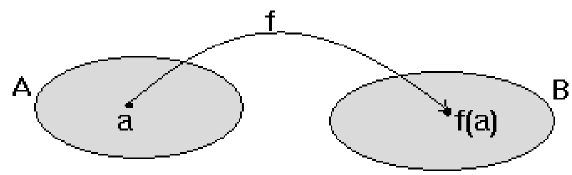
LECTURE 4

Real-Valued Functions

Recall that a function is simply a rule for associating with each element of a set A an element of another set B. We often represent such an abstract function pictorially as



We can represent the same situation with the notation

$$f : A \to B \quad : \quad a \mapsto f(a)$$

which indicates that f is a function that takes elements of the set A to elements of the set B according to the rule $a \in A$ maps to $f(a) \in B$. The set A is referred to as the *domain* of the function f and the set B is referred to as the *target* space and the set

$$\{b \in B \mid b = f(a) \text{ for some } a \in A\}$$

is called the *image* (or *range*) of f.

When the set A is actually a subset of $\mathbb{R}^n$ then we say that $f : \mathbb{R}^n \to B$ is a *function of several variables* (even if the dimension n is huge). If the target space B is $\mathbb{R}$, the set of real numbers, then we say that $f : A \to \mathbb{R}$ is a *real-valued function (of several variables)*.

In this course we shall be studying primarily real-valued functions of several variables. Such functions appear naturally in a plethora of applications (indeed, that's probably why you're supposed to take this course). For example, if we take the temperature of each point (x, y, z) in this room, we would end up with an association

$$\begin{aligned} \text{point with coordinates } (x,y,z) \quad &\Rightarrow \quad \text{the temperature at that point} \\ (x,y,z) \quad &\Rightarrow \quad T(x,y,z) \end{aligned}$$

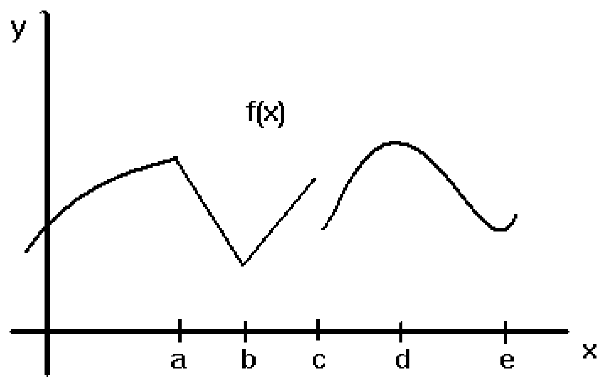
and so a real-valued function of three variables

$$T : \mathbb{R}^3 \to \mathbb{R} \quad : \quad (x,y,z) \mapsto T(x,y,z)$$

The special case of real-valued functions of a single variable was the focus of discussions in Calculus I, and there we saw that graphical methods do much to enhance our intuition about the behavior of such functions. Recall that the graph of a function $f : \mathbb{R} \to \mathbb{R} \ : \ x \mapsto f(x)$ is the set of points (x, y) in the plane satisfying $y = f(x)$

$$\text{graph of } f = \left\{ (x,y) \in \mathbb{R}^2 \mid y = f(x) \right\}$$

Just by looking at the graph of a function we can readily identify properties of that function: for example from the graph



we see that $f(x)$ has

- a discontinuity at $x = c$
- derivatives everywhere except at the points a, b, and c
- local maxima at the points a and d
- local minima at the points b and e

As we shall see graphical methods are also very useful in elucidating the behavior of real-valued functions of several variables.

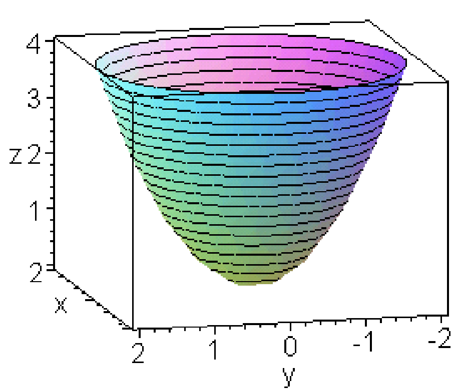
1. Graphs of Real-Valued Functions of Several Variables

DEFINITION 4.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function of several variables. The **graph** of f is the set of points in $\mathbb{R}^{n+1}$ of the form*

$$\left\{(x_1, x_2, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = f(x_1, x_2, \ldots, x_n)\right\}$$

EXAMPLE 4.2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R} \;:\; (x, y) \mapsto x^2 + y^2$. The graph of f is the set of points

$$\left\{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\right\}$$



Unfortunately, the use of graphs to visual functions is really only effective for functions with 1 or 2 variables. To draw the graph of a function of 3 variables requires a 4-dimensional space to draw the picture (and this is pretty hard to even imagine for most people).

2. Level Curves and Surfaces

There is another method for visualizing functions that does not require us to leap to a higher dimension. This method use the notion of level surfaces which I'll now define:

DEFINITION 4.3. *A **level surface** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a set of points in the domain $\mathbb{R}^n$ of f of the form*

$$S_k = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid f(x_1, x_2, \dots, x_n) = k\}$$

In the case where $n = 2$ (i.e., when f is a function of two variables) level surfaces is just the solution set for a equation of the form

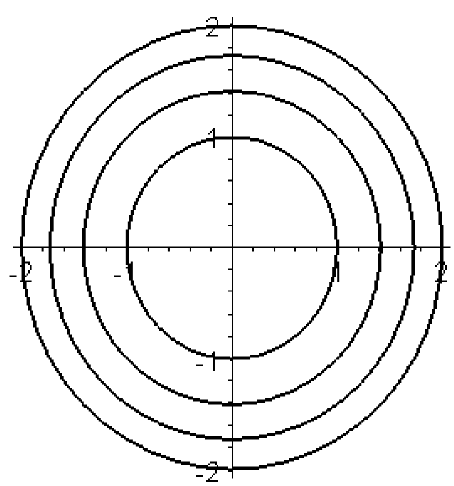
$$f(x, y) = k$$

which will just be some curve in $\mathbb{R}^2$. In general, the solution set of an equation of the form $f(x_1, x_2, \dots, x_n) = k$ will be a surface of dimension $n - 1$ in $\mathbb{R}^n$.

Suppose f is the function $f(x, y) = x^2 + y^2$. Let's sketch the level curves S_k for $k = 0, 1, 2, 3, 4, \dots$. Clearly the solutions of

$$k = f(x, y) = x^2 + y^2$$

are just circles of radius $\sqrt{k}$. And so the level curves of f look like



Note that the spacing between the curves gets smaller as we move away from the origin. This is because the function $f(x, y) = x^2 + y^2$ is increasing more and more rapidly the further we go from the origin (it takes less and less distance for the value of f to increase by 1 as we move away from the origin). This gives us a clue as to what the graph of $f(x, y)$ must look like. If we make a plot the level curves S_k in such a way that the change in k between successive curves is constant, then the places where the neighboring curves are close together will correspond to places where f is changing most rapidly. In fact what we obtain in this fashion is a **contour map** of the graph of f.