

LECTURE 2

Vectors and Vector Spaces, Cont'd

1. Equations of Lines and Planes

1.1. Equation of a Line in \mathbb{R}^3 . There are two common geometrical ways of describing a straight line in a 3-dimensional space.

- Given two distinct points $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^3$, there is a unique line passing through both \mathbf{p}_1 and \mathbf{p}_2 .
- Given one point $\mathbf{p} \in \mathbb{R}^3$ and a direction \mathbf{v} , there is a unique line passing through \mathbf{p}_0 with the direction \mathbf{v} .

In this course, we shall think of a lines sets of points of the following form

$$(2.1) \quad \mathbf{L} = \{ \mathbf{p} \in \mathbb{R}^3 \mid \mathbf{p} = \mathbf{p}_0 + \mathbf{v}t \quad , \quad t \in \mathbb{R} \}$$

The connection with the second geometrical description of a line is evident from the notation. To make the connection with the first geometrical description, all we have to do is set $\mathbf{p}_0 = \mathbf{p}_1$ and set $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$.

If we express the vectors \mathbf{p}, \mathbf{p}_0 , and \mathbf{v} in terms of components; e.g.

$$\begin{aligned} \mathbf{p} &= (x, y, z) \\ \mathbf{p}_0 &= (x_0, y_0, z_0) \\ \mathbf{v} &= (v_x, v_y, v_z) \end{aligned}$$

the we obtain from (??) the following *parametric equation* for a line

$$\begin{aligned} x &= x_0 + v_x t \\ y &= y_0 + v_y t \\ z &= z_0 + v_z t \end{aligned}$$

In terms of the components of two points $\mathbf{p}_1 = (x_1, y_1, z_1)$ and $\mathbf{p}_2 = (x_2, y_2, z_2)$ lying on the line we have corresponding to the first geometrical description of a line the following *parametric equation*

$$\begin{aligned} x &= x_1 + (x_2 - x_1)t \\ y &= y_1 + (y_2 - y_1)t \\ z &= z_1 + (z_2 - z_1)t \end{aligned}$$

1.2. Equation of a Plane in \mathbb{R}^3 . Just as a line can be prescribed by specifying its direction and a single point on the line; a *plane* can be prescribed by specifying a single point \mathbf{p}_0 lying in the plane and two distinct directions \mathbf{v}, \mathbf{u} lying in the plane. In vector notation such a prescription takes the form

$$\mathbf{P} = \{ \mathbf{p} \in \mathbb{R}^3 \mid \mathbf{p} = \mathbf{p}_0 + \mathbf{u}s + \mathbf{v}t \quad , \quad s, t \in \mathbb{R} \}$$

If we set

$$\begin{aligned}\mathbf{p} &= (x, y, z) \\ \mathbf{u} &= (u_x, u_y, u_z) \\ \mathbf{v} &= (v_x, v_y, v_z)\end{aligned}$$

then the relation $\mathbf{p} = \mathbf{p}_0 + \mathbf{u}s + \mathbf{v}t$ leads to the following *parametric representation* of a plane

$$\begin{aligned}x &= x_0 + u_x s + v_x t \\ y &= y_0 + u_y s + v_y t \\ z &= z_0 + u_z s + v_z t\end{aligned}$$

Another way of prescribing a plane is to specify one point \mathbf{p}_0 lying in the plane and the direction of a vector \mathbf{n} that is perpendicular to the plane. If another point \mathbf{p}_1 is to lie in the plane, the vector from \mathbf{p}_0 to \mathbf{p}_1 must be perpendicular to \mathbf{n} , since \mathbf{n} is perpendicular to every direction in the plane. In terms of vector notation we must have

$$0 = \mathbf{n} \cdot (\mathbf{p}_1 - \mathbf{p}_0)$$

If we set

$$\begin{aligned}\mathbf{n} &= (n_x, n_y, n_z) \\ \mathbf{p}_1 &= (x, y, z) \\ \mathbf{p}_0 &= (x_0, y_0, z_0)\end{aligned}$$

then we have

$$0 = \mathbf{n} \cdot (\mathbf{p}_1 - \mathbf{p}_0) = n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0)$$

1.3. Applications.

EXAMPLE 2.1. Find the line passing through the point $(3, 1, -2)$ that intersects the line \mathbf{l}_0

$$\begin{aligned}x &= -1 + t \\ y &= -2 + t \\ z &= -1 + t\end{aligned}$$

perpendicularly.

- The vector equation for the line \mathbf{l}_0 is

$$\mathbf{l}_0 = (-1, -2, -1) + t(1, 1, 1)$$

and so the direction of the line \mathbf{l}_0 is $\mathbf{v}_0 = (1, 1, 1)$. If \mathbf{l} is a line through the point $(3, 1, -2)$ then it has an equation of the form

$$\mathbf{l} = (3, 1, -2) + t\mathbf{v}$$

Now if \mathbf{l} intersects \mathbf{l}_0 perpendicularly the direction \mathbf{v} of \mathbf{l} must be perpendicular to the direction $(1, 1, 1)$ of \mathbf{l}_0 . Therefore

$$\begin{aligned}0 &= \mathbf{v} \cdot (1, 1, 1) \\ &= v_x + v_y + v_z\end{aligned}$$

We know also have a point (x, y, z) common to both lines so

$$\begin{aligned}-1 + t &= x = 3 + v_x s \\ -2 + t &= y = 1 + v_y s \\ -1 + t &= z = -2 + v_z s\end{aligned}$$

Note that we can set $s = 1$ if we simultaneously rescale the direction vector \mathbf{v} . We thus arrive at four equations for four unknowns

$$\begin{aligned}v_x + v_y + v_z &= 0 \\v_x - t &= -4 \\v_y - t &= -3 \\v_z - t &= 1\end{aligned}$$

If we sum the last three equations we get

$$v_x + v_y + v_z - 3t = -6$$

or, using the first equation,

$$-3t = -6 \quad \Rightarrow \quad t = 2$$

We then find

$$\begin{aligned}v_x &= -4 + t = -2 \\v_y &= -3 + t = -1 \\v_z &= 1 + t = 3\end{aligned}$$

Thus, $\mathbf{v} = (-2, -1, 3)$ and the equation of the line \mathbf{l} is

$$\mathbf{l} = (3, 1, -2) + t(-2, -1, 3)$$

□

EXAMPLE 2.2. Find the equation of the plane that contains the point $(2, -1, 3)$ and is perpendicular to the line

$$\mathbf{l} = (1, -1, 3) + t(3, -2, 4)$$

- If $\mathbf{p} = (x, y, z)$ is a point on this plane, then line from the point $\mathbf{p}_0 = (2, -1, 3)$ to \mathbf{p} will also lie in the plane and so must be perpendicular to the direction $\mathbf{v} = (3, -2, 4)$ of \mathbf{l} . This leads to the condition

$$\begin{aligned}0 &= \mathbf{v} \cdot (\mathbf{p} - \mathbf{p}_0) \\&= (3, -2, 4) \cdot (x - 2, y - 1, z - 3) \\&= 3x - 6 - 2y + 2 + 4z - 12 \\&= 3x - 2y + 4z - 16\end{aligned}$$

The equation of the plane is thus

$$3x - 2y + 4z = 16$$

□

EXAMPLE 2.3. Find the equation of the plane containing the lines

$$\begin{aligned}\mathbf{l}_1 &= (0, 1, 1) + t(1, 2, 1) \\ \mathbf{l}_2 &= (0, 1, 1) + s(1, -1, 1)\end{aligned}$$

- The direction of the first line is $\mathbf{v}_1 = (1, 2, 1)$, the direction of the second line is $\mathbf{v}_2 = (1, -1, 1)$, and the direction that is perpendicular to both these lines is

$$\begin{aligned}\mathbf{n} &= \mathbf{v}_1 \times \mathbf{v}_2 \\&= (1, 2, 1) \times (1, -1, 1) \\&= ((2)(1) - (1)(-1), (1)(1) - (1)(1), (1)(-1) - (2)(1)) \\&= (3, 0, 3)\end{aligned}$$

Every other vector in the plane must also be perpendicular to \mathbf{n} .

Since the point $(0, 1, 1)$ lies in the line \mathbf{l}_1 which in turn lies in the plane, $\mathbf{p}_0 = (0, 1, 1)$ is a point lying in the plane. If $\mathbf{p} = (x, y, z)$ is any other point in the plane, then the displacement vector $\mathbf{p} - \mathbf{p}_0 = (x, y - 1, z - 1)$ must also lie in the plane and must be perpendicular to \mathbf{n} . Therefore

$$\begin{aligned} 0 &= \mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) \\ &= (3, 0, 3) \cdot (x, y - 1, z - 1) \\ &= 3x + 3z - 3 \end{aligned}$$

The equation of the plane is thus

$$3x + 3z = 3$$

□