pure* if all its facets (maximal faces) are of the same dimension. Most of the results in this section can be found in [4] or [3].

Definition 1.2.1 *A pure complex Δ is shellable if there exists an ordering of its facets, $F_1, F_2, \ldots, F_n$, such that $F_i \cap (\bigcup_{j=1}^{i-1} F_j)$ is a non-empty union of facets of ∂F_i whenever $1 < i \leq n$. If such a sequence of facets exists, it is called a shelling of Δ.*

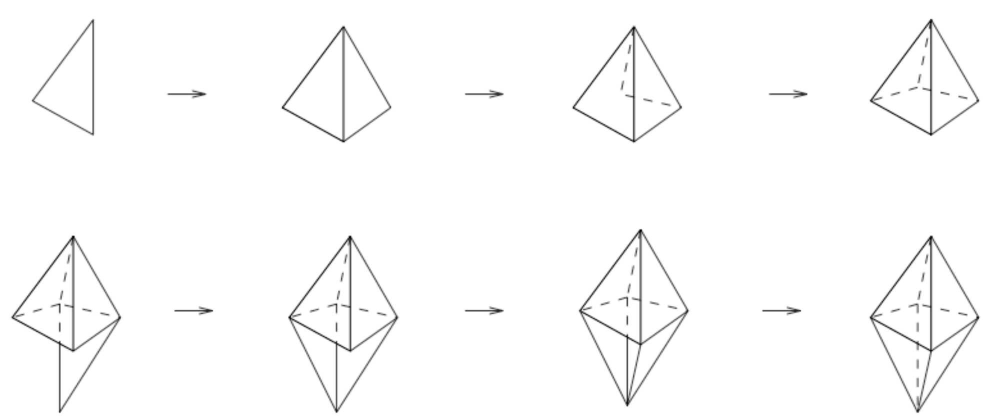

Figure 1.2: A shelling of the octahedron's boundary.

Although Björner and Wachs have defined a notion of shellability for non-pure simplicial complexes (see [5]), all shellable complexes considered herein will be pure. In practice, the following alternate definition is often used to show that a particular ordering of the facets of Δ is a shelling.

Proposition 1.2.2 *The facet ordering $F_1, F_2, \ldots, F_n$ is a shelling of Δ if and only if, for all j and k with $j < k$, there exists a $k' < k$ such that $|F_{k'} \cap F_k| = |F_k| - 1$ and $F_j \cap F_k \subseteq F_{k'} \cap F_k$.*

Now fix a facet ordering $F_1, F_2, \ldots, F_n$ of the complex Δ, and write Δ_i to denote the subcomplex of Δ generated by the first i facets of the ordering. The following is perhaps a more intuitive characterization of shellings:

Proposition 1.2.3 *The ordering $F_1, F_2, \ldots, F_n$ is a shelling of Δ if and only if, for all i with $2 \leq i \leq n$, the set of faces $\Delta_i \setminus \Delta_{i-1}$ contains a unique minimal element (with respect to inclusion).*

In practice, one usually thinks of $\Delta_i \setminus \Delta_{i-1}$ as the set of "new" faces obtained by adding F_i to Δ_{i-1}. When the facet ordering is a shelling, the unique minimal element of this set guaranteed by the previous proposition is called the *unique minimal new face* (or *u.m.n.f.*) associated to F_i and is written $r(F_i)$. The next few results illustrate the parallel between shellable complexes and connected graphs. Throughout, let Δ be a $(d-1)$-dimensional simplicial complex with shelling $F_1, F_2, \ldots, F_n$.

Proposition 1.2.4 *Let Δ' be the subcomplex of Δ generated by the set of facets $\{F_i : r(F_i) \neq F_i\}$. Then Δ' is contractible.*

The complex Δ' can be thought of as a higher-dimensional analogue of a connected graph's spanning tree. Now let F_i be a facet not included in Δ'. Then by definition $|r(F_i)| = d$, meaning that F_i is attached to $\bigcup_{j=1}^{i-1} F_j$ along its entire boundary. Thus, contracting Δ' yields the following:

Proposition 1.2.5 *The complex Δ is homotopy equivalent to a wedge of $|\{F_i : r(F_i) = F_i\}|$-many $(d-1)$-spheres.*

Thus a shellable complex is either contractible or homotopy equivalent to a bouquet of spheres. Shellings also tell us information about subcomplexes, as shown by the following theorem of Danaraj and Klee ([8]):

Theorem 1.2.6 *Let Σ be a pure, full-dimensional proper subcomplex of a d-sphere, and suppose that Σ is shellable. Then Σ is a d-ball.*

One benefit of a shelling is that it allows us to examine the change to a complex's f-vector at each step as follows: It is clear that $f_{\Delta_i}(t)$ counts all faces of Δ_{i-1}, plus the "new" faces obtained by adding F_i to Δ_{i-1}. Because the given facet ordering is a shelling, any new face must contain the face $r(F_i)$. Thus, attaching F_i to Δ_{i-1} contributes $(t+1)^{d-|r(F_i)|}$ to the f-polynomial of Δ_{i-1}, and $f_{\Delta_i}(t) = f_{\Delta_{i-1}}(t) + (t-1)^{d-|r(F_i)|}$.

Let Δ_0 be the empty complex, and set $f_{\Delta_0}(t) = 0$. Keeping track of the change to the f-polynomial during each step of the shelling of Δ, $f_\Delta(t) = \sum_{i=1}^{n}(t+1)^{d-|r(F_i)|}$. Because $h_\Delta(t+1) = f_\Delta(t) = \sum_{i=1}^{n}(t+1)^{d-|r(F_i)|}$, the h-vector of a shellable complex Δ has the following combinatorial interpretation:

Proposition 1.2.7 *Fix a shelling $F_1, F_2, \ldots, F_t$ of Δ. The h-vector of Δ is given by $h_i = |\{F_j : |r(F_j)| = i\}|$.*

While a shelling can be thought of piecing together a complex from its facets, the coarser concept of a *convex-ear decomposition* can be thought of as building a complex out of subcomplexes of simplicial polytope boundaries:

Definition 1.2.8 *A complex Δ has a convex-ear decomposition if there exist pure $(d-1)$-dimensional subcomplexes $\Sigma_1, \ldots \Sigma_t$ such that:*

(i) $\bigcup_{1=1}^{t} \Sigma_i = \Delta$.

(ii) Σ_1 *is the boundary complex of a simplicial d-polytope, and for $i > 1$ there exists a simplicial d-polytope Δ_i so that Σ_i is a pure, full-dimensional subcomplex of $\partial\Delta_i$.*

(iii) *For $i > 1$, Σ_i is a simplicial ball.*

(iv) *For $i > 1$, $(\bigcup_{j=1}^{i-1} \Sigma_j) \cap \Sigma_i = \partial\Sigma_i$.*

As an example, let Δ be the 2-dimensional simplicial complex with the vertex set $\{1, 2, 3, 4, 5, 6\}$ and facets $123, 124, 126, 134, 135, 145, 156, 234, 236, 345$, and 356. Let Σ_1 be the subcomplex with facets $123, 124, 134$, and 234, let Σ_2 be the subcomplex with facets $135, 145$, and 345, and let Σ_3 be the subcomplex with facets $126, 156, 236$, and 356. The sequence $\Sigma_1, \Sigma_2, \Sigma_3$ is a convex-ear decomposition of Δ. In Figures 1.3 and 1.4, Σ_2 is shown being attached to Σ_1, and then Σ_3 is shown being attached to $\Sigma_1 \cup \Sigma_2$.

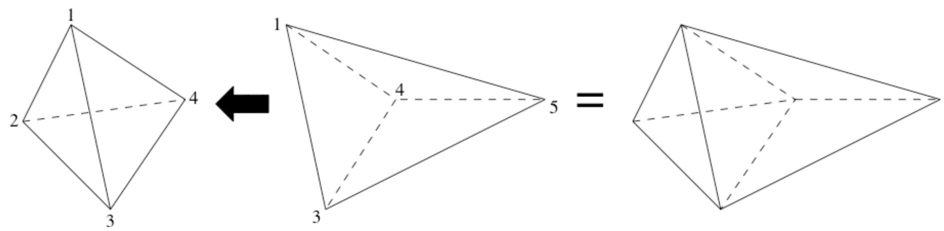

Figure 1.3: The first step in a convex-ear decomposition.

It is easy to verify this ordering is a convex-ear decomposition. The reader should note, however, that $\Sigma_1, \Sigma_3, \Sigma_2$ is *not* a convex-ear decomposition, as $\Sigma_3 \cap \Sigma_1 \neq \partial\Sigma_3$.

The following proposition is proven by a straightforward induction argument:

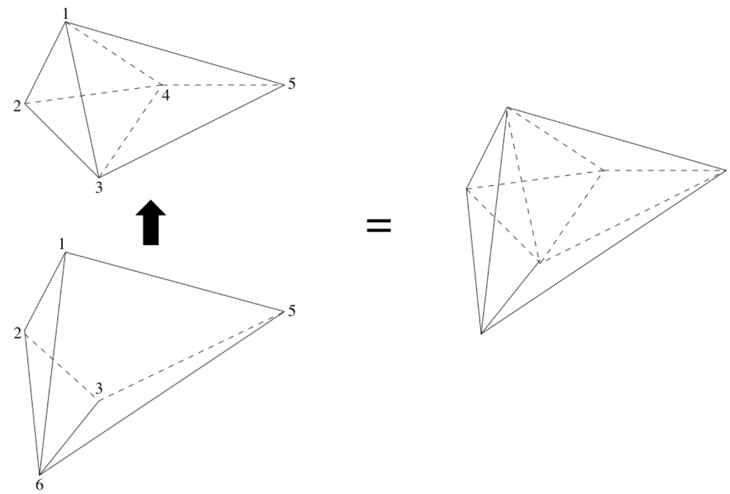

Figure 1.4: The second step in a convex-ear decomposition.

Proposition 1.2.9 *Let Δ be a $(d-1)$-dimensional complex, and let $\Sigma_1, \Sigma_2, \ldots, \Sigma_t$ be a convex-ear decomposition of Δ. Then Δ is homotopy equivalent to a wedge of t-many $(d-1)$-spheres.*

Convex-ear decompositions were first introduced by Chari in [7], where he proved the following:

Theorem 1.2.10 *Let Δ be a $(d-1)$-dimensional simplicial complex that admits a convex-ear decomposition. Then, for $i < d/2$, the h-vector of Δ satisfies:*

(i) $h_i \leq h_{d-i}$, *and*

(ii) $h_i \leq h_{i+1}$.

Thus, the h-vector of a complex admitting a convex-ear decomposition bears some resemblance to the h-vector of a simplicial polytope boundary. This similarity is deepened by the following result of Swartz, proven in [25]:

Theorem 1.2.11 *Let Δ be a $(d-1)$-dimensional complex admitting a convex-ear decomposition. Then the g-vector of Δ, $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$, is an M-vector.*

1.3 Poset Order Complexes and EL-Labelings

This first definition is an important one, ubiquitous throughout combinatorics:

Definition 1.3.1 *A partially ordered set, or poset, $(P, \leq)$ consists of a point set P and a relation $\leq$ satisfying:*

(i) *$x \leq x$ for all $x \in P$,*

(ii) *$x \leq y$ and $y \leq z \Rightarrow x \leq z$, and*

(iii) *$x \leq y$ and $y \leq x \Rightarrow x = y$.*

Although infinite posets have been the subject of much study, every poset considered herein will be finite.

For points x, y of a partially ordered set $(P, \leq)$, $x < y$ means $x \leq y$ and $x \neq y$. We say *y covers x* if $x < y$ and there is no $z \in P$ such that $x < z < y$. If $x \in P$ does not cover any element of P, it is called a *minimal element*. If it is covered by no element of P, it is called a *maximal element*. When P has a unique minimal element and/or a unique maximal element, these are sometimes referred to as $\hat{0}$ and $\hat{1}$, respectively. When the partial order in question is clear, sometimes we write P as short for $(P, \leq)$.

One poset that we will be studying a great deal is the *Boolean lattice*, defined as follows:

Definition 1.3.2 *For a positive integer d, the Boolean lattice B_d is the partially ordered set of subsets of* $[d] = \{1, 2, 3, \ldots, d\}$*, ordered by inclusion.*

To picture a partially ordered set P, one often uses its *Hasse diagram*. The Hasse diagram of P is the graph with vertex set $\{v_x : x \in P\}$ and edge set $\{(v_x, v_y) : y \text{ covers } x\}$, drawn so that v_x is lower than v_y whenever $x \leq y$. Figure 1.5 shows the Hasse diagram of B_3, the Boolean lattice on 3 elements:

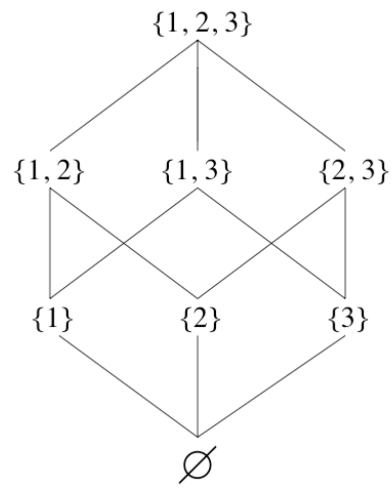

Figure 1.5: The Hasse diagram of B_3

If $x_1, x_2, \ldots, x_n$ are elements of a partially ordered set P and $x_1 < x_2 < \ldots < x_n$, we call this a *chain* in P. If $\{i_1, i_2, \ldots, i_m\} \subseteq [n]$, $i_1 < i_2 < \ldots < i_m$, and $m < n$, we say $x_{i_1} < x_{i_2} < \ldots < x_{i_m}$ is a *proper subchain* of our first chain. A chain that is not a proper subchain of any other chain is called *maximal*. If $x_1 < x_2 < \ldots < x_n$ is not a proper subchain of any chain starting at x_1 and ending at x_n, it is called *saturated*. For example, $\emptyset < \{1, 2\} < \{1, 2, 3\}$ is a chain in B_3 that is a proper subchain of two maximal chains: $\emptyset < \{1\} < \{1, 2\} < \{1, 2, 3\}$, and $\emptyset < \{2\} < \{1, 2\} < \{1, 2, 3\}$. It is neither a saturated chain nor a maximal one, although its subchain $\{1, 2\} < \{1, 2, 3\}$ is saturated.

Now suppose P is a poset with a $\hat{0}$ such any two of its maximal chains have the same number of elements, let $z \in P$, and let $\hat{0} < x_1 < x_2 < \ldots < x_m = z$ and $\hat{0} < y_1 < y_2 < \ldots < y_n = z$ be two saturated chains. It follows that $m = n$, and we define the *rank* of x, written $rank(x)$, to be this common integer. A poset whose maximal chains all have the same number of elements is called *ranked* or *graded*. The Boolean lattice B_d, for example, is a graded poset: For any $x \in B_d$, $rank(x) = |x|$.

When P is a graded poset of rank d (that is, every maximal chain of P contains $d + 1$ elements) with a $\hat{0}$ and a $\hat{1}$, several new posets can be constructed: For any $S \subseteq [d - 1]$, let P_S be the poset on the points $\{x \in P : rank(x) \in S\} \cup \{\hat{0}, \hat{1}\}$ with partial order inherited from P. Taking our running example B_3 and $S = \{1\}$, the poset $(B_3)_S$ is as shown in Figure 1.6.

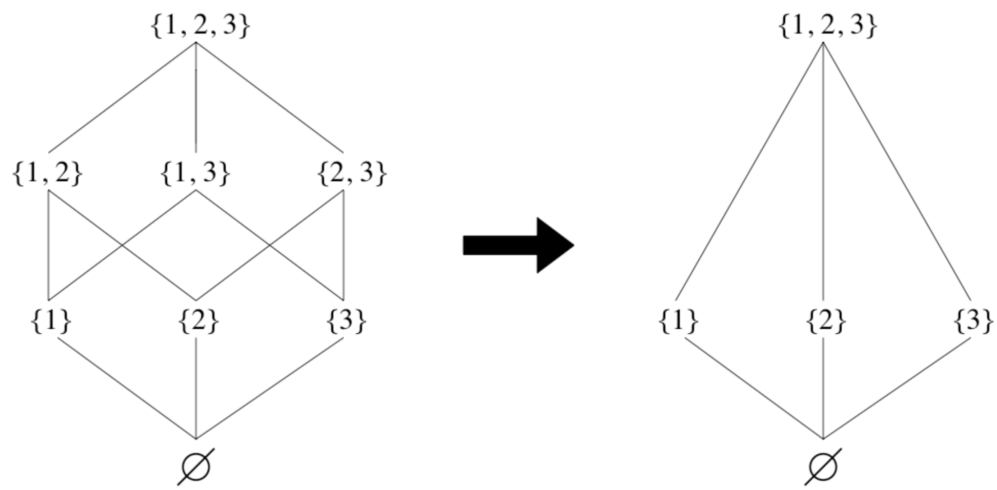

Figure 1.6: Moving from B_3 to $(B_3)_{\{1\}}$

For any subset $S \subseteq [d - 1]$ and any maximal chain $\mathbf{c}$ of P, let $\mathbf{c}_S$ denote the subchain of $\mathbf{c}$ consisting of all elements in $\mathbf{c}$ whose ranks are in $S \cup \{0, d\}$. In particular, we write $\mathbf{c}_j$ as shorthand for $\mathbf{c}_{\{j\}}$, the element of $\mathbf{c}$ of rank j with $\hat{0}$ and $\hat{1}$ adjoined. For any S, $\mathbf{c}_S$ is a maximal chain in P_S.

Implicit in any poset is a simplicial complex, known as its *order complex*:

Definition 1.3.3 *Let P be a poset. The order complex of P, written $\Delta(P)$, is the simplicial complex with vertex set $\{v_x : x \in P\}$ such that $\{v_{x_1}, v_{x_2}, \ldots, v_{x_n}\}$ is a face of $\Delta(P)$ if and only if $x_{\sigma(1)} < x_{\sigma(2)} < \ldots < x_{\sigma(n)}$ is a chain in P for some permutation $\sigma \in \mathcal{S}_n$.*

When a poset P has a $\hat{0}$ or a $\hat{1}$, we will often investigate the order complex of the *proper* part of P, namely $P \setminus \{\hat{0}, \hat{1}\}$, rather than the whole poset. The reason for this is simple: When P has a least or greatest element (or both), then $\Delta(P)$ is contractible, and therefore not very topologically interesting. If P has either a greatest or a least element, we write $\overline{P}$ to denote its proper part.

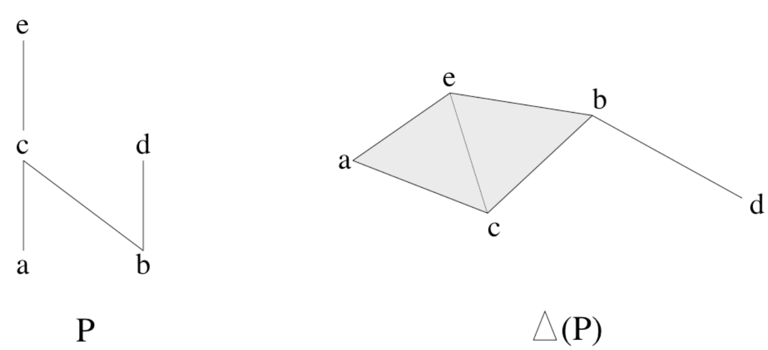

Figure 1.7: A poset and its order complex

Definition 1.3.4 *Given a simplicial complex Σ, the face poset of Σ, written P_Σ, is the poset of faces of Σ, ordered by inclusion.*

For example, if Σ is a single 2-dimensional simplex then $P_\Sigma = B_3$. The following proposition shows that no topological information about Σ is lost in passing to its face poset:

Proposition 1.3.5 *Let Σ be a simplicial complex. Then the order complex $\Delta(P_\Sigma \setminus \emptyset)$ is the first barycentric subdivision of Σ.*

Because facets of an order complex $\Delta(\overline{P})$ correspond to maximal chains in P, $\Delta(\overline{P})$ is pure if and only if P is graded. If P is graded, the next natural question to ask is whether $\Delta(\overline{P})$ is shellable.

Definition 1.3.6 *A labeling of a poset P is a function $\lambda : \{(x, y) \in P^2 : y \text{ covers } x\} \to \mathbb{Z}$. In other words, λ is a way of writing an integer on each edge of the Hasse diagram of P.*

Now let λ be a labeling of a poset P. If $x, y \in P$ and y covers x, write $\lambda(x, y)$ as short for $\lambda((x, y))$. If $\mathbf{c} := x_0 < x_1 < x_2 < \ldots < x_n$ is a saturated chain in P, we write $\lambda(\mathbf{c})$ to mean the word $\lambda(x_0, x_1)\lambda(x_1, x_2)\ldots\lambda(x_{n-1}, x_n)$. This word is called the *label* or *λ-label* of $\mathbf{c}$.

Definition 1.3.7 *Let P be a graded poset with a $\hat{0}$ and $\hat{1}$, and let λ be a labeling of P. We call λ an EL-labeling if:*

(i) *In each interval $[x, y]$ of P, there is a unique saturated chain with strictly increasing λ-label, and*

(ii) *The label of this chain is lexicographically first among all labels of saturated chains in $[x, y]$.*

Whenever we say that a poset P admits an EL-labeling, we will also assume that P is graded and has a $\hat{0}$ and $\hat{1}$. The following theorem, proven by Björner and Wachs in [4], provides the motivation for EL-labelings:

Theorem 1.3.8 *Let P be a poset admitting an EL-labeling λ. Then $\Delta(\overline{P})$ is shellable. Moreover, lexicographic order (with respect to their λ-labels) of the maximal chains of P yields a shelling of this complex.*

If $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ is a word of integers, recall that the *weak descent set* of σ is the set $\{i : \sigma(i) \geq \sigma(i+1)\}$. As the following shows, EL-labelings also provide information about an order complex's h-vector.

Proposition 1.3.9 *Let P be a poset admitting an EL-labeling λ. Then $h_i(\Delta(\overline{P}))$ is the number of maximal chains of P whose λ-labels' weak descent sets are of cardinality i.*

Let P be a poset. The *Möbius function* $\mu : P \times P \to \mathbb{Z}$ is recursively defined by $\mu(x, x) = 1$ for all $x \in P$, $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$, and $\mu(x, y) = 0$ if $x \nleq y$. The only property of the Möbius function used here is given by the following proposition:

Proposition 1.3.10 *Let P be a poset with an EL-labeling λ, and let $x, y \in P$ with $x < y$. Then $|\mu(x, y)|$ is equal to the number of saturated chains in $[x, y]$ with weakly decreasing λ-label.*

In particular, $|\mu(\hat{0}, \hat{1})|$ counts the number of maximal chains of P with non-increasing labels. We end this section with a few necessary definitions and an easy lemma which will prove useful.

Definition 1.3.11 *Let λ be an EL-labeling of a graded poset P, and let **c** be a non-maximal chain in P. The completion of **c**, written com(**c**), is the maximal chain in P that results from filling in each gap in **c** with the unique chain in that interval with an increasing λ-label.*

Notice that *com*(**c**) depends on the labeling λ, so we sometimes write $com_\lambda(\mathbf{c})$ to avoid ambiguity. The following helpful lemma follows immediately from the definition of an EL-labeling (we say that a subposet P' of a graded poset P is *full-rank* if it is graded and $rank(P') = rank(P)$):

Lemma 1.3.12 *Let P be as above, let P' be a full-rank subposet of P such that λ restricted to P' is an EL-labeling, and let $\mathbf{c}$ be a chain in P'. Then com($\mathbf{c}$) (as defined in P) is a maximal chain in P'.*

Finally, if $\mathbf{c}$ is a chain containing an element of rank j, we write $\mathbf{c}_{-j}$ to denote the chain that results from removing that element.

1.4 Matroids and Geometric Lattices

A *matroid* is a combinatorial axiomatization of linear independence, defined as follows:

Definition 1.4.1 *A matroid $M = M(E, \mathcal{I})$ is a finite set E along with a nonempty set $\mathcal{I}$ of subsets of E satisfying the following constraints:*

(i) *If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$, and*

(ii) *If $A, B \in \mathcal{I}$ and $|A| < |B|$, then there exists some $x \in B \setminus A$ so that $A \cup \{x\} \in \mathcal{I}$.*

The sets in $\mathcal{I}$ are called *independent sets*. If $E' \subseteq E$, a *basis* of E' is a set $B \in \mathcal{I}$ such that no $A \in \mathcal{I}$ satisfies $B \subsetneq A \subseteq E'$. It is easy to see that any two bases of E' must have the same cardinality. This cardinality is called the *rank* of E'. In particular, if $E' \in \mathcal{I}$, then $\mathit{rank}(E') = |E'|$. A basis of the matroid M is simply a basis of the ground set E.

For a matroid M with point set E, its *dual*, written M^*, is the matroid with ground set E and bases $E \setminus B$, where B is a basis of M.

Possibly the most intuitive example of a matroid is as follows: Let V be a vector space, and let E be a finite set of vectors in V. For a subset $A \subseteq E$, say that A is independent if and only if it is linearly independent in V. It is easy to see that this produces a matroid.

However, this is just one of the myriad settings in which matroids arise. For another, let $G = G(V, E)$ be a finite graph. The *graphic matroid* associated to G, written $M(G)$, is the matroid on the set E of edges of G, where $A \subseteq E$ is independent if and only if it contains no circuits. Under this correspondence, a set $B \subseteq E$ is a basis of $M(G)$ if and only if it is the set of edges of some maximal spanning forest. Matroids were first introduced by Hassler Whitney in [27]. The following proposition, whose proof is straightforward, was one of the motivations for such an object:

Proposition 1.4.2 *Let G be a planar graph, and let G^* be its planar dual. Then $M(G^*) = M^*(G)$*

Now let $M = M(E, \mathcal{I})$ be a matroid. A set $F \subseteq E$ is called a *flat* of M if $rank(F) < rank(F \cup \{x\})$ for any $x \in E \setminus F$. For instance, if M is a matroid specified by a set of vectors in a vector space V as described above, then $F \subseteq E$ is a flat if and only if it is the intersection of a subspace of V with E. That is, any vector in $E \setminus F$ lies outside of the subspace spanned by F.

The *lattice of flats* of a matroid M, written $\mathcal{L}(M)$, is the poset of all flats of M, ordered by inclusion.

Although geometric lattices are usually defined in poset-theoretic terms, the following is equivalent by a theorem of Birkhoff ([2]):

Definition 1.4.3 *Let L be a poset. Then L is a geometric lattice if and only if there exists a matroid M such that $L \simeq \mathcal{L}(M)$.*

For a matroid M with point set E, the *independence complex* of M, $\Delta(M)$, is the simplicial complex with vertex set E, where $A \subseteq E$ is a face of $\Delta(M)$ if and only it is an independent set of the matroid.

Matroid independence complexes were first shown to be shellable by Provan and Billera in [18], but here we use a technique first employed by Björner. Fix a total order $e_1 < e_2 < \ldots < e_n$ of the elements of E. For a basis B, let $\hat{B}$ denote the word of elements of B, written in increasing order.

Theorem 1.4.4 *(See, for instance, [3]) Let $\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_t$ be all bases of M, listed in lexicographic order. This ordering is a shelling of the independence complex $\Delta(M)$.*

Now let B_i be a basis in the above ordering. If $x \in B_i$ and $B_i \setminus \{x\}$ is not contained in any B_j for $j < i$, then x is called *internally active*. For a basis B, let $i(B)$ denote the set of internally active elements of B. The next proposition appears in [3]:

Proposition 1.4.5 *Let M be a rank r matroid, and let $B_1, B_2, \ldots, B_t$ be an ordering of the above type. Then the h-vector of $\Delta(M)$ is given by $h_i = |\{B_j : |i(B_j)| = r - i\}|$.*

A point $x \in E$ of a matroid is called a *coloop* if it is in every basis of the matroid. Note that if M has a coloop then $\Delta(M)$ is a cone, since every one if its facets must contain the vertex corresponding to the coloop. The first application of convex-ear decompositions was given by Chari in [7]:

Theorem 1.4.6 *The independence complex of a coloop-free matroid admits a convex-ear decomposition.*

Nyman and Swartz have also shown the following ([17]):

Theorem 1.4.7 *Let L be a geometric lattice. Then the order complex $\Delta(\overline{L})$ admits a convex-ear decomposition.*

We will generalize this theorem in Section 3.1 to rank-selected subposets of geometric lattices.

Now let M be a matroid with point set E, and fix a bijection $\omega : [n] \to E$. A *circuit* of M is a dependent set C such that $C \setminus \{e\}$ is independent for any $e \in C$. Given the above order, a *broken circuit* is a set of the form $C \setminus \{e\}$ where C is a circuit and e is the least element of C under the ordering ω.

The *broken-circuit complex* of M, $BC_\omega(M)$, is the simplicial complex of all subsets of M that do not contain a broken circuit. Facets of $BC_\omega(M)$ are called *nbc-bases*. It is easy to see that every nbc-basis of M must contain $\omega(1)$. For this reason, the complex $BC_\omega(M)$ is a cone with apex $\omega(1)$, and the *reduced* broken-circuit complex, $\overline{BC}_\omega(M)$ is defined to be $BC_\omega(M) \setminus \{\omega(1)\}$.

It is clear that the complex $BC_\omega(M)$ (and the corresponding reduced complex) depends on the ordering ω. The following can be found in [3]:

Proposition 1.4.8 *The f-vector (and therefore the h-vector) of the broken circuit complex $BC_\omega(M)$ is independent of the ordering ω.*

1.5 Cohen-Macaulay Complexes

For definitions of the algebraic objects mentioned in this section, see [23].

Let Δ be a $(d-1)$-dimensional simplicial complex with vertex set $\{v_1, v_2, \ldots, v_n\}$, and let k be a field. The *face ring* of Δ (sometimes called the *Stanley-Reisner ring* of Δ) is the quotient $k[x_1, x_2, \ldots, x_n]/I_\Delta$, where I_Δ is the ideal of the polynomial ring $k[x_1, x_2, \ldots, x_n]$ generated by all monomials of the form $x_{i_1}x_{i_2}\ldots x_{i_m}$ such that $\{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\}$ is *not* a face of Δ. We write $k[\Delta]$ to denote this ring.

For a monomial $\eta = x_1^{e_1}x_2^{e_2}\ldots x_m^{e_m}$, define $\lambda(\eta) = \lambda_1^{e_1}\lambda_2^{e_2}\ldots\lambda_m^{e_m}$. The *Hilbert series* of $k[\Delta]$ (under the fine grading) is $F(k[\Delta], \lambda) = \sum \lambda(\eta)$, where the sum is over all non-zero monomials of $k[\Delta]$. The following is shown in [23]:

Proposition 1.5.1 *For any simplicial complex Δ with vertex set $\{v_1, v_2, \ldots, v_n\}$,*

$$F(k[\Delta], \lambda) = \sum_{F\in\Delta}\prod_{v_i\in F}\frac{\lambda_i}{1-\lambda_i}$$

Definition 1.5.2 *A simplicial complex Δ is Cohen-Macaulay if its face ring $k[\Delta]$ is a Cohen-Macaulay ring.*

Recall that, for a face $F \in \Delta$, the *link* of F is the subcomplex of Δ given by $lk(F) = \{G \in \Delta : F \cup G \in \Delta \text{ and } F \cap G = \emptyset\}$ (so in particular $lk(\emptyset) = \Delta$). The following result, known as Reisner's Theorem ([19]), gives a topological characterization of Cohen-Macaulay complexes:

Theorem 1.5.3 *Let Δ be a simplicial complex. Then Δ is Cohen-Macaulay if and only if, for every face F of Δ, $\tilde{H}_i(lk(F)) = 0$ whenever $i < \dim(lk(F))$.*

Definition 1.5.4 *A Cohen-Macaulay complex Δ is q-Cohen-Macaulay (or q-CM) if the deletion of any set of $q-1$ vertices from Δ results in a Cohen-Macaulay complex of the same dimension as Δ.*

The following was shown by Swartz in [25]:

Theorem 1.5.5 *Let Δ be a complex that admits a convex-ear decomposition. Then Δ is* 2*-Cohen-Macaulay.*

Given Theorem 1.2.11, the above Theorem provides further motivation towards the following conjecture of Björner and Swartz, posed in [25]:

Conjecture 1.5.6 *Let Δ be a* 2*-Cohen-Macaulay complex. Then the g-vector of Δ is an M-vector.*

Chapter 2

A Convex-Ear Decomposition for Rank-Selected Supersolvable Lattices

The goal of this chapter is to prove the following theorem (necessary definitions are provided in the following section):

Theorem 2.0.7 *Let L be a supersolvable lattice of rank d with nowhere-zero Möbius function, and let $S \subseteq [d-1]$. Then the order complex $\Delta(\overline{L}_S)$ admits a convex-ear decomposition.*

Given the work of Chari and Swartz, the above yields:

Corollary 2.0.8 *Let L be as in the statement of the previous theorem. The h-vector of $\Delta(\overline{L})$ satisfies, for $i < |S|/2$,*

(i) $h_i \leq h_{|S|-i}$, *and*

(ii) $h_i \leq h_{i+1}$

Furthermore, the g-vector of Δ is an M-vector.

Our approach is a bit unorthodox: We first provide convex-ear decompositions for supersolvable lattices with nowhere-zero Möbius functions, then for rank-selected subposets of Boolean lattices, and finally for rank-selected subposets of supersolvable lattices with nowhere-zero Möbius functions. As the reader unfamiliar with these topics will soon see, the first two classes of posets are each special cases of the third. However, the first two decompositions will make it easy to state the third, while proving the third by itself would be quite cumbersome. The first decomposition, which is by far the simplest, will also serve as an easy introduction to convex-ear decompositions.

2.1 Preliminaries

We begin this chapter by showing that the order complex of the proper part of a supersolvable lattice with nowhere-zero Möbius function admits a convex ear decomposition. It should be noted that our search for such a decomposition was motivated by Welker's result that such complexes are 2-Cohen-Macaulay ([26]).

Let P be a poset with d-many elements. An *order completion* of P is a total ordering of its elements: $x_1 < x_2 < \ldots < x_d$ such that if $x_i < x_j$ in P then $i < j$. An *order ideal* of P is a subset $I \subseteq P$ such that if $y \in I$ and $x < y$, then $x \in I$. Let $\mathcal{I}(P)$ be the poset of order ideals of P ordered by inclusion.

The following definition is not the standard one, but is equivalent by the fundamental theorem of finite distributive lattices (see, for instance, [24, Theorem 3.4.1]):

Definition 2.1.1 *A finite lattice L is distributive if there exists a poset P such that L is isomorphic to $\mathcal{I}(P)$.*

All distributive lattices admit EL-labelings. To see this, let I and J be two order ideals of some d-element poset P and note that J covers I in $\mathcal{I}(P)$ if and only if $J = I \cup \{x\}$ for some minimal element x of $P \setminus I$. Thus there is a 1-1 correspondence between maximal chains in $\mathcal{I}(P)$ and order completions of P (and so $\mathcal{I}(P)$ is pure of rank d). Now let $x_1 < x_2 < \ldots < x_d$ be an order completion of P, and define the labeling λ

by $\lambda(I, J) = n$, where $J = I \cup \{x_n\}$. It is an easy exercise to show that λ is in fact an EL-labeling.

In fact, λ has an interesting extra property: Each maximal chain in $\mathcal{I}(P)$ is labeled with a permutation of $[d]$. This leads to the following definition:

Definition 2.1.2 *Let P be a graded poset of rank d, and let λ be an EL-labeling of P. We say that λ is an $\mathcal{S}_d$-EL-labeling if every maximal chain of P is labeled by an element of S_d (when viewed as a word on the alphabet $[d]$).*

Lemma 2.1.3 *Let L be a distributive lattice of rank d, and let P be the poset for which L is the lattice of order ideals. Then every $\mathcal{S}_d$-EL-labeling λ of L is obtained from P in the fashion described above. That is, for every $\mathcal{S}_d$-EL-labeling λ, there exists a bijection $\nu : P \to [d]$ such that $\lambda(I, J) = n$ if and only if $J = I \cup \nu^{-1}(n)$, where I and J are order ideals of P.*

Proof: The proof is by induction on d, the rank of L (and therefore also the number of elements in the poset P). If $d = 1$, then P has only one element x and L is the two element chain $\emptyset < \{x\}$, and the result is trivial. Suppose $d > 1$. Let $\hat{1}$ be the top element of L (that is, the order ideal consisting of the entire poset P), and let λ be an S_d-EL-labeling of L. There are two cases to consider: Either $\hat{1}$ covers just one element of L, or it covers more than one element. In the first case, P must have a unique maximal element x, and so $L \setminus \hat{1}$ is the lattice of order ideals for $P \setminus x$. Now λ restricted to $L \setminus \hat{1}$ is an $\mathcal{S}_{d-1}$-EL-labeling. By induction, there exists a bijection $\nu : P \setminus x \to [d-1]$ that gives rise to the labeling λ. Extending ν so that $\nu(x) = d$ completes the proof in this first case.

For the second case, let $J_1, J_2, \ldots, J_n$ be the maximal elements of $L \setminus \hat{1}$. Then each J_i, viewed as an order ideal, contains every element of P except some maximal one. Call this corresponding element x_i. λ restricted to each interval $[\hat{0}, J_i]$ is an S_{d-1}-EL-labeling (but with the alphabet $[d] \setminus \{\lambda(J_i, \hat{1})\}$ rather than $[d-1]$). By induction, λ restricted to each interval $[\hat{0}, J_i]$ corresponds to some labeling $\nu_i : P \setminus x_i \to [d] \setminus \lambda(J_i, \hat{1})$. It remains to be shown that these labelings agree (i.e., that they piece together to form a labeling $\nu : P \to [d]$). Let ν_i and ν_k be two labelings, and choose some x that is neither x_i nor x_k. Let I and J be two order ideals of P, with $J = I \cup \{x\}$. Since the chain $I < J$ is contained in both the intervals $[\hat{0}, J_i]$ and $[\hat{0}, J_k]$, $\lambda(I, J) = \nu_i(x) = \nu_k(x)$. □

We are now ready to define this chapter's basic object of study:

Definition 2.1.4 *Let L be a lattice. L is supersolvable if there exists a maximal chain $\boldsymbol{c}_M$ of L, called the M-chain, such that the sublattice of L generated by $\boldsymbol{c}_M$ and any other (not necessarily maximal) chain of L is a distributive lattice.*

Supersolvable lattices were first introduced by Stanley in [20]. Recall that a finite group G is *solvable* if there exists a chain of subgroups $\{1\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_n = G$ such that, for all i, G_i is normal in G_{i+1} and G_{i+1}/G_i is abelian. A group is *supersolvable* if, for all i, G_i is normal in G and the quotient group G_{i+1}/G_i is cyclic. Supersolvable lattices are so named because the subgroup lattice of a supersolvable group is such a lattice ([20]).

The next result shows that supersolvable lattices are interesting from a purely combinatorial perspective:

Theorem 2.1.5 *(McNamara, [15]) Let P be a poset of rank d. Then P is a supersolvable lattice if and only if it admits an $\mathcal{S}_d$-EL-labeling.*

Given the $\mathcal{S}_d$-EL-labeling constructed earlier for distributive lattices, Theorem 2.1.5

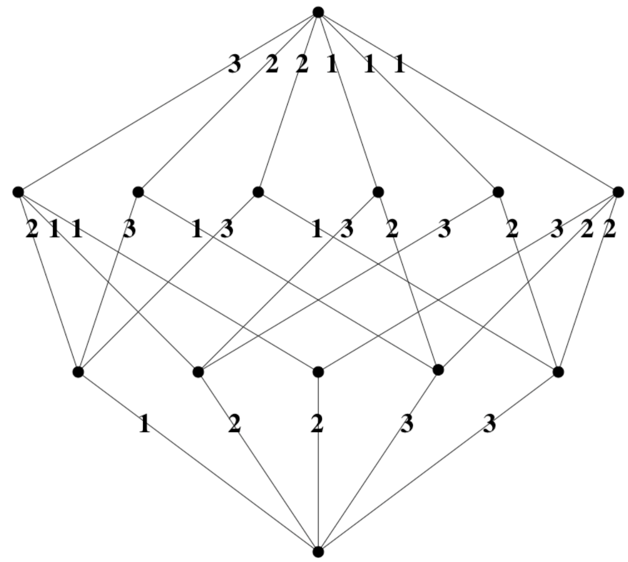

Figure 2.1: A rank 3 supersolvable lattice with an $\mathcal{S}_3$-EL-labeling.

shows that any distributive lattice is also supersolvable. We will also need the following theorem of Stanley, implicitly shown in [20]:

Theorem 2.1.6 *Let L be a rank d supersolvable lattice with S_d-EL-labeling λ and M-chain $\boldsymbol{c}_M$, let $\boldsymbol{d}$ be a chain in L, and let L' be the (distributive) sublattice of L generated by $\boldsymbol{c}_M$ and $\boldsymbol{d}$. Then λ restricted to L' is an $\mathcal{S}_d$-EL-labeling.*

Also in [20], Stanley proves that, under an $\mathcal{S}_d$-EL-labeling of a supersolvable lattice L, the unique maximal chain with increasing label is an M-chain.

The Boolean lattice B_d is an example of a supersolvable lattice. To construct an $\mathcal{S}_d$-EL-labeling of B_d, simply let $\lambda(x, y) = n$, where $y = x \cup \{n\}$. It is easily seen that λ is such a labeling. In fact, B_d is a distributive lattice: It is the lattice of order ideals for the d-element poset in which *every* set of elements is an order ideal, and thus no two elements are comparable.

In the following sections we will need the concept of *lexicographic order*, defined as follows:

Definition 2.1.7 *Let $\sigma = \sigma(1)\sigma(2)\ldots\sigma(d)$ and $\tau = \tau(1)\tau(2)\ldots\tau(d)$ be words of integers. The word σ is lexicographically less than τ, written $\sigma <_{lex} \tau$, if there exists an i such that $\sigma(j) = \tau(j)$ for $j < i$ and $\sigma(i) < \tau(i)$.*

Lexicographic order is a total order. That is, for two distinct words of integers σ and τ, each of length d, either $\sigma <_{lex} \tau$ or $\tau <_{lex} \sigma$. We say that a sequence $\sigma_1, \sigma_2, \ldots, \sigma_n$ of words of integers of length d is in *reverse* lexicographic order if $\sigma_j <_{lex} \sigma_i$ whenever $i < j$. (This differs from many authors' definition of reverse lexicographic order.)

For $\sigma \in \mathcal{S}_d$ and $i < j$, write $\sigma(i, j)$ to mean the set $\{\sigma(i), \sigma(i+1), \ldots, \sigma(j)\}$. We close this section with a helpful Lemma:

Lemma 2.1.8 *If $\sigma, \tau \in \mathcal{S}_d$ and $\sigma <_{lex} \tau$, there exists an $m \in [d-1]$ such that $\sigma(m) < \sigma(m+1)$ and $\sigma(1, m) \neq \tau(1, m)$.*

Proof: Let $i_1 < i_2, < \ldots < i_n$ be all integers in $[d]$ satisfying $\sigma(1, i_k) = \tau(1, i_k)$ (so, in particular, $i_n = d$). Then for all k with $1 \leq k < n$, $\sigma(i_k + 1, i_{k+1}) = \sigma(1, i_{k+1}) \setminus \sigma(1, i_k) = \tau(1, i_{k+1}) \setminus \tau(1, i_k) = \tau(i_k + 1, i_{k+1})$. It is clear that the lexicographically last permutation $\delta \in \mathcal{S}_d$ with $\delta(i_k + 1, i_{k+1}) = \tau(i_k + 1, i_{k+1})$ for all k is the permutation satisfying $\delta(i_k + 1) > \delta(i_k + 2) > \ldots > \delta(i_{k+1})$. Because $\sigma <_{lex} \tau$, it cannot be the case that $\sigma = \delta$, and there must be some $m \in [d-1] \setminus \{i_1, i_2, \ldots, i_n\}$ with $\sigma(m) < \sigma(m+1)$. Since m is not equal to any i_k, $\sigma(1, m) \neq \tau(1, m)$. $\square$

2.2 The Supersolvable Lattice

Let P be a poset. Throughout, we say that P has a *nowhere-zero Möbius function* if $\mu(x, y) \neq 0$ whenever $x, y \in P$ and $x \leq y$. The main result in this section is the following:

Theorem 2.2.1 *Let L be a supersolvable lattice of rank d with nowhere-zero Möbius function. Then $\Delta(\overline{L})$ admits a convex-ear decomposition.*

For the rest of this section, let L be a supersolvable lattice of the above type and let λ be an $\mathcal{S}_d$-EL-labeling as guaranteed by Theorem 2.1.5. To construct the ears of the decomposition, let $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_t$ be all maximal chains of L with decreasing labels. (The order of the list is arbitrary, but fixed from here on.) This list is non-empty, since $\mu(\hat{0}, \hat{1}) \neq 0$. For each i, let L_i be the sublattice of L generated by $\mathbf{d}_i$ and $\mathbf{c}_M$, and let Σ_i be the simplicial complex whose facets are given by maximal chains in $\overline{L}_i$ that are not chains in L_j for any $j < i$. We let the Σ_i's do double-duty, simultaneously representing the complex mentioned above and the set of (not necessarily maximal) chains in L that correspond to faces of that complex. Given the order below, it is sometimes helpful to think of maximal chains (i.e., facets) of Σ_i as "new," and maximal chains of $\overline{L}_i$ that are not in Σ_i as "old."

We claim that $\Sigma_1, \Sigma_2, \ldots, \Sigma_t$ is a convex-ear decomposition of $\Delta(L)$. As this will be our easiest convex-ear decomposition, we give a somewhat pedantic treatment here in order to acclimate the reader to the process.

Proof of property (ii): By definition, each L_i is a distributive lattice. Fix i, and let P be the poset such that $\mathcal{I}(P) \simeq L_i$. By Theorem 2.1.6 and Lemma 2.1.3, the chain $\mathbf{c}_M$ in L_i gives us an order completion of P: $x_1 < x_2 < \ldots < x_d$. Similarly, the chain $\mathbf{d}_i$ gives another order completion of P: $x_d < x_{d-1} < \ldots < x_1$. So for any $x_j, x_k \in P$, one of the above order completions gives $x_j < x_k$, while the other gives $x_k < x_j$. Thus no two elements in P are comparable, and any subset of elements is an order ideal of P. So L_i is isomorphic to B_d, the Boolean lattice on d elements. Since the order complex of B_d is the first barycentric subdivision of the boundary of the d-dimensional simplex, and since $\Sigma_1 = L_1$ and $\Sigma_i \subsetneq L_i$ for $i > 1$ (because $\mathbf{c}_M$ is in every L_i), this completes our proof of property (ii) of the decomposition. □

In Figure 2.2 we show one of the subposets L_i of the supersolvable lattice pictured in Figure 2.1.

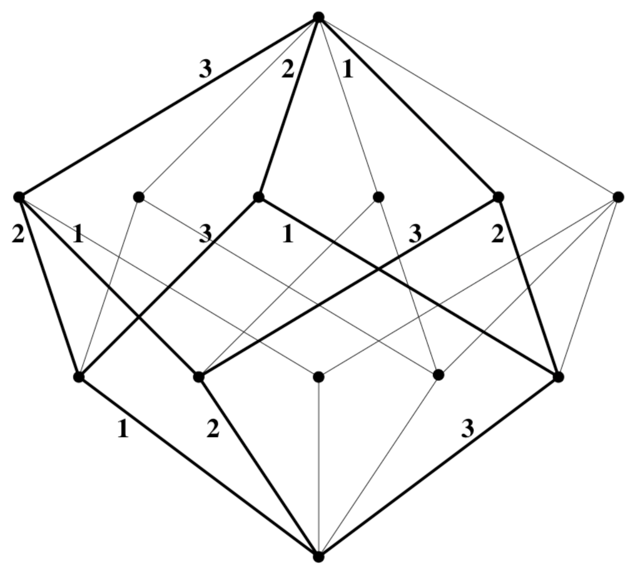

Figure 2.2: One of the subposets $L_i \simeq B_3$.

Proof of property (i): Let $\mathbf{c} := \hat{0} = x_0 < x_1 < \ldots < x_d = \hat{1}$ be a maximal chain of L. We must show that $\mathbf{c}$ is a chain in L_i for some i, and we do this by induction on the number of ascents of the λ-label of $\mathbf{c}$. If $\lambda(\mathbf{c})$ has no ascents, then $\mathbf{c} = \mathbf{d}_i$ for some i, and is a chain in L_i, by definition. Otherwise, $\mathbf{c}$ has at least one ascent, say at position j. Since L has nowhere-zero Möbius function, the interval (x_{j-1}, x_{j+1}) has at least one element other than x_j. Let $\mathbf{c}'$ be the chain that results from replacing x_j in $\mathbf{c}$ with one of these other elements, and note that $\lambda(\mathbf{c}')$ now has a descent in its jth position. Since

$\mathbf{c}'$ has one fewer ascent than $\mathbf{c}$, it belongs to some L_i by induction. Because λ is an EL-labeling on L_i (Theorem 2.1.6), $com((\mathbf{c}')_{-j}) = \mathbf{c}$ is a chain in L_i by Lemma 1.3.12. □

Proof of property (iii): To prove that Σ_i is a ball for all $i > 2$, we show that reverse lexicographic order of the maximal chains in Σ_i is a shelling. Applying Theorem 1.2.6 will then complete the proof. Let $\mathbf{c} := \hat{0} = x_0 < x_1 < \ldots < x_d = \hat{1}$ and $\mathbf{e}$ be two chains in Σ_i, with $\mathbf{e}$ lexicographically later (and therefore earlier in the shelling) than $\mathbf{c}$. Because $\lambda(\mathbf{c}) <_{lex} \lambda(\mathbf{e})$, Lemma 2.1.8 guarantees an $m \in [d-1]$ such that $\lambda(\mathbf{c})$ has an ascent in the mth position. That is, $\lambda(x_{m-1}, x_m) < \lambda(x_m, x_{m+1})$. Since $\lambda(\mathbf{c})(1, m) \neq \lambda(\mathbf{e})(1, m)$, $\mathbf{e}_m \neq x_m$.

Now let $\mathbf{c}'$ be the unique maximal chain of L_i that coincides with $\mathbf{c}$ everywhere but the mth position. Then, by definition of an EL-labeling, $\mathbf{c}'$ is lexicographically later than $\mathbf{c}$ (and thus earlier in the shelling), $|\mathbf{c} \setminus \mathbf{c}'| = 1$, and $\mathbf{c} \cap \mathbf{e} \subseteq \mathbf{c}'$. It remains to be shown that $\mathbf{c}'$ is in Σ_i. If $\mathbf{c}'$ were not a chain in Σ_i, it would be a chain in L_k for some $k < i$, meaning $(\mathbf{c}')_{-m}$ is a chain in L_k. But then, again by Lemma 1.3.12, $com((\mathbf{c}')_{-m}) = \mathbf{c}$ is a chain in L_k. This would imply that $\mathbf{c}$ is not a chain in Σ_i, which is a contradiction. □

Property (iv) remains to be proven. Because we will use the same method to prove this property for later decompositions, we outline the method in full here in order to refer back to it later.

Proof of property (iv): Fix $i > 1$, and note that a chain $\mathbf{c}$ in Σ_i is in $\partial\Sigma_i$ if and only if there exist two maximal chains containing it, $\mathbf{c}_{old}$ and $\mathbf{c}_{new}$, such that $\mathbf{c}_{old}$ is a maximal chain of L_i but not Σ_i, and $\mathbf{c}_{new}$ is a maximal chain in Σ_i.

From the above description of chains in the boundary of Σ_i, $\partial\Sigma_i \subseteq (\bigcup_1^{i-1} \Sigma_j) \cap \Sigma_i$. To see the reverse inclusion, let $\mathbf{c}$ be a chain in $(\bigcup_1^{i-1} \Sigma_j) \cap \Sigma_i$. Then $\mathbf{c}$ is, by definition, a subchain of some facet of Σ_i. This chain is the required $\mathbf{c}_{new}$. To complete the proof, we must find a suitable $\mathbf{c}_{old}$. However, since $\mathbf{c}$ is a chain in $\bigcup_1^{i-1} \Sigma_j$, it must be a chain in some

L_j for $j < i$. Then Lemma 1.3.12 guarantees that $com(\mathbf{c})$ is in L_j, so set $\mathbf{c}_{old} = com(\mathbf{c})$. □

Our next step is to show that rank-selected subposets of Boolean lattices admit convex-ear decompositions. Because each L_i is a Boolean lattice, we will then be able to use this decomposition to obtain this chapter's main result.

2.3 Rank-Selected Boolean Lattices

Recall that B_d denotes the rank d Boolean lattice, the poset of all subsets of $[d]$ ordered by inclusion. This section is devoted to proving the following:

Theorem 2.3.1 *For any subset $S \subseteq [d-1]$, the order complex $\Delta((\overline{B}_d)_S)$ admits a convex-ear decomposition.*

We fix λ to be the "natural" $\mathcal{S}_d$-EL-labeling of B_d, defined as follows: If $x, y \in B_d$ and y covers x then $\lambda(x, y) = m$, where $y = x \cup \{m\}$ when x and y are viewed as subsets of $[d]$. It is clear that λ is an $\mathcal{S}_d$-EL-labeling.

Now fix a subset $S \subseteq [d-1]$ for the remainder of this section, and write S as a disjoint union of intervals, where $a_1 < a_2 < \ldots < a_s$:

$$S = [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_s, b_s]$$

Let the above union be such that no $a_i - 1$ or $b_i + 1$ is a member of S and $b_i < a_{i+1}$ for all i. Where appropriate, we also set $b_0 = 0$ and $a_{s+1} = d$.

Because maximal chains in B_d, under their λ-labels, are in bijection with permutations of $[d]$, we do much of our work in the context of $\mathcal{S}_d$, where we write permutations in word form: $\sigma = \sigma(1)\sigma(2)\ldots\sigma(d)$. When $1 \leq m < n \leq d$, we write $\sigma(i, j)$ to mean the set $\{\sigma(i), \sigma(i+1), \ldots, \sigma(j)\}$.

Let $\mathbf{c}$ be a maximal chain in B_d with $\lambda(\mathbf{c}) = \sigma \in \mathcal{S}_d$. We wish to characterize the labels of all chains that coincide with $\mathbf{c}$ at ranks in S. This will turn out to be the coincidence set $C(\sigma)$ described below. Similarly, the set $Sp(\sigma)$ defined below is the set of labels of chains that coincide with $\mathbf{c}$ at ranks *not* in S.

For a permutation $\sigma \in \mathcal{S}_d$, define the *coincidence set* of σ, written $C(\sigma)$, as the set of all $\tau \in \mathcal{S}_d$ such that $\tau(m) = \sigma(m)$ for all $m \in S \setminus \{a_1, a_2, \ldots, a_s\}$ and $\sigma(b_i + 1, a_{i+1}) = \tau(b_i + 1, a_{i+1})$ for all i. To visualize the set $C(\sigma)$, define the bracketed word σ^C to be the word of σ with a left bracket inserted before each $\sigma(b_i + 1)$ and a right bracket inserted after each $\sigma(a_i)$ (as usual, we let $b_0 = 0$ and $a_{s+1} = d$). Then $C(\sigma)$ is the set of permutations that can be obtained by permuting the elements between the brackets of σ^C.

For example, suppose $d = 7$, $S = \{2, 3, 4, 6\}$, and $\sigma = 5\,3\,7\,4\,1\,6\,2$. Then $S = [2, 4] \cup [6, 6]$, and the bracketed word defined above is:

$$\sigma^C = [\,5\,3\,]\,7\,4\,[\,1\,6\,]\,[\,2\,]$$

Thus the set $C(\sigma)$ consists of four permutations: $3\,5\,7\,4\,1\,6\,2$, $3\,5\,7\,4\,6\,1\,2$, $5\,3\,7\,4\,1\,6\,2 = \sigma$, and $5\,3\,7\,4\,6\,1\,2$.

Now define the *span* of σ, written $Sp(\sigma)$, to be the set of all permutations $\tau \in \mathcal{S}_d$ such that $\tau(m) = \sigma(m)$ whenever $b_i + 1 < m < a_i$ for some i, and $\tau(a_i, b_i + 1) = \sigma(a_i, b_i + 1)$ for all i. Here, we do not follow our convention that $b_0 = 0$ and $a_{s+1} = d$. As before, define a bracketed word σ^{Sp} as follows: insert a left bracket before each $\sigma(a_i)$ and a right bracket after each $\sigma(b_i + 1)$. Then $Sp(\sigma)$ consists of all permutations obtained from σ

by permuting the elements between the brackets of σ^{Sp}. Continuing with our example,

$$\sigma^{Sp} = 5\,[\,3\,7\,4\,1\,]\,[\,6\,2\,]$$

Thus a permutation in $Sp(\sigma)$ is given by permuting the set $\{1, 3, 4, 7\}$ within the first bracket and the set $\{2, 6\}$ within the second. (When no confusion can result, we say 'bracket' to mean the word specified by a pair of brackets.)

Note that the above definitions of $C(\sigma)$ and $Sp(\sigma)$ depend on our choice of the set $S \subseteq [d-1]$. However, as we have fixed one choice of S for the entire section, we suppress 'S' from our notation. Given the bracket interpretations of the sets $C(\sigma)$ and $Sp(\sigma)$, the following lemma is obvious:

Lemma 2.3.2 *Fix two permutations $\sigma, \tau \in \mathcal{S}_d$. Then $\sigma \in C(\tau)$ if and only if $C(\sigma) = C(\tau)$, and $\sigma \in Sp(\tau)$ if and only if $Sp(\sigma) = Sp(\tau)$.*

For a permutation $\sigma \in \mathcal{S}_d$, let $\mathbf{c}^\sigma$ denote the unique maximal chain in B_d with σ as its λ-label. That is,

$$\mathbf{c}^\sigma := \hat{0} = x_0 < x_1 < \ldots < x_{d-1} < x_d = \hat{1}$$

and $\sigma(m) = \lambda(x_{m-1}, x_m)$ for all m. For a subset $T \subseteq [d-1]$, we write $\mathbf{c}^\sigma_T$ as shorthand for $(\mathbf{c}^\sigma)_T$. The following is our reason for introducing the sets $C(\sigma)$ and $Sp(\sigma)$:

Lemma 2.3.3 *Let $\sigma, \tau \in \mathcal{S}_d$. Then $C(\sigma) = C(\tau)$ if and only if $\mathbf{c}^\sigma_S = \mathbf{c}^\tau_S$, and $Sp(\sigma) = Sp(\tau)$ if and only if $\mathbf{c}^\sigma_{[d-1]\setminus S} = \mathbf{c}^\tau_{[d-1]\setminus S}$.*

Proof: Viewing elements of B_d as subsets of $[d]$, the result easily follows. $\square$

In Figure 2.3, we show (between the chain with increasing label and the chain with decreasing label) the four maximal chains in B_7 whose labels are permutations in $C(\sigma)$, where σ and S are as in our running example (elements whose ranks are in $S \cup \{0,7\}$ are filled in).

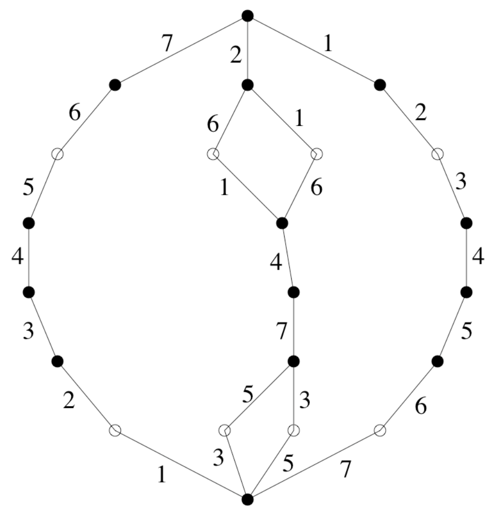

Figure 2.3: Maximal chains whose labels are in $C(\sigma)$.

Let P be any graded poset of rank d that admits an EL-labeling. Then the order complex of P_S is shellable and homotopy equivalent to t-many spheres (see [4]), where t is the number of maximal chains of P whose labels have weak descent set S. In the case we treat, where $P = B_d$, each label is a permutation and so t is the number of permutations in $\mathcal{S}_d$ with descent set S. It makes sense, then, that our convex-ear decomposition is constructed from the set $D = \{\delta \in \mathcal{S}_d : D(\delta) = S\}$.

For any $\sigma \in \mathcal{S}_d$, define a permutation δ_σ as follows: first, let π_σ be the permutation obtained by replacing each bracket in σ^C with the increasing word in those letters. In keeping with our running example,

$$\pi_\sigma^C = [\,3\,5\,]\,7\,4\,[\,1\,6\,]\,[\,2\,]$$

where we have written π_σ^C rather than just π_σ in hopes of better readability. Next, obtain δ_σ by replacing the contents of each bracket in π_σ^{Sp} with the decreasing word in those letters. Continuing with our example, $\pi_\sigma^{Sp} = 3\,[\,5\,7\,4\,1\,]\,[\,6\,2\,]$, and so $\delta_\sigma^{Sp} =$

3 [7 5 4 1] [6 2]

Note that, by construction, π_σ is in both $C(\sigma)$ and $Sp(\delta_\sigma)$, and so $C(\sigma) \cap Sp(\delta_\sigma) \neq \emptyset$.

Proposition 2.3.4 *For any $\sigma \in \mathcal{S}_d$, $\delta_\sigma \in D$.*

Proof: Let $n \in S$. Then $\delta_\sigma(n)$ and $\delta_\sigma(n+1)$ are in the same bracket of δ_σ^{Sp}. Because δ_σ is obtained from π_σ by putting the contents of each bracket of π_σ^{Sp} in decreasing order, it must be the case that $\delta_\sigma(n) > \delta_\sigma(n+1)$. Thus $S \subseteq D(\delta_\sigma)$. Suppose $S \neq D(\delta_\sigma)$, and choose some $m \in D(\delta_\sigma) \setminus S$. Then $m = a_j - 1$ or $m = b_j + 1$ for some j. Suppose $m = a_j - 1$. Because $\pi_\sigma(a_j - 1)$ is in the same bracket of π_σ^C as $\pi_\sigma(a_j)$, $\pi_\sigma(a_j - 1) < \pi_\sigma(a_j)$. Furthermore, $\pi_\sigma(a_j)$ is the leftmost element of some bracket of π_σ^{Sp}, and so by construction $\delta_\sigma(a_j) \geq \pi_\sigma(a_j)$. Similarly, $\pi_\sigma(a_j - 1)$ is either not in any bracket of π_σ^{Sp} or is the rightmost element in some bracket, so $\delta_\sigma(a_j - 1) \leq \pi_\sigma(a_j - 1)$. Stringing these inequalities together,

$$\delta_\sigma(m) = \delta_\sigma(a_j - 1) \leq \pi_\sigma(a_j - 1) < \pi_\sigma(a_j) \leq \delta_\sigma(a_j) = \delta_\sigma(m+1),$$

which is a contradiction. The proof for the case in which $m = b_j + 1$ for some j is symmetric. Thus $D(\delta_\sigma) = S$, so $\delta_\sigma \in D$. □

Now choose $\sigma, \delta, \tau \in \mathcal{S}_d$, with $\tau \in C(\sigma) \cap Sp(\delta)$. By Lemma 2.3.3, $\mathbf{c}_S^\tau = \mathbf{c}_S^\sigma$ and $\mathbf{c}_{[d-1]\setminus S}^\tau = \mathbf{c}_{[d-1]\setminus S}^\delta$. Because only one maximal chain in B_d can satisfy both these constraints, it follows that the permutation τ is uniquely determined. Thus for any $\sigma, \delta \in \mathcal{S}_d$, $|C(\sigma) \cap Sp(\delta)| \leq 1$.

Lemma 2.3.5 *Let $\sigma \in \mathcal{S}_d$ and $\delta \in D$, and suppose that $C(\sigma) \cap Sp(\delta) = \{\tau\}$. Then $\delta = \delta_\sigma$ if and only if the contents of each bracket of τ^C is increasing.*

Proof: Suppose each bracket of τ^C is increasing. As $\tau \in C(\sigma)$, it follows that $\tau = \pi_\sigma$, as defined in the proof of Proposition 2.3.4. Since δ_σ is obtained by permuting elements in the brackets of $\pi_\sigma^{Sp} = \tau^{Sp}$, $\tau \in Sp(\delta_\sigma)$. By assumption, $\tau \in Sp(\delta)$, and so by Lemma 2.3.2 $Sp(\delta_\sigma) = Sp(\delta)$. Because both δ and δ_σ are members of D, each bracket of δ^{Sp} and δ_σ^{Sp} must be decreasing, so $\delta = \delta_\sigma$.

Now suppose some bracket of τ^C is non-increasing. Put another way, the word $\tau(b_j + 1)\tau(b_j + 2)\ldots\tau(a_{j+1})$ is non-increasing for some j. Choose an m with $b_j + 1 \leq m \leq a_{j+1} - 1$ and $\tau(m) > \tau(m + 1)$. If it were the case that $b_j + 1 < m < a_{j+1} - 1$, then we would necessarily have $\delta(m) = \tau(m)$ and $\delta(m + 1) = \tau(m + 1)$, since both entries are outside the brackets of δ^{Sp} and $\tau \in Sp(\delta)$. But then $m \in des(\delta) = S$, a contradiction. Therefore, either $m = b_j + 1$ or $m = a_{j+1} - 1$. We treat only the first case, the proof of the second being similar.

Note that $\tau \in C(\sigma) = C(\pi_\sigma)$, and so $\pi_\sigma = \pi_\tau$. Because π_τ is obtained by putting the brackets of τ^C in increasing order, $\tau(b_j + 1) > \tau(b_j + 2)$ and so $\pi_\tau(b_j + 1) < \tau(b_j + 1)$. It follows that $Sp(\pi_\tau) \neq Sp(\tau)$. Putting this together,

$$Sp(\delta_\sigma) = Sp(\pi_\sigma) = Sp(\pi_\tau) \neq Sp(\tau) = Sp(\delta)$$

and so $\delta \neq \delta_\sigma$. □

Proposition 2.3.6 *For $\sigma \in \mathcal{S}_r$, δ_σ is the lexicographically least permutation in the set* $\{\delta \in D : C(\sigma) \cap Sp(\delta) \neq \emptyset\}$.

Proof: Fix $\delta \in D \setminus \{\delta_\sigma\}$ such that $C(\sigma) \cap Sp(\delta) = \{\tau\}$ for some $\tau \in \mathcal{S}_r$. By the previous proposition, some bracket of τ^C is non-increasing, meaning the word $\tau(b_j + 1)\tau(b_j + 2)\ldots\tau(a_{j+1})$ is non-increasing for some j. So in forming the permutation π_τ, this bracket is put in increasing order. It follows that $\delta_\tau = \delta_\sigma$ is lexicographically less than δ. □

We now use our work in $\mathcal{S}_d$ to construct a convex-ear decomposition for the order complex of $(B_d)_S$. Let $\delta_1, \delta_2, \ldots, \delta_t$ be all permutations in D, listed in lexicographic order of their labels. For each i let $\mathbf{d}_i = \mathbf{c}^{\delta_i}$ (in other words, $\mathbf{d}_i$ is the unique maximal chain in B_d with δ_i as its λ-label). Also let L_i be the poset generated by all maximal chains in $(B_d)_S$ of the form $\mathbf{c}_S$, where $\mathbf{c}$ is a maximal chain in B_d such that $\mathbf{c}_{[r-1]\setminus S} = (\mathbf{d}_i)_{[r-1]\setminus S}$. Finally, let Σ_i be the simplicial complex whose facets are given by maximal chains in $\overline{L}_i$ that are not chains in L_j for any $j < i$. As in the previous section, we use Σ_i to refer both to the simplicial complex above and the poset whose chains correspond to (not necessarily maximal) chains in $(B_d)_S$.

Proposition 2.3.7 $\Sigma_1, \Sigma_2, \ldots, \Sigma_t$ *is a convex-ear decomposition of the order complex* $\Delta((\overline{B}_d)_S)$.

To every maximal chain $\mathbf{e}$ in $(B_d)_S$, associate an equivalence class of maximal chains in B_d, namely all maximal chains $\mathbf{c}$ such that $\mathbf{c}_S = \mathbf{e}$. By Lemma 2.3.3, this equivalence class can be viewed as the set $\{\mathbf{c}^\tau : \tau \in C(\sigma)\}$ for some $\sigma \in \mathcal{S}_r$. We refer to $C(\sigma)$ as the class corresponding to $\mathbf{e}$.

Next let $\mathbf{c}$ be a maximal chain in B_d such that $\mathbf{c}_S$ is a maximal chain in L_i. $\mathbf{c}_{[d-1]\setminus S} = (\mathbf{d}_i)_{[d-1]\setminus S}$, and so, by Proposition 2.3.3, $\lambda(\mathbf{c}) \in Sp(\delta_i)$. Let $\sigma = \lambda(\mathbf{c})$. The chain $\mathbf{c}_S$ then corresponds to the equivalence class $C(\sigma)$, and we have proven half of the following lemma:

Lemma 2.3.8 *Let* $\sigma \in \mathcal{S}_d$ *and let* $\boldsymbol{e}$ *be a maximal chain in* $(B_d)_S$ *corresponding to the equivalence class* $C(\sigma)$*. Then* $\boldsymbol{e}$ *is a maximal chain in* L_i *if and only if* $C(\sigma) \cap Sp(\delta_i) \neq \emptyset$*.*

Proof: We have already proven the "only if" direction above. For the other direction, suppose $C(\sigma) \cap Sp(\delta_i) \neq \emptyset$. Choose the unique τ in this intersection. By Lemma 2.3.3,

$\mathbf{c}_S^\tau = \mathbf{e}$ and $\mathbf{c}^\tau_{[r-1]\setminus S} = (\mathbf{d}_i)_{[r-1]\setminus S}$, and so $\mathbf{e}$ is a maximal chain in L_i. □

Now let $\mathbf{e}$ and σ be as in the statement of the above lemma, and suppose $\mathbf{e}$ is a facet in Σ_i. Then δ_i is the lexicographically first permutation δ in D such that $C(\sigma) \cap Sp(\delta) \neq \emptyset$, and so, by Proposition 2.3.6, $\delta_i = \delta_\sigma$. Summarizing,

Lemma 2.3.9 *Let* $\mathbf{e}$ *be a maximal chain in* $(B_d)_S$ *corresponding to the class* $C(\sigma)$ *for some* $\sigma \in \mathcal{S}_d$. *Then* $\mathbf{e}$ *represents a facet in* Σ_i *if and only if* $\delta_i = \delta_\sigma$.

We are now ready to prove the properties of our convex-ear decomposition.

Proof of property (i): We must show that any maximal chain $\mathbf{e}$ in $(B_d)_S$ is a maximal chain in some L_i. By Lemma 2.3.8, we must find some $\delta \in D$ such that $C(\sigma) \cap Sp(\delta) \neq \emptyset$, where $C(\sigma)$ is the class corresponding to $\mathbf{e}$. But Lemma 2.3.4 guarantees such a permutation, namely δ_σ. □

Proof of property (ii): Fix $\mathbf{d}_i$, and write $\mathbf{d}_i := \hat{0} = x_0 < x_1 < \ldots < x_d = \hat{1}$. A maximal chain in L_i is determined by a choice of maximal chain in each open interval (x_{a_j-1}, x_{b_j+1}). Each of these intervals is isomorphic to $\overline{B}_{b_j-a_j+2}$. As noted before, the order complex of $\overline{B}_n$ is $b(\partial\Delta_{n-1})$, where 'b' denotes the first barycentric subdivision and Δ_{n-1} denotes the $(n-1)$-dimensional simplex. Thus the order complex of L_i is the product:

$$b(\partial\Delta_{b_1-a_1+1}) * b(\partial\Delta_{b_2-a_2+1}) * \ldots * b(\partial\Delta_{b_s-a_s+1})$$

where '$*$' denotes simplicial join (see [10] for background on this operation, and [28] for its application to polytopes). It follows that the order complex of each L_i is the boundary complex of a simplicial polytope. Since Σ_1 is the order complex of L_1, it remains to be shown that Σ_i is a *proper* subcomplex of the order complex of L_i when $i > 1$.

Fix δ_i with $i > 1$, and define a permutation $\sigma \in Sp(\delta_i)$ by putting each bracket of δ_i^{Sp} in increasing order. There are two cases to consider: first, suppose that $\sigma = 1\,2\ldots d$.

In this case, we leave it to the reader to show that $\delta_i = \delta_1$, the lexicographically first permutation in $\mathcal{S}_d$ with descent set S, contradicting our assumptions. Now suppose otherwise. Since each bracket of σ^{Sp} is increasing, it must be the case that some bracket of σ^C is non-increasing. Then, by Lemma 2.3.5, $\delta_i \neq \delta_\sigma$, since $C(\sigma) \cap Sp(\delta_i) = \{\sigma\}$. Finally, by Proposition 2.3.6, δ_σ precedes δ_i lexicographically, and so $\delta_\sigma = \delta_j$ for some $j < i$. □

Proof of property (iii): Fix $i > 1$, and let $\mathbf{e}$ be a maximal chain representing a facet in Σ_i. Let $\mathbf{c}^{\mathbf{e}}$ be the unique maximal chain of B_d satisfying $\mathbf{c}^{\mathbf{e}}_S = \mathbf{e}$ and $\mathbf{c}^{\mathbf{e}}_{[d-1]\setminus S} = (\mathbf{d}_i)_{[d-1]\setminus S}$, and define $\sigma_{\mathbf{e}}$ to be the label $\lambda(\mathbf{c})$. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the maximal chains of $(B_d)_S$ corresponding to facets of Σ_i. Writing σ_j as shorthand for $\sigma_{\mathbf{e}_j}$, let the above order be so that σ_j is lexicographically greater than σ_k whenever $j < k$. In particular, $\sigma_1 = \delta_i$. We claim that this ordering is a shelling of Σ_i.

Let $j < k$. By Lemma 2.1.8, there must be some ascent $\sigma_k(m) < \sigma_k(m+1)$ such that $\sigma_k(1, m) \neq \sigma_j(1, m)$. Because $\mathbf{c}^{\mathbf{e}_j}$ and $\mathbf{c}^{\mathbf{e}_k}$ coincide at ranks not in S, $\sigma_j(1, r) = \sigma_k(1, r)$ for all $r \notin S$, by Lemma 2.3.3. Thus $m \in S$. Let σ'_k be the permutation obtained from σ_k by switching $\sigma_k(m)$ and $\sigma_k(m+1)$.

Note that σ'_k is lexicographically later than σ_k. We need to show that $\mathbf{c}^{\sigma'_k}_S$ is a chain in Σ_i, which is equivalent to showing that $\delta_{\sigma'_k} = \delta_i$. Since $m \in S$, the contents of each bracket of $(\sigma'_k)^{Sp}$ are the same as the contents of the corresponding bracket of σ_k^{Sp}. Because $\sigma_k \in Sp(\delta_i)$, $\sigma'_k \in Sp(\delta_i)$, meaning $C(\sigma'_k) \cap Sp(\delta_i) = \{\sigma'_k\}$. Now consider a bracket in σ_k^{Sp}:

$$\sigma_k(b_p + 1)\sigma_k(b_p + 2)\ldots\sigma_k(a_{p+1})$$

By Lemma 2.3.5, this bracket is increasing (since $\delta_{\sigma_k} = \delta_i$, by assumption). Because $m \in S$, the only way this bracket in $(\sigma'_k)^{Sp}$ can differ is if $b_p + 1 = m + 1$ or $a_{p+1} = m$. In the first case, $\sigma_k(b_p + 1) = \sigma_k(m + 1)$ is replaced by $\sigma_k(m)$. In the second case,

$\sigma_k(a_{p+1}) = \sigma_k(m)$ is replaced by $\sigma_k(m+1)$. However, since $\sigma_k(m) < \sigma_k(m+1)$, this bracket is still increasing in $(\sigma'_k)^{Sp}$ in both cases, and so $\delta_{\sigma'_k} = \delta_i$ by Lemma 2.3.5.

To complete the proof, we have to show that $\mathbf{e}_j \cap \mathbf{e}_k \subseteq \mathbf{e}'_k \cap \mathbf{e}_k$. Since $\mathbf{e}_k$ coincides with $\mathbf{e}'_k$ everywhere except at rank m, it is enough to show that $\mathbf{e}_j$ and $\mathbf{e}_k$ do not intersect at that rank. But this follows immediately, since $\sigma_j(1, m) \neq \sigma_k(1, m)$. □

Proof of property (iv): We take our cue from the proof of property (iv) in the previoius section, since the Σ_i's are defined analogously. That is, let $\mathbf{e}_i$ and $\mathbf{e}_j$ be facets of Σ_i and Σ_j, respectively, where $i < j$. Let $\mathbf{e} = \mathbf{e}_i \cap \mathbf{e}_j$. By the discussion in the proof of property (iv) in the previous section, it suffices to find a facet $\mathbf{e}_{old}$ of some Σ_k with $k < j$ such that $\mathbf{e}_{old}$ contains $\mathbf{e}$.

Define the maximal chain $\mathbf{e}_{old}$ by $\mathbf{e}_{old} = (com(\mathbf{e}))_S$, and let σ be the λ-label of $com(\mathbf{e})$. By construction, $\pi_\sigma = \sigma$. Now let τ be the λ-label of some maximal chain $\mathbf{c}$ in B_d with $\mathbf{c}_S = \mathbf{e}_i$. It is clear that π_τ is independent of the choice of maximal chain $\mathbf{c}$, and that π_σ is lexicographically less than or equal to π_τ. It follows that δ_σ is lexicographically less than or equal to δ_τ, which means that $\mathbf{e}_{old}$ is a facet of Σ_k for some $k \leq i < j$. □

Now that we have constructed the most intricate of our convex-ear decompositions, we can prove a theorem in the next section (Theorem 2.4.2) that allows us to provide decompositions for posets that are essentially composed of Boolean lattices.

2.4 The Rank-Selected Supersolvable Lattice

It is implicit in our earlier work that supersolvable lattices are composed of Boolean lattices that are pieced together in an orderly fashion. Using the previous sections, we can prove the following:

Theorem 2.4.1 *Let L be a rank d supersolvable lattice with nowhere-zero Möbius function, and let $S \subseteq [d-1]$. Then the order complex $\Delta(\overline{L}_S)$ admits a convex-ear decomposition.*

Because the techniques used here will be based on our decomposition in Section 2.3, and because we will use similar constructions in the following sections, we do most of the work in the following theorem:

Theorem 2.4.2 *Let P be a rank d poset with a $\hat{0}$ and $\hat{1}$, and suppose that P is a union of subposets $P_1, P_2, \ldots, P_q$, each isomorphic to the Boolean lattice B_d. That is, every chain in P is a chain in P_i for some i. Let $S \subseteq [d-1]$. The order complex $\Delta(\overline{P}_S)$ admits a convex-ear decomposition if, for each $i > 1$ there exists an $\mathcal{S}_d$-EL-labeling λ_i of P_i satisfying the following property: If $\boldsymbol{e}$ is a non-maximal chain in P_i that is also a chain in P_j for some $j < i$, then $(com_{\lambda_i}(\boldsymbol{e}))_S$ is a chain in P_k for some $k < i$.*

Proof: The proof is by induction on q. If $q = 1$, then $P = P_1$ is the Boolean lattice B_d, and the proof reduces to Theorem 2.3.1. Now suppose that $q > 1$ and, by induction, let $\Sigma'_1, \Sigma'_2, \ldots, \Sigma'_r$ be a convex-ear decomposition of $\Delta(\overline{P}^{q-1}_S)$, where P^{q-1} is the subposet of P generated by $P_1, P_2, \ldots, P_{q-1}$. Since we focus our attention on the subposet P_q, set $\lambda = \lambda_q$. Following Section 2.3, let $\delta_1, \delta_2, \ldots, \delta_t$ be all permutations in $\mathcal{S}_d$ with descent set S, listed in lexicographic order. For each i, let L_i denote the subposet of $(P_q)_S$ generated by all maximal chains of the form $\mathbf{c}_S$ for some maximal chain $\mathbf{c}$ satisfying $\mathbf{c}_{[d-1]\setminus S} = (\mathbf{d}_i)_{[d-1]\setminus S}$. Let $\hat{\Sigma}_i$ denote the complex whose facets are given by maximal chains in $\overline{L}_i$ that are not chains in L_j for any $j < i$, and let Σ_i be the subcomplex of $\hat{\Sigma}_i$ whose facets are maximal chains in Σ_i that are not chains in P^{q-1}. We claim that the sequence $\Sigma'_1, \Sigma'_2, \ldots, \Sigma'_r, \Sigma_1, \Sigma_2, \ldots, \Sigma_t$, once all $\Sigma_i = \emptyset$ are removed, is a convex-ear decomposition of $\Delta(\overline{P}_S)$.

Property (i) is verified immediately, based on Section 2.3: $\Delta(\overline{P}_S^{q-1}) = \bigcup_{i=1}^{q-1} \Sigma_i$, by induction, while $\Delta((\overline{P}_q)_S) = \bigcup_{i=1}^{t} \Sigma_i$ by Theorem 2.3.1.

Property (ii) is almost verified as well: We know that each $\hat{\Sigma}_i$ for $i > 1$ is a proper subcomplex of the boundary of some d-dimensional simplicial polytope, and that the same holds for each Σ_i because each is a subcomplex of $\hat{\Sigma}_i$. What remains to be shown is that Σ_1 is a *proper* subcomplex of some simplicial d-polytope boundary, as $\hat{\Sigma}_1$ *is* a simplicial d-polytope boundary. In other words, it must be shown that some maximal chain of $\hat{\Sigma}_1$ is a chain of P^{q-1}. Since $\delta_1 = \delta_{\mathbf{1}}$, where $\mathbf{1} = 1\,2\ldots d$ is the identity permutation, the chain $\mathbf{c}_S$ is in $\hat{\Sigma}_1$, where $\mathbf{c}$ is the maximal chain in P_q with $\lambda(\mathbf{c}) = \mathbf{1}$. Since $\mathbf{c} = com(\hat{0} < \hat{1})$ and $\hat{0} < \hat{1}$ is a chain in P^{q-1}, $\mathbf{c}_S$ is a chain in P^{q-1}.

For property (iii), fix some i and let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be all maximal chains of Σ_i. For each j, let $\mathbf{c}^{\mathbf{e}_j}$ be the unique maximal chain of P_q that coincides with $\mathbf{d}_i$ at ranks not in S and coincides with $\mathbf{e}_j$ at ranks in S. Let $\sigma_j = \lambda(\mathbf{c}^{\mathbf{e}_j})$, and let the above order of maximal chains be such that σ_j is lexicographically greater than σ_k whenever $j < k$. As in Section 2.3, we claim that this ordering is a shelling of Σ_i. To this end, choose $\mathbf{e}_j$ and $\mathbf{e}_k$, with $j < k$. By Lemma 2.1.8, there must be some $m \in [d-1]$ with $\sigma_k(m) < \sigma_k(m+1)$ and $\sigma_j(1,m) \neq \sigma_k(1,m)$. Because the chains $\mathbf{c}^{\mathbf{e}_j}$ and $\mathbf{c}^{\mathbf{e}_k}$ coincide at ranks outside of S, it must be the case that $m \in S$. Let σ'_k be the permutation obtained from σ_k by interchanging $\sigma_k(m)$ and $\sigma_k(m+1)$, and let $\mathbf{e}'_k$ be the associated chain. In the proof of property (iii) in Section 2.3, it is shown that $\mathbf{e}_j \cap \mathbf{e}_k \subseteq \mathbf{e}'_k \cap \mathbf{e}_k$, and that $\mathbf{e}'_k$ is not a chain in $\hat{\Sigma}_\ell$ for any $\ell < i$. Thus, the only way the same proof could fail to work in the case considered here is if $\mathbf{e}'_k$ were a chain in P^{q-1}.

Indeed, suppose this were the case. In the proof of property (iii) in Section 2.3, it is shown that $\delta_i = \delta_{\sigma'_k}$. Thus, by Lemma 2.3.5, each bracket in $(\sigma'_k)^C$ is increasing, meaning $\mathbf{c}^{\mathbf{e}'_k} = com(\mathbf{e}'_k)$. Therefore, $\mathbf{c}^{\mathbf{e}'_k}$ would be a chain in P^{q-1}, as would $(\mathbf{c}^{\mathbf{e}'_k})_{-m}$. Since

$\mathbf{c}^{\mathbf{e}_k} = com((\mathbf{c}^{\mathbf{e}'_k})_{-m})$, $(\mathbf{c}^{\mathbf{e}_k})_S$ would be a chain in P^{q-1}, contradicting the assumption that $\mathbf{e}_k$ is a maximal chain in Σ_i.

Finally, to prove property (iv), let $\mathbf{e}$ be a *non*-maximal chain in $\Sigma_i \cap ((\bigcup_{j=1}^{i-1} \Sigma_j) \cup (\bigcup_{k=1}^{r} \Sigma'_k))$. As in our previous proofs of property (iv), it suffices to find a maximal chain $\mathbf{e}_{old}$ in the union of all the earlier ears such that $\mathbf{e}'$ contains $\mathbf{e}$ as a subchain. Here, we must consider two cases: First, suppose that $\mathbf{e}$ is a chain in Σ_j for some $j < i$. Because Σ_j is a subcomplex of $\hat{\Sigma}_j$, the proof of this property in Section 2.3 produces a maximal chain in $\cup_{k=1}^{j} \hat{\Sigma}_k$ of which $\mathbf{e}$ is a subchain. Because this maximal chain is either a chain in $\cup_{k=1}^{j} \Sigma_k$ or P^{q-1}, this completes our proof in this case. For the second case, suppose that $\mathbf{e}$ is a chain in P^{q-1}. Then $(com(\mathbf{e}))_S$ is a chain in P^{q-1}. □

Proof of Theorem 2.4.1: With most of our work done by Theorem 2.4.2 and Section 2.2, the proof here is fairly painless: Let L be a rank d supersolvable lattice with $\mathcal{S}_d$-EL-labeling λ, let $S \subseteq [d-1]$, and let $L_1, L_2, \ldots, L_t$ be as in Section 2.2. That is, let $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_t$ be all maximal chains in L with decreasing λ-labels, and for each i let L_i be the sublattice of L generated by $\mathbf{c}_M$ and $\mathbf{d}_i$, where $\mathbf{c}_M$ is the M-chain (i.e., the unique maximal chain of L with increasing λ-label). In Section 2.2 it is shown that each L_i is isomorphic to the Boolean lattice B_d. Now let $\mathbf{e}$ be a non-maximal chain of some L_i, that is also a chain in L_j for some $j < i$. Since λ restricts to an $\mathcal{S}_d$-EL-labeling on L_j (by Theorem 2.1.6), Lemma 2.3.3 tells us that $com(\mathbf{e})$ is a maximal chain in L_j, meaning $(com(\mathbf{e}))_S$ is as well. Invoking Theorem 2.4.2 completes the proof. □

Theorem 2.4.2 also allows us to provide homology bases for poset order complexes of the above type:

Theorem 2.4.3 *Using the notation and hypothesis of Theorem 2.4.2, let $\boldsymbol{d}^i_1, \boldsymbol{d}^i_2, \ldots, \boldsymbol{d}^i_{t_i}$ be all maximal chains in P_i whose λ_i-labels have descent set S, and such that no $\boldsymbol{d}^i_j$ is a* *chain in P_k for any $k < i$. For each $\mathbf{d}^i_j$, let Δ^i_j be the associated polytope in $\Delta(\overline{P}_i)$. Then the set $\{\Delta^i_j\}$, for all relevant i and j, is a homology basis for $\Delta(\overline{P})$.*

Proof: Because $\Delta(\overline{P})$ is homotopy equivalent to $(\sum t_i)$-many spheres (Proposition 1.2.9), and because no $\mathbf{d}^i_j$ is in Δ^k_ℓ for (k, ℓ) lexicographically less than (i, j), the result follows. □

Chapter 3

More Convex-Ear Decompositions

In this chapter we construct two more convex-ear decompositions. The first expands upon a result of Nyman and Swartz (Theorem 1.4.7), while the second will allow us to prove enumerative properties for a class of flag h-vectors in Section 4.2.4

Combined, the two main results in this chapter read as follows:

Theorem 3.0.4 *Let P be a graded poset of rank d, let $S \subseteq [d-1]$, and suppose that P is either a geometric lattice or the face poset of a $(d-1)$-dimensional shellable simplicial complex. Then $\Delta(\overline{P}_S)$ admits a convex-ear decomposition, and thus its h-vector satisfies, for $i < |S|/2$,*

(i) $h_i \leq h_{|S|-i}$, *and*

(ii) $h_i \leq h_{i+1}$.

Furthermore, its g-vector is an M-vector.

3.1 The Rank-Selected Geometric Lattice

Let L be a geometric lattice of rank d. In [17], Nyman and Swartz show that $\Delta(\overline{L})$ admits a convex-ear decomposition. We open this section by briefly describing their technique.

Let $a_1, a_2, \ldots, a_\ell$ be a fixed linear ordering of the atoms of L. The *minimal labeling* λ labels the edges of the Hasse diagram of L as follows: If y covers x, then $\lambda(x, y) = \min\{i : x \vee a_i = y\}$. Björner proved that λ is an EL-labeling ([3]).

Viewing L as the lattice of flats of a simple matroid M, let $B_1, B_2, \ldots, B_t$ be all the nbc-bases of M listed in lexicographic order. For a fixed $i \leq t$, let $B_i = \{a_{i_1}, a_{i_2}, \ldots, a_{i_d}\}$ where $a_{i_1} < a_{i_2} < \ldots < a_{i_d}$ under the fixed ordering of the atoms of L. Fix a permutation $\sigma \in \mathcal{S}_d$, and define $\mathbf{c}^i_\sigma$ to be the maximal chain $a_{i_{\sigma(1)}} < a_{i_{\sigma(1)}} \vee a_{i_{\sigma(2)}} < \ldots < a_{i_{\sigma(1)}} \vee a_{i_{\sigma(2)}} \vee \ldots \vee a_{i_{\sigma(d)}}$. Define the basis labeling $\nu_i(\mathbf{c}^i_\sigma)$ of $\mathbf{c}^i_\sigma$ to be the word $i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(d)}$.

For each i with $1 \leq i \leq t$, let L_i be the poset generated by the maximal chains $\{\mathbf{c}^i_\sigma : \sigma \in \mathcal{S}_d\}$, and let Σ_i be the simplicial complex whose facets are given by maximal chains in $\overline{L}_i$ that are not chains in any L_j for $j < i$.

Theorem 3.1.1 *[17] $\Sigma_1, \Sigma_2, \ldots, \Sigma_t$ is a convex-ear decomposition of $\Delta(L)$.*

The next lemma is shown in [17] and provides the key tool in proving the above theorem.

Lemma 3.1.2 *A chain $\boldsymbol{c}$ in L_i is in Σ_i if and only if $\nu_i(\boldsymbol{c}) = \lambda(\boldsymbol{c})$*

The main theorem of this section is the following:

Theorem 3.1.3 *If L is a rank d geometric lattice and $S \subseteq [d-1]$, the order complex $\Delta(\overline{L}_S)$ admits a convex-ear decomposition.*

Proof: Each L_i is clearly isomorphic to the Boolean lattice B_d under the mapping $a_{i_{\sigma(1)}} \vee a_{i_{\sigma(2)}} \vee \ldots \vee a_{i_{\sigma(m)}} \to \{\sigma(1), \sigma(2), \ldots, \sigma(m)\}$. Moreover, the basis labeling ν_i is the standard $\mathcal{S}_d$-EL-labeling of L_i (though with the alphabet $\{i_1, i_2, \ldots, i_d\}$ rather than $[d]$).

Now fix some i, and suppose that $\mathbf{e}$ is a non-maximal chain in L_i that is also a chain in L_j for some $j < i$. Suppose that j is the least such integer, and consider the maximal chain $\mathbf{c} = com_{\nu_j}(\mathbf{e})$. This chain can clearly not be in L_k for any $k < j$, because then $\mathbf{e}$ would be a chain in L_k, contradicting the minimality of j. Thus $\nu_j(\mathbf{c}) = \lambda(\mathbf{c})$ by Lemma 3.1.2, meaning $\mathbf{c} = com_\lambda(\mathbf{e})$. Now consider the chain $\mathbf{c}' = com_{\nu_i}(\mathbf{e})$. If $\mathbf{c}'$ is not a chain in L_k for any $k < i$ then, again by Lemma 3.1.2, $\lambda(\mathbf{c}') = \nu_i(\mathbf{c}')$. Thus $\mathbf{c}' = com_\lambda(\mathbf{e})$, which is a contradiction since the chain $com_\lambda(\mathbf{e})$ is uniquely determined. Thus $com_{\nu_i}(\mathbf{e})$ must be a chain in L_k for some $k < i$, and so its subchain $(com_{\nu_i}(\mathbf{e}))_S$ is a chain in L_k. Applying Theorem 2.4.2 completes the proof. □

3.2 The Rank-Selected Face Poset of a Shellable Complex

The main result of this section can be seen as motivated by Hibi's result ([11]) that the codimension-1 skeleton of a shellable complex Σ is 2-Cohen-Macaulay:

Theorem 3.2.1 *Let Σ be a $(d-1)$-dimensional shellable complex with face poset P_Σ, and let $S \subseteq [d-1]$. Then the order complex $\Delta((\overline{P}_\Sigma)_S)$ admits a convex-ear decomposition.*

Proof: Fix a shelling $F_1, F_2, \ldots, F_t$ of Σ, and for each i let P_i denote the face poset of F_i. Then each P_i is isomorphic to the Boolean lattice B_d. Our proof relies on the following obvious fact: Let $\mathbf{e}$ be a non-maximal chain in P_i, and let x be its element of highest rank. Then $\mathbf{e}$ is not a chain in L_j for any $j < i$ if and only if, when viewed as a face of F_i, x contains the unique minimal new face $r(F_i)$.

Now fix some i. Any bijection $\phi : F_i \to [d]$ induces an $\mathcal{S}_d$-EL-labeling λ_ϕ of P_i in the obvious way: For $x, y \in P_i$ with $y = x \cup \{v\}$ for some vertex v of F_i, set $\lambda_\phi(x, y) = \phi(v)$. Let $\phi : F_i \to [d]$ be any bijection that labels vertices in $r(F_i)$ last. That is, if $v \in r(F_i)$ and $w \in F_i \setminus r(F_i)$ then $\phi(w) < \phi(v)$. Set $\lambda = \lambda_\phi$. Suppose $\mathbf{e}$ is a non-maximal chain in P_i that is also a chain in P_j for some $j < i$, and let x be the element of $\mathbf{e}$ of highest rank.

By the above observation, $r(F_i) \nsubseteq x$. If v is the vertex in $F_i \setminus x$ with the greatest ϕ-label then, by definition of ϕ, $v \in r(F_i)$. Let $\mathbf{c} = com_\lambda(\mathbf{e})$, and let $\sigma = \lambda(\mathbf{c})$. Then $\sigma(d) = \phi(v)$. Since $d \notin S$ (because $S \subseteq [d-1]$), the element of $\mathbf{c}_S$ of highest rank, when viewed as a face of F_i, does not contain the vertex v. Because $v \in r(F_i)$, $\mathbf{c}_S$ must be a chain in P_j for some $j < i$.

We are almost in a position to apply Theorem 2.4.2. The only possible snag is that, unless Σ consists of a single facet, P_Σ has no greatest element. However, the only place this property is used in the proof of the theorem is in showing property (ii) of the decomposition. That is, for any $i > 1$ it must be shown that $\mathbf{c}_S$ is a chain in P_j for some $j < i$, where $\mathbf{c}$ is the unique maximal chain in P_i with increasing λ-labels. As before, let x be the element in $\mathbf{c}_S$ of highest rank. Since the rank of x must be less than d (because $S \subseteq [d-1]$), x cannot contain the vertex $\phi^{-1}(d)$ (when viewed as a face of F_i). Since $r(F_i) \neq \emptyset$ because $i > 1$ and $\phi^{-1}(d) \in r(F_i)$, $r(F_i) \nsubseteq x$. Thus $\mathbf{c}_S$ must be a chain in P_j for some $j < i$. We can now apply Theorem 2.4.2 to obtain our result. □

The above theorem does not hold if $d \in S$. Indeed, if Σ is the shellable complex consisting of two 2-dimensional simplices joined at a common boundary facet and $S = \{2, 3\} \subseteq [3]$, then $\Delta((\overline{P}_\Sigma)_S)$ does not admit a convex-ear decomposition, as it is a tree.

Let Σ' be the 3-dimensional complex given by two 3-dimensional simplices joined at boundary facets, and let Σ be its 2-skeleton. Then Σ admits a convex-ear decomposition and, moreover, so does the complex $\Delta((\overline{P}_\Sigma)_S)$ for any choice of $S \subseteq [3]$. Figure 3.1 shows the case when $S = \{2, 3\}$.

Thus, we can conjecture the following:

Conjecture 3.2.2 *When Σ is a $(d-1)$-dimensional complex admitting a convex-ear decomposition and $S \subseteq [d]$, the complex $\Delta((\overline{P}_\Sigma)_S)$ admits a convex-ear decomposition.*

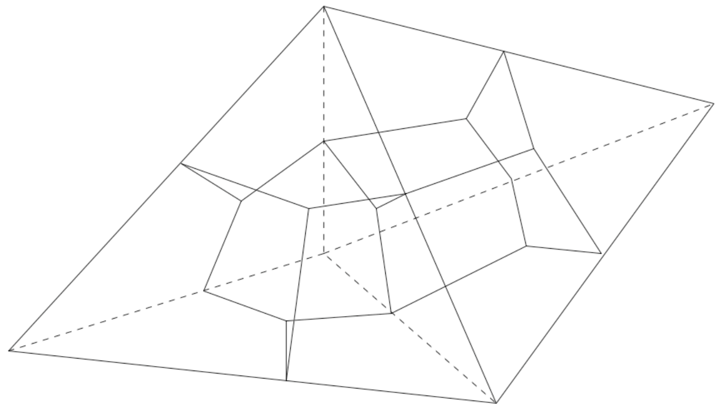

Figure 3.1: The complex $\Delta((\overline{P}_\Sigma)_{\{2,3\}})$ as a subcomplex of Σ.

A $(d-1)$-dimensional complex Σ with vertex set V is called *balanced* if there exists a $\phi : V \to [d]$ such that $\phi(v) \neq \phi(w)$ whenever v and w are in a common face of Σ. The function ϕ is called a *coloring* of Σ.

The order complex of any graded poset P is always balanced: For a vertex v of $\Delta(P)$, simply let $\phi(v)$ be the rank of v when considered as an element of the poset P. Thus the barycentric subdivision of any simplicial complex is balanced, since it is the order complex of its face poset.

If Σ is a $(d-1)$-dimensional balanced complex with coloring ϕ and $S \subseteq [d]$, define the Σ_S to be the subcomplex of Σ with faces $\{F \in \Sigma : \phi(v) \in S \text{ for all } v \in F\}$. With these new definitions, we can rephrase Theorem 3.2.1 in a more geometric tone.

Theorem 3.2.3 *Let Σ' be a $(d-1)$-dimensional shellable complex, and let Σ be the first barycentric subdivision of its codimension-1 skeleton. Then, for any coloring ϕ of the vertices of Σ and any $S \subseteq [d-1]$, the complex Σ_S admits a convex-ear decomposition.*

Chapter 4

Applications to the Flag h-Vector

4.1 Preliminaries

Let P be a graded poset of rank d with order complex $\Delta = \Delta(\overline{P})$. For $S \subseteq [d-1]$, let $f_S = f_S(P)$ be the number of maximal chains of the rank selected subposet P_S. This gives a natural refinement of the f-vector of Δ since $f_i(\Delta) = \sum_{|S|=i} f_S(P)$.

We also define the flag h-vector of Δ by $h_S = \sum_{T \subseteq S} (-1)^{|S-T|} f_T$. Equivalently, by inclusion-exclusion (see [24]), $f_S = \sum_{T \subseteq S} h_T$. If the poset P admits an EL-labeling, the flag h-vector has a nice enumerative interpretation (see, for instance, page 133 of [24]):

Proposition 4.1.1 *Let P be a poset that admits an EL-labeling. Then h_S counts the number of maximal chains of P whose labels have descent set S.*

The above proposition in conjunction with Proposition 1.3.9 implies that the flag h-vector of P satisfies:

$$h_i(\Delta) = \sum_{|S|=i} h_S(P)$$

whenever P admits an EL-labeling. In fact, this is true for any graded poset, and so the flag h-vector is a refinement of the usual h-vector in the same way that the flag f-vector is a refinement of the usual f-vector.

Now let $\sigma \in S_d$ be a permutation written as a word in $[d]$: $\sigma = \sigma(1)\sigma(2)\dots\sigma(d)$. If we interchange $\sigma(i)$ and $\sigma(i+1)$ for some $i \notin D(\sigma)$, we call this a *switch*. The word σ is less than τ in the *weak order* (sometimes called the *weak Bruhat order*), written $\sigma <_w \tau$, if τ can be obtained from σ by a sequence of switches.

For example, $1\,3\,2\,4 <_w 1\,4\,2\,3$, since $1\,3\,2\,4 <_w 1\,3\,4\,2 <_w 1\,4\,3\,2$.

When $S \subseteq [d]$, let $D^d_S = \{\sigma \in S_d : D(\sigma) = S\}$. For two subsets $S, T \subseteq [d]$, we say that S *dominates* T if there exists an injection $\phi : D^d_T \to D^d_S$ such that $\tau <_w \phi(\tau)$ for all $\tau \in D^d_T$. If $d = 4$, the set $\{1, 3\}$ dominates the set $\{1\}$. To see this, define a function $\phi : D^4_{\{1\}} \to D^4_{\{1,3\}}$ by $\phi(\tau) = \tau(1)\tau(2)\tau(4)\tau(3)$. Then $\phi(\tau) \in D^4_{\{1,3\}}$ for any $\tau \in D^4_{\{1\}}$, and ϕ is clearly injective.

For a study of which pairs of subsets $S, T \subseteq [d]$ satisfy this dominance relation, see [9] or [17].

4.2 The Flag h-Vector of a Face Poset

In [17], the authors prove the following:

Theorem 4.2.1 *If L is a geometric lattice of rank d, $S, T \subseteq [d-1]$, and S dominates T, the flag h-vector of $\Delta(\overline{L})$ satisfies $h_T \leq h_S$.*

Using the convex-ear decomposition from Section 2.2, the proof of the above theorem carries over verbatim:

Theorem 4.2.2 *Let L be a rank d supersolvable lattice with nowhere-zero Möbius function. Let $S, T \subseteq [d-1]$, and suppose that S dominates T. Then the flag h-vector of Δ satisfies $h_T \leq h_S$.*

Proof: Let λ be an $\mathcal{S}_d$-EL-labeling of L. For any subset $A \subseteq [d-1]$, h_A is the number of maximal chains of L whose λ-labels have descent set A. Fix an ear Σ_i of the decomposition given in Section 2.2. Let $\mathbf{c}$ be a maximal chain in Σ_i with $\sigma = \lambda(\mathbf{c})$ and $D(\sigma) = T$. Suppose $\sigma(m) < \sigma(m+1)$ is an ascent, let σ' be the permutation obtained from σ by interchanging $\sigma(m)$ and $\sigma(m+1)$, and let $\mathbf{c}'$ be the unique chain of L with $\lambda(\mathbf{c}') = \sigma'$. Then $\mathbf{c}'$ is a chain in Σ_i (if $\mathbf{c}'$ were in some earlier L_j, then $\mathbf{c} = com(\mathbf{c}'_{-m})$ would be as well). Now let $\phi : D^d_T \to D^d_S$ be an injection with $\sigma <_w \phi(\sigma)$ for all $\sigma \in D^d_T$. Then for every $\sigma \in D^d_T$ such that a maximal chain in Σ_i has σ as its label, the above shows that there is a maximal chain in Σ_i with $\phi(\sigma)$ as its label. Because this is true for each step in the convex-ear decomposition, the result follows. □

Our goal in this section is to find an analogue of the above theorems for face posets of Cohen-Macaulay simplicial complexes.

Let P be a graded poset of rank d with a $\hat{0}$ and $\hat{1}$, and let $\Delta = \Delta(\overline{P})$ be the order complex of its proper part. Under the fine grading of the face ring $k[\Delta]$, $F(k[\Delta], \lambda) = \sum_{F\in\Delta}\prod_{x_i\in F}\frac{\lambda_i}{1-\lambda_i}$. We specialize this grading to accommodate the flag h-vector as follows: identify λ_i and λ_j whenever the vertices in Δ to which they correspond have the same rank r (as elements of P). Call this new variable ν_r. This specialized grading yields:

$$F(k[\Delta], \nu) = \sum_{S\subseteq[d-1]} f_S \prod_{i\in S}\frac{\nu_i}{1-\nu_i}$$

We put this over the common denominator of $\prod_{i\in[d-1]}(1-\nu_i)$ to obtain:

$$F(k[\Delta], \nu) = \sum_{S\subseteq[d-1]} \frac{f_S\prod_{i\in S}\nu_i\prod_{i\notin S}(1-\nu_i)}{\prod_{i\in[d-1]}(1-\nu_i)} = \sum_{S\subseteq[d-1]}\frac{h_S\prod_{i\in S}\nu_i}{\prod_{i\in[d-1]}(1-\nu_i)}$$

Now suppose Δ triangulates a ball, and let $\Delta' = \Delta - \partial\Delta - \emptyset$. The following equation is Corollary II.7.2 from [23]:

$$(-1)^d F(k[\Delta], 1/\lambda) = (-1)^{d-1}\tilde{\chi}(\Delta) + \sum_{F\in\Delta'}\prod_{x_i\in F}\frac{\lambda_i}{1-\lambda_i}$$

Letting f'_S be the flag f-vector for Δ', noting that $\tilde{\chi}(\Delta) = 0$, plugging in $1/\lambda$ in place of λ, and specializing to the ν-grading, the previous expression becomes:

$$(-1)^d F(k[\Delta], \nu) = \sum_{S \subseteq [d-1]} f'_S \prod_{i \in S} \frac{1}{\nu_i - 1}$$

Putting the above over the common denominator of $\prod_{i \in [d-1]} (\nu_i - 1)$ and multiplying by $(-1)^d$ gives us:

$$F(k[\Delta], \nu) = \sum_{S \subseteq [d]} \frac{f'_S \prod_{i \notin S} (\nu_i - 1)}{\prod_1^d (1 - \nu_i)}$$

Comparing with our earlier expression for $F(k[\Delta], \nu)$ and noting that the denominators are equal,

$$\sum_{S \subseteq [d-1]} h_S \prod_{i \in S} \nu_i = \sum_{S \subseteq [d-1]} f'_S \prod_{i \notin S} (\nu_i - 1)$$

In general, the flag f- and h-vectors satisfy the equation

$$\sum_{S \subseteq [d-1]} f_S \prod_{i \notin S} (\nu_i - 1) = \sum_{S \subseteq [d-1]} h_S \prod_{i \notin S} \nu_i$$

So, we can write the above equation as:

$$\sum_{S \subseteq [d-1]} f'_S \prod_{i \notin S} (\nu_i - 1) = \sum_{S \subseteq [d-1]} h_{[d-1]-S} \prod_{i \notin S} \nu_i$$

We now apply this above equation to obtain a set of inequalities for the flag h-vector of the face poset of a shellable simplicial complex.

Let Σ be a d-dimensional shellable complex with face poset P_Σ and shelling order $F_1, F_2, \ldots, F_t$, and for each i let P_i be the face poset of F_i. Let $A = [d-1]$, and set $\Delta = \Delta\ ((P_K)_A)$. Note that Δ is simply the order complex of P_Σ once we remove the elements corresponding to the facets of Σ and the element corresponding to the empty set. Let $\Sigma_1, \Sigma_2, \ldots, \Sigma_t$ be the convex-ear decomposition of Δ given in Section 3.2.1.

Lemma 4.2.3 *Let $S, T \subseteq [d-1]$. If S dominates T then, for any i, there are at least as many maximal chains in Σ_i with descent set S (under the labeling described in the proof of Theorem 6.2) as there are with descent set T.*

Proof: Because the poset P_Σ satisfies the hypotheses of Theorem 2.4.2, the proof of this lemma is identical to the proof of Theorem 4.2.2. □

Theorem 4.2.4 *Let $S, T \subseteq [d-1]$, and suppose that S dominates T. Then the flag h-vector of Δ satisfies $h_T \leq h_S$.*

Proof: The argument here is based on the one given in [17] for geometric lattice order complexes. $h_T(\Sigma_1) \leq h_S(\Sigma_1)$, since the poset associated to Σ is just the Boolean lattice B_d. In general, suppose the result holds for $\Sigma_1 \cup \Sigma_2 \cup \ldots \Sigma_{k-1}$. Let $\Omega = \Sigma_1 \cup \Sigma_2 \cup \ldots \Sigma_{k-1}$, and let $\Sigma'_k = \Sigma_k - \partial\Sigma_k - \emptyset$. Because Σ_k triangulates a ball, we can now use our earlier expression for the flag h-vector of a ball and invoke an argument similar to Chari's in [7]:

$$\begin{aligned}
\sum_{S\subseteq[d]} h_S(\Omega \cup \Sigma_k) \prod_{i \notin S} \nu_i &= \sum_{S\subseteq[d]} f_S(\Omega \cup \Sigma_k) \prod_{i \notin S} (\nu_i - 1) \\
&= \sum_{S\subseteq[d]} f_S(\Omega) \prod_{i \notin S} (\nu_i - 1) + \sum_{S\subseteq[d]} f_S(\Sigma'_k) \prod_{i \notin S} (\nu_i - 1) \\
&= \sum_{S\subseteq[d]} h_S(\Omega) \prod_{i \notin S} \nu_i + \sum_{S\subseteq[d]} h_{[d]-S}(\Sigma_k) \prod_{i \notin S} \nu_i \\
&= \sum_{S\subseteq[d]} (h_S(\Omega) + h_{[d]-S}(\Sigma_k)) \prod_{i \notin S} \nu_i
\end{aligned}$$

Reverse lexicographic order of the maximal chains of Σ_k is a shelling (as shown in Theorem 2.4.2), so it follows that $h_S(\Sigma_k)$ is the number of maximal chains of Σ_k whose labels have *ascent* set S. Thus $h_{[d]-S}$ counts the number of maximal chains in P_k with descent set S. Since we add at least as many maximal chains whose labels have descent set S as we do maximal chains whose labels have descent set T (Lemma 4.2.3), the result follows. □

The previous theorem can be generalized to face posets of Cohen-Macaulay complexes.

Theorem 4.2.5 *Let K be a d-dimensional Cohen-Macaulay simplicial complex with face poset P, and let $\Delta = \Delta(P)$. Let $S, T \subseteq [d-1]$, and suppose that S dominates T. Then the flag h-vector of Δ satisfies $h_T \leq h_S$.*

Proof: First, note that any linear inequality of the flag h-vector of a complex is equivalent to some linear inequality of the flag f-vector of that complex. Next we note that, in the case when Δ is the order complex of the face poset of a complex K, a linear inequality of the flag f-vector is equivalent to a linear inequality of the standard f-vector of K. To see why this is true, let $S \subseteq [d]$ and write S as a decreasing word: $a_1, a_2, \ldots, a_m$. Then f_S is simply the product $b_1 b_2 \ldots b_m$, where $b_1 = f_{a_1}(K)$ and for $i > 1$ $b_i = b_{i-1}\binom{a_{i-1}}{a_i}$. Since Stanley has shown that all linear inequalities involving the f-vector of Cohen-Macaulay complexes are of the form $\sum_i c_i h_i \geq 0$ where each $c_i \geq 0$, and since every Cohen-Macaulay h-vector is the h-vector of some shellable complex (see, for instance, [23]), it must be the case that Theorem 4.2.4 amounts to an inequality of the above form. Because $h_i(\Delta) \geq 0$ for all i when Δ is Cohen-Macaulay (see for instance [23], pg. 57), it follows that Cohen-Macaulay complexes satisfy the conclusion of Theorem 4.2.4. □

We now show that Theorem 4.2.5 cannot be extended to include posets whose order complexes are Cohen-Macaulay (or 2-CM, for that matter):

A graded poset P is *Eulerian* if for all $x, y \in P$ with $x < y$ $\mu(x, y) = (-1)^k$, where $k = \text{rank(y)} - \text{rank(x)}$. An Eulerian poset whose order complex is Cohen-Macaulay is called *Gorenstein**. It can be shown that the order complex of a Gorenstein* poset is 2-Cohen-Macaulay. For $S \subseteq [n]$, define $w(S)$ to be the set of all $i \in [n]$ such that exactly one of i and $i+1$ is in S. For instance, if $S = \{2,3\} \subseteq [4]$ then $w(S) = \{1,3\}$. Since Conjecture 2.3 from [22] was proved by Karu in [13], we can rephrase Proposition 2.8 from [22] as:

Proposition 4.2.6 *If $S, T \subseteq [n]$ are such that $h_T(\Delta) \leq h_S(\Delta)$ whenever Δ is the order complex of a Gorenstein* poset then $w(T) \subseteq w(S)$.*

Now consider $S, T \subseteq [4]$ given by $S = \{1,2\}$ and $T = \{1\}$. In [17], it is shown that S dominates T. However, $w(S) = \{2\}$ and $w(T) = \{1\}$, so $w(T) \nsubseteq w(S)$ and it is clear that we cannot weaken the assumptions of Theorem 4.2.5 to include the wider class of Cohen-Macaulay posets (or even 2-CM posets).

We close this section by mentioning an interesting consequence to Theorem 4.2.5.

Corollary 4.2.7 *For each pair of subsets $S, T \subseteq [d-1]$, where S dominates T, there exist nonnegative integers $a_1^{S,T}, a_2^{S,T}, \ldots, a_d^{S,T}$ such that, for any d-dimensional Cohen-Macaulay K with face poset P and face poset order complex $\Delta = \Delta(P - \emptyset)$,*

$$h_S(\Delta) - h_T(\Delta) = \sum_{i=1}^{d} a_i^{S,T} h_i(K)$$

Proof: In the proof of Theorem 4.2.5, we see how linear inequalities of the flag h-vector of Δ translate to linear inequalities of the h-vector of K. The conclusion of the theorem tells us that $h_S(\Delta) - h_T(\Delta) \geq 0$ whenever S dominates T. Since all linear inequalities of the h-vector of K must be of the form $\sum_{i=0}^{d} a_i h_i(K) \geq 0$, where each $a_i \geq 0$, the corollary follows. □

Question 4.2.8 *Can one find a combinatorial interpretation of the coefficients $\{a_i^{S,T}\}$?*

Chapter 5

The h-Vector of a Lattice Path Matroid

5.1 Preliminaries

We begin this section with a common matroid construction:

Definition 5.1.1 *Let $A_1, A_2, \ldots, A_t$ be a collection of finite sets. The transversal matroid corresponding to this collection is the matroid with ground set $\bigcup_{i=1}^{t} A_i$ and independent sets $\{S \subseteq \bigcup_{i=1}^{t} A_i : |S \cap A_i| \leq 1$ for all $i\}$.*

The collection $\{A_i\}$ is known as a *presentation* of the transversal matroid M. Note that distinct presentations can give rise to the same transversal matroid.

Definition 5.1.2 *Let M be a transversal matroid. If M has a presentation $[a_1, c_1], [a_2, c_2], \ldots, [a_r, c_r]$ where each $[a_i, c_i]$ is an interval in the integers, $a_1 < a_2 < \ldots < a_r$, and $c_1 < c_2 < \ldots < c_r$, then M is called a lattice path matroid.*

Anytime we speak of a lattice path matroid M, we assume (unless explicitly stated otherwise) that it has a presentation of the above type, where $a_1 = 1$. We thus identify points of the matroid with the set $[c_r]$. When M is a lattice path matroid, it admits an alternate geometric interpretation. Before describing this interpretation, however, we need a few definitions:

A *lattice path* to the point (n, r) is a sequence, beginning at the origin, of unit length steps, each either directly north or directly east. For a lattice path p to (n, r), define a set

$B_p \subseteq [n+r]$ by $B_p = \{i :$ the ith step of p is north$\}$. Similarly, for a basis B, associate a lattice path p_B such that $B_{p_B} = B$. That is, p_B is the lattice path whose ith step is north if and only if $i \in B$.

Proposition 5.1.3 *Suppose M is a lattice path matroid with a presentation of the above type. Let B_a be the basis $\{a_1, a_2, \ldots, a_r\}$, and define B_c analogously. Then all bases of M are of the form B_p, where p is some lattice path within the region specified by p_{B_a} and p_{B_c}.*

For example, let M be the lattice path matroid with presentation $[1,3], [2,5], [4,7]$, and $[6,8]$. Then the region specified by p_{B_a} and p_{B_c} is as shown in Figure 5.2.3. The lattice path shown within the region corresponds to the basis $\{2,3,4,7\}$.

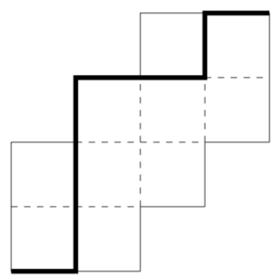

Figure 5.1: A lattice path matroid with basis $\{2,3,4,7\}$.

Lattice path matroids were first introduced by Bonin, de Mier, and Noy in [6], where it is shown that an element of a basis B of M is internally active if and only if the corresponding north step of p_B coincides with p_{B_a}. For example, the basis in the figure above has only one internally active element: 4.

Given Proposition 1.4.5, the h-vector of a lattice path matroid carries some interesting geometric information:

Corollary 5.1.4 *Let M be a rank r lattice path matroid as above, and let $\Delta = \Delta(M)$ be* *its independence complex. For all i, $h_i(\Delta)$ is the number of lattice paths between p_{B_a} and p_{B_c} that coincide with p_{B_a} exactly $r - i$ times.*

5.2 A Conjecture of Stanley

Let Γ be a monomial order ideal (see Definition 1.1.3). An element $\alpha \in \Gamma$ is called *maximal* if $\alpha | \beta$ and $\beta \in \Gamma$ implies $\alpha = \beta$.

Definition 5.2.1 *A monomial order ideal is pure if all its maximal elements are of the same degree. A pure M-vector is a degree sequence of some pure monomial order ideal.*

Not all M-vectors are pure: The sequence $(1, 3, 1)$ is an M-vector (since it is the degree sequence of the order ideal $\{1, x, y, z, xy\}$), but it is not pure since a degree 2 monomial can have at most two degree 1 divisors. The following was conjectured by Stanley (see, for instance, [23]):

Conjecture 5.2.2 *The h-vector of a matroid is a pure M-vector.*

In [16], Merino shows this to be true in the case when M is a *cographic* matroid (when $M^* = M(G)$ for some graph G).

Theorem 5.2.3 *Stanley's conjecture holds in the case when M is a lattice path matroid.*

Proof: If M is any matroid with a coloop e, then the h-vector of M is the h-vector of $M - e$ with an extra 0 appended. Thus, we may assume that M is coloop-free. Let $[a_1, c_1], [a_2, c_2], \ldots, [a_r, c_r]$ be a presentation of M with $a_1 < a_2 < \ldots < a_r$ and $c_1 <$

$c_2 < \ldots < c_r$, and let M_0 be the lattice path matroid with presentation $[a_1+1, c_1], [a_2+1, c_2], \ldots, [a_r+1, c_r]$. Since M has no coloops, $a_i < c_i$ for all i, hence M_0 is well-defined. It follows that the number of bases of M_0 is the number of bases of M with 0 external activity, which is $h_r(M)$. Let n denote the nullity of M, and note that $n = c_r - r$. For $b_i \in [a_i, c_i]$, let $m(b_i) = 1$ if $b_i = a_i$ and $x_{b_i - i}$ otherwise. For a basis $B = b_1 < b_2 < \ldots < b_r$ of M_0, let η_B be the degree $r - k$ monomial $m(b_1)m(b_2)\ldots m(b_r)$, where k is the internal activity of B.

It is easy to visualize the monomial $\eta(B)$: Above each column of the lattice path representation of M, place a variable. Then $\eta(B)$ has one occurence of every variable corresponding to a north step of B, except when this step coincides with the leftmost path. Figure 5.2 gives an example of this: Every north step in the basis B contributes to the monomial $\eta(B)$, except for the third one (since it coincides with the leftmost path).

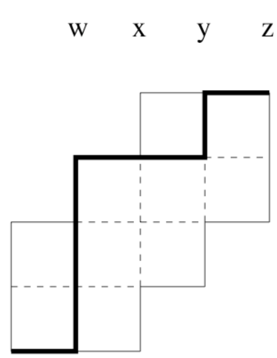

Figure 5.2: A basis B with $\eta(B) = w^2y$.

Finally, let Δ_{M_0} be the monomial order ideal generated by the set $\{\eta_B : B$ is a basis of $M_0\}$. Δ_{M_0} is pure, since it is generated by degree-r monomials.

Let $B = b_1 < b_2 < \ldots < b_r$ be a basis with t-many internally active elements. Then B contributes 1 to h_{r-t}, and η_B is necessarily a degree-$(r-t)$ monomial. To see that η_B divides some monomial in Δ_{M_0}, let $B' = b'_1 < b'_2 < \ldots < b'_r$ by $b'_i = a_i + 1$ if $b_i = a_i$ and $b'_i = b_i$ otherwise. Since $b_i + 1 = b_{i+1}$ implies that $b_{i+1} = a_{i+1}$, it follows that this process

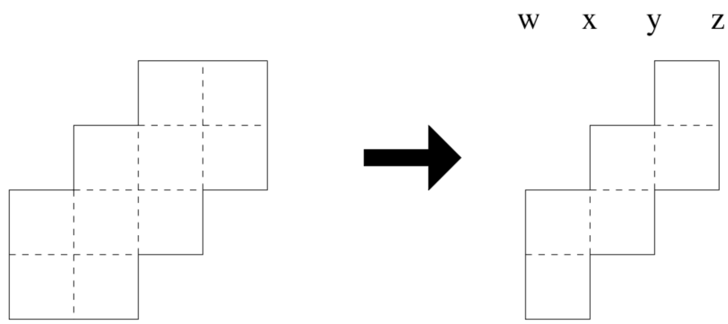

Figure 5.3: Producing M_0 from M.

returns a basis. Furthermore, B' is a basis of M_0, and $\eta_B | \eta_{B'}$, as desired.

We now wish to show that if B is a basis with t-many internally active elements and ν is a monomial dividing η_B with degree(ν) =degree(η_B) $- 1$, then $\nu = \eta_{B'}$ for some basis B' of M. Let $x\nu = \eta_B$, and let $b_i \in B$ be the least $b_j \in B$ such that $m(b_j) = x$. Let $k = \max A$, where $A = \{j : j \leq i \text{ and } b_{j-1} \leq a_j - 1\}$ (we later address the case in which $A = \emptyset$). Define a new basis $B' = b'_1 < b'_2 < \ldots < b'_r$ by $b'_j = b_j$ if $j < k$ or $j > i$, $b'_k = a_k$, and $b'_j = b_{j-1} + 1$ otherwise. First, note that B' is a basis, since $b_{k-1} = b'_{k-1} < a_k = b'_k$. Next, note that the internally active elements of B' are exactly those of B plus b_k. This is because the only elements changed in B' are those b_j with $k \leq j \leq i$, and whenever $k < j \leq i$ $b'_j = b_{j-1} + 1 > a_j$, meaning b'_j is not internally active. For all j with $k < j \leq i$, $m(b'_j) = x_{b'_j - j} = x_{(b_{j-1}+1)-j} = x_{b_{j-1}-(j-1)} = m(b_{j-1})$. It follows that $\eta_{B'} = \nu$. Now suppose that $A = \emptyset$. Simply define $B' = b'_1 < b'_2 < \ldots < b'_r$ by $b'_1 = a_1$, $b'_j = b_{j-1}$ whenever $1 < j \leq i$, and $b'_j = b_j$ for $j > i$. A similar argument shows that $\eta_B = \nu$ in this case as well.

Thus there is a $1-1$ correspondence between degree-d monomials in Δ_{M_0} and bases of M with $(r-d)$-many internally active elements, which are exactly the bases contributing to $h_d(M)$. This proves our theorem. □

The above argument has an easy geometric interpretation: Let ν and η_B be as above. We wish to produce a lattice path corresponding to some B' such that $\nu = \eta_{B'}$. As above, let $x = \eta_B/\nu$. Take the lattice path corresponding to η_B, and remove the lowest north step of this path that contributes x to η_B.

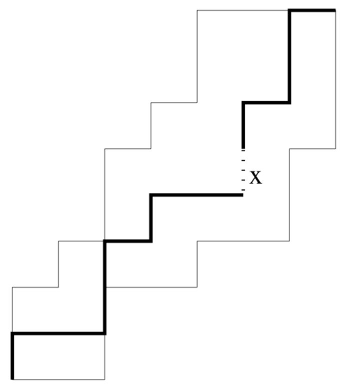

Figure 5.4: Removing the lowest step in B that corresponds to x in η_B.

What results is an incomplete lattice path (since we have removed a step). Now make a note of the highest point beneath the removed step at which the path touches (not necessarily coincides with) the leftmost path. In Figure 5.5, this point is noted with an arrow.

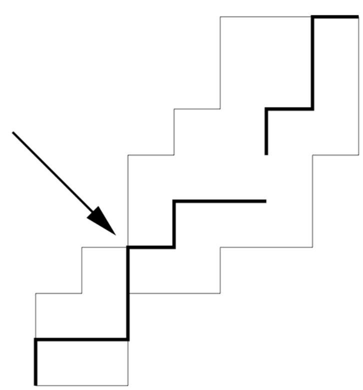

Figure 5.5: The point in question.

Cut the path at this point, and move the top piece up one unit step, as shown in Figure 5.6.

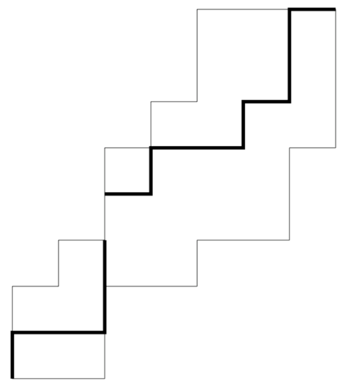

Figure 5.6: Moving the piece up one step.

Finally, fill in the missing step in the resulting path. Since this new path coincides with the leftmost path at the filled-in step, its monomial is ν, as desired.

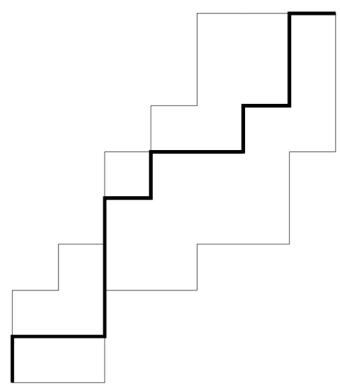

Figure 5.7: The resulting path, whose associated monomial is ν.

Definition 5.2.4 *If M is a lattice path matroid, let Δ_M be the associated pure monomial order ideal, as defined in the proof of Theorem 5.2.3. We call Δ_M the lpm-ideal associated to M.*

The proof of Theorem 5.2.3 actually shows slightly more than Stanley's conjecture: It shows that the h-vector of any coloop-free lattice path matroid is the M-vector of some lpm-ideal (namely Δ_{M_0}). Macaulay's Theorem (see [23]) provides an arithmetic characterization of all M-vectors, yet no such characterization of all *pure* M-vectors is known. This prompts the following question:

Question 5.2.5 *Can an arithmetic characterization of all M-vectors associated to lpm-ideals be found?*

5.3 The Relationship to Shifted Complexes

Definition 5.3.1 *A simplicial complex Δ is shifted if there exists an ordering $v_1 < v_2 < \ldots < v_n$ of its vertex set such that $F \setminus \{v_j\} \cup \{v_i\}$ is a face of Δ whenever F is a face and $i < j$. That is, swapping a vertex in a face of Δ for a vertex earlier in the ordering always results in another face.*

One main reason for the introduction of shifted simplicial complexes is the following theorem of Kalai (see, for instance, [12]):

Theorem 5.3.2 *Let Δ be a simplicial complex. Then there exists a shifted complex Δ' with the same f-vector (and thus the same h-vector) as Δ.*

For the remainder of this section, we work exclusively with pure complexes, and we fix a vertex set V with ordering $v_1 < v_2 < \ldots < v_n$. The *poset of shifted d-sets*, which we write as Φ_d, is the poset of all subsets of V of cardinality d, where B covers A if $A = B \setminus \{v_j\} \cup \{v_i\}$ and $i < j$. Φ_d obviously depends on the set V, but since we work with the same V for this whole section, we suppress it from the notation. It is easy to see that this cover relation gives rise to a partial order. Indeed, the only problem would be if there exist $A_1, A_2, \ldots, A_m$ such that $A_1 < A_2 < \ldots < A_m < A_1$. For $A \in \Phi_d$ with

$A = \{v_{i_1}, v_{i_2}, \ldots, v_{i_d}\}$, let $\sum A = \sum_{j=1}^{d} i_j$. Then $\sum A < \sum B$ whenever B covers A, so the above situation cannot happen.

For any set $\mathcal{A} = \{A_i\}$ of d-sets of V, let $tr(\mathcal{A})$ denote the transitive closure of $\mathcal{A}$. That is, $tr(\mathcal{A}) = \{B : B \subseteq A \text{ for some } A \in \mathcal{A}\}$. By definition, $tr(\mathcal{A})$ is a simplicial complex whose vertex set is contained in V.

If P is a poset with a least element $\hat{0}$, a *principal* order ideal of P is an interval of the form $[\hat{0}, x]$ for some $x \in P$. Klivans proved the following in [14]:

Theorem 5.3.3 *Let M be a rank d matroid with point set V whose independence complex is shifted. Then $\Delta(M)$ is isomorphic to $tr(I)$ for some principal order ideal $I \subseteq \Phi_d$. Conversely, $tr(I)$ is a matroid complex for any principal order ideal $I \subseteq \Phi_d$.*

In order to talk about a principal order ideal of Φ_d, we need to know that it has a least element. But this is clearly the set $\{v_1, v_2, \ldots, v_d\}$. Call this set $F_{\hat{0}}$. Similarly, Φ_d has a largest element, $F_{\hat{1}} = \{v_{n-d+1}, v_{n-d+2}, \ldots, v_n\}$.

The above theorem shows that, to every $F \in \Phi_d$, we can associate a shifted matroid complex $tr([F_{\hat{0}}, F])$, where $[F_{\hat{0}}, F]$ is the principal order ideal in Φ_d generated by F. Let M_F be the associated shifted matroid. In fact, every shifted matroid is a lattice path matroid. To see this, let M_F be a shifted matroid, with $F = \{v_{i_1}, v_{i_2}, \ldots, v_{i_d}\}$ and $i_1 < i_2 < \ldots < i_d$. Let the basis $B_a = \{1, 2, \ldots, d\}$, and let $B_c = \{i_1, i_2, \ldots, i_d\}$. We leave to the reader the easy task of verifying that M_F is the lattice path matroid specified by p_{B_a} and p_{B_c}.

Similarly, let $F \in \Phi_d$. Then the complex $tr([F, F_{\hat{1}}])$ is isomorphic to the lattice path matroid specified by the paths p_{B_a} and p_{B_b}, where B_a is the basis corresponding to F and B_c is the basis corresponding to $F_{\hat{1}}$. Call such a complex a *reverse*-shifted matroid. An example of such a matroid is given in Figure 5.9.

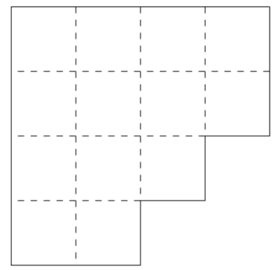

Figure 5.8: The shifted matroid determined by the facet $\{3,5,7,8\}$.

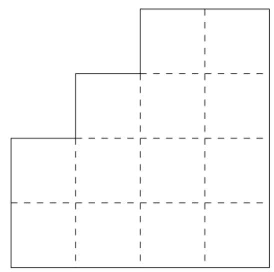

Figure 5.9: The reverse shifted matroid determined by the facet $\{1,2,4,6\}$.

Finally, consider an interval $[F,F']$ in Φ_d. Because $[F,F'] = [F_{\hat{0}},F'] \cap [F,F_{\hat{1}}]$, the complex $tr([F,F'])$ is isomorphic to the lattice path matroid specified by p_{B_a} and p_{B_c}, where B_a corresponds to F and B_c corresponds to F'.

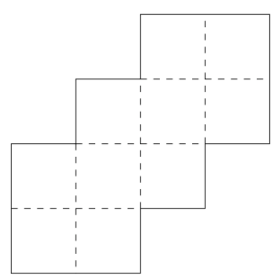

Figure 5.10: The lattice path matroid corresponding to $[\{1,2,4,6\},\{3,5,7,8\}]$.

Recalling Definition 5.2.4, we obtain the following corollary to the above characterization of lattice path matroids:

Corollary 5.3.4 *A sequence* $(h_0, h_1, \ldots, h_d)$ *is the M-vector of some lpm-ideal if and only if there exists some interval* $[F, F'] \subseteq \Phi_d$ *such that the above sequence is the h-vector of* $tr([F, F'])$*.*

Thus, understanding the combinatorics of lattice path matroids is a natural direction in the study of shifted simplicial complexes.

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