Distribution of zeros for random Laurent rational functions

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Abstract

We study the asymptotic distribution of zeros for the random rational functions that can be viewed as partial sums of a random Laurent series. If this series defines a random analytic function in an annulus A, then the zeros accumulate on the boundary circles of A, being equidistributed in the angular sense, with probability one. We also show that the equidistribution phenomenon holds if the annulus of convergence degenerates to a circle. Moreover, equidistribution of zeros still persists when the Laurent rational functions diverge everywhere, which is new even in the deterministic case. All results hold under two types of general conditions on random coefficients. The first condition is that the random coefficients are non-trivial i.i.d. random variables with finite $\log^+$ moments. The second condition allows random variables that need not be independent or identically distributed, but only requires certain uniform bounds on the tails of their distributions.

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1 Introduction

Zeros of the polynomials $P_n(z) = \sum_{k=0}^n A_k z^k$ with random coefficients have been extensively studied since 1930s. The early history of this subject includes work of Bloch and Pólya, Littlewood and Offord, Erdős and Offord, Rice, Kac, and many others; see, e.g., Bharucha-Reid and Sambandham [9], and Farahmand [12]. It is well known that the bulk of zeros for these polynomials accumulate near the unit circumference, being equidistributed in the angular sense, under mild conditions on the probability distribution of the coefficients. Let $\{Z_k\}_{k=1}^n$ be the zeros of a polynomial P_n of degree n, and define the *zero counting measure*

$$\tau_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{Z_k}.$$

The fact of equidistribution for the zeros of random polynomials is expressed via the weak convergence of τ_n to the normalized arclength measure $\mu_{\mathbb{T}}$ on the unit circumference $\mathbb{T}$, where $d\mu_{\mathbb{T}}(e^{it}) := dt/(2\pi)$. Namely, we have that $\tau_n \xrightarrow{w} \mu_{\mathbb{T}}$ with probability 1 (abbreviated as a.s. or almost surely). Ibragimov and Zaporozhets [16] proved that if the coefficients are independent and identically distributed non-trivial random variables, then the condition $\mathbb{E}[\log^+ |A_0|] < \infty$ is necessary and sufficient for $\tau_n \xrightarrow{w} \mu_{\mathbb{T}}$ almost surely. Here, $\mathbb{E}[X]$ denotes the expectation of a random variable X, and X is called non-trivial if $\mathbb{P}(X = 0) < 1$. Further related results are found in papers of Kabluchko and Zaporozhets [17, 18], etc.

The study of asymptotic distribution of zeros for deterministic polynomials dates back to early 1900s, including well known results of Jentzsch and Szegő, see Andrievskii and Blatt [1] for an overview. In this paper, we use the knowledge accumulated in the study of zeros for deterministic polynomials and rational functions. In particular, we apply the ideas of Edrei [11] and their developments in [20] and [19] to study random Laurent rational functions.

We also mention a related and well developed topic of almost sure equidistribution of zeros for random orthogonal polynomials. These questions were considered by Shiffman and Zelditch [25]-[26], Bloom [5] and [6], Bloom and Shiffman [8], Bloom and Levenberg [7], Bayraktar [3] and others. Bayraktar [4] recently studied zero distribution of the multivariate random Laurent polynomials associated with a fixed Newton polytope. The author [21, 22] also considered zero distribution for random polynomials spanned by general bases.

2 Zeros of Random Laurent Rational Functions

Let $\{a_k\}_{k=0}^\infty$ and $\{b_k\}_{k=1}^\infty$ be deterministic sequences of complex numbers such that

$$\lim_{k\to\infty} |a_k|^{1/k} = 1/R \quad \text{and} \quad \lim_{k\to\infty} |b_k|^{1/k} = r. \tag{2.1}$$

We consider complex random variables $\{A_k\}_{k=0}^\infty$ and $\{B_k\}_{k=1}^\infty$ that satisfy the same conditions as used by Ibragimov and Zaporozhets [16]:

$$\{A_k\}_{k=0}^\infty \text{ and } \{B_k\}_{k=1}^\infty \text{ are i.i.d. with } \mathbb{P}(A_0 = 0) < 1 \text{ and } \mathbb{E}[\log^+ |A_0|] < \infty. \tag{2.2}$$

The main goal of this paper is the study of zeros for the random Laurent rational functions

$$L_{m,n}(z) = \sum_{k=0}^{m} A_k a_k z^k + \sum_{k=1}^{n} B_k b_k z^{-k}, \quad m, n \in \mathbb{N}. \tag{2.3}$$

In particular, we establish the almost sure weak limits of normalized zero counting measures

$$\tau_{m,n} = \frac{1}{m+n} \sum_{k=1}^{m+n} \delta_{Z_k} \tag{2.4}$$

for such Laurent rational functions.

Let μ_s be the normalized angular measure $d\theta/(2\pi)$ on the circle $\{z \in \mathbb{C} : |z| = s > 0\}$.

Theorem 2.1. *Suppose that the coefficients of the Laurent rational functions* (2.3) *satisfy* (2.1) *and* (2.2). *If* $0 < r < R < \infty$ *and* $\lim_{m,n\to\infty} m/(m+n) = \alpha$ *along a sequence of Laurent rational functions, then their zero counting measures satisfy* $\tau_{m,n} \xrightarrow{w} \alpha\mu_R + (1-\alpha)\mu_r$ *as* $m, n \to \infty$ *with probability one.*

A sequence of rational functions with m/n having a limit is often called *a ray sequence*. We actually deal with two sequences $\{m_j\}_{j=0}^{\infty}$ and $\{n_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} m_j/(m_j+n_j) = \alpha$, but the index j will be suppressed in our paper to simplify the notation. This theorem is an analog of earlier deterministic results on the zeros of partial sums of Laurent series due to Edrei [11]. Our proof is based on the extensions of Edrei's results to general Laurent-type rational functions obtained in [20]. The sequence of Laurent rational functions considered in Theorem 2.1 can also be viewed as a sequence of partial sums of random Laurent series convergent in the annulus $A := \{z \in \mathbb{C} : r < |z| < R\}$ to a random analytic function. It is possible to show along the lines of Ibragimov and Zaporozhets [16] that if $\mathbb{E}[\log^+ |A_0|] = \infty$, then the limiting measure for $\tau_{m,n}$ may be different from $\alpha\mu_R + (1-\alpha)\mu_r$. In fact, $\mathbb{E}[\log^+ |A_0|] = \infty$ holds if and only if $\limsup_{n\to\infty} |A_n|^{1/n} = \infty$ with probability one, see [2], [16], etc. This leads to divergent power series (with radius of convergence equal to zero). Zeros of divergent random power series were studied in [16] and [17], while the classical deterministic case was considered by Rosenbloom [24] (cf. Theorem XVIII on pages 40–41), and by Dilcher and Rubel [10].

We also have the following interesting companion for the above result, where two parts of the limiting measure for the zeros of Laurent rational functions merge into one.

Theorem 2.2. *If under the assumption of Theorem 2.1 we have* $r = R$, *then* $\tau_{m,n} \xrightarrow{w} \mu_R$ *with probability one.*

The latter case is closely related to the zero distribution of random trigonometric polynomials. It is clear that

$$L_{m,n}(e^{i\theta}) = \sum_{k=0}^{m} A_k a_k e^{ik\theta} + \sum_{k=1}^{n} B_k b_k e^{-ik\theta} = T_{m,n}(\theta), \quad m, n \in \mathbb{N},$$

are trigonometric polynomials in $\theta \in [0, 2\pi)$. For example, if $a_k = b_k = 1$ and $A_0 \in \mathbb{R}$, $A_k = B_k \in \mathbb{R}$ for all $k \in \mathbb{N}$, then

$$L_{n,n}(e^{i\theta}) = A_0 + 2\sum_{k=1}^{n} A_k \cos\theta =: t_n(\theta), \quad n \in \mathbb{N}.$$

Thus the zeros of t_n in $[0, 2\pi)$ correspond to the zeros of $L_{n,n}(z)$ on the unit circumference. It was first proved by Dunnage (see [12] for history and further results) that random cosine polynomials t_n with real Gaussian coefficients have asymptotically $2n/\sqrt{3}$ expected zeros on $[0, 2\pi)$. Hence the bulk of zeros of $L_{n,n}(z)$ is equidistributed near $\mathbb{T}$, while the mentioned fraction of zeros is expected to lie exactly on $\mathbb{T}$.

We remark that there is an overlap between Theorem 2.2 and Theorem 1.2 in [4].

We now consider a degenerate case when the deterministic coefficients a_k, b_k satisfy (2.1) with $r > R$. Our random Laurent rational functions diverge with probability one everywhere in $\mathbb{C}$ under such assumptions. In fact, this case was not studied even for the zeros of partial sums of the classical Laurent series. The random Laurent rational functions (2.3) reduce to the partial sums of a standard Laurent series by setting $A_k = 1$ and $B_k = 1$ for all k with probability one.

Theorem 2.3. *Suppose that the coefficients of the Laurent rational functions* (2.3) *satisfy* (2.1) *and* (2.2). *If $r > R$ are finite and $\lim_{m,n\to\infty} m/(m+n) = \alpha$, then $\tau_{m,n} \xrightarrow{w} \mu_\rho$ as $m, n \to \infty$ with probability one, where $\rho = R^\alpha r^{1-\alpha}$.*

It is of interest that zeros preserve very regular behavior despite divergent nature of Laurent rational functions. But now the circle of accumulation for zeros depends on the choice of a "ray sequence," i.e., on the asymptotic proportion of positive and negative powers in a particular sequence of Laurent rational functions.

We note that the conditions on random variables in (2.2) can be relaxed in several ways, while all results of this section remain valid. Thus our proofs use joint independence of the sequence $\{|A_k|\}_{k=0}^\infty$, and separately joint independence of $\{|B_k|\}_{k=1}^\infty$. Moreover, while both $\{|A_k|\}_{k=0}^\infty$ and $\{|B_k|\}_{k=1}^\infty$ are required to be identically distributed, the common distributions for these two sequences may be different.

Another kind of condition, preserving all our results as stated above, allow dependence and different distributions for random variables. Let the distribution function of $|A_k|$ be defined by $F_k(x) = \mathbb{P}(\{|A_k| \le x\}),\ x \in \mathbb{R}$. We can replace (2.2) with the following assumptions:

There is $N \in \mathbb{N}$, a decreasing function $f : [a,\infty) \to [0,1],\ a > 1$, and an increasing function $g : [0,b] \to [0,1],\ 0 < b < 1$, such that

$$\int_a^\infty \frac{f(x)}{x}\,dx < \infty \text{ and } 1 - F_k(x) \le f(x),\ \forall\, x \in [a,\infty), \tag{2.5}$$

and

$$\int_0^b \frac{g(x)}{x}\,dx < \infty \text{ and } F_k(x) \le g(x),\ \forall\, x \in [0,b], \tag{2.6}$$

hold for all $k \ge N$. We further assume that (2.5) *and* (2.6) *also hold for the distribution functions of the sequence $\{|B_k|\}_{k=1}^\infty$.*

If $F(x)$ is the distribution function of $|X|$, where X is a complex random variable, then

$$\mathbb{E}[\log^+ |X|] < \infty \quad \Leftrightarrow \quad \int_a^\infty \frac{1 - F(x)}{x}\,dx < \infty,\ a \ge 0,$$

and

$$\mathbb{E}[\log^- |X|] < \infty \quad \Leftrightarrow \quad \int_0^b \frac{F(x)}{x}\,dx < \infty,\ b > 0,$$

see, e.g., Theorem 12.3 of Gut [15, p. 76]. Hence when all random variables $|A_k|,\ k = 0, 1, \dots$, are identically distributed, one can state assumptions (2.5)-(2.6) in a more compact equivalent form

$$\mathbb{E}[|\log |A_0||] < \infty. \tag{2.7}$$

Theorem 2.4. *Theorems 2.1, 2.2 and 2.3 remain valid if assumption* (2.2) *on the random coefficients is replaced with* (2.5) *and* (2.6).

3 Proofs

We need several facts about limiting behavior of random coefficients. The first result is well known, and can be found in many papers, see [2], [16], [21], etc. We include its short proof for convenience of the reader.

Lemma 3.1. *If $\{A_k\}_{k=0}^\infty$ are non-trivial, independent and identically distributed complex random variables that satisfy $\mathbb{E}[\log^+ |A_0|] < \infty$, then*

$$\limsup_{n\to\infty} |A_n|^{1/n} = 1 \qquad a.s. \tag{3.1}$$

and

$$\limsup_{n\to\infty} \left(\max_{0\le k\le n} |A_k|\right)^{1/n} = 1 \qquad a.s. \tag{3.2}$$

This follows from the Borel-Cantelli Lemmas stated below (see, e.g., [15, p. 96]) in a standard way.

Borel-Cantelli Lemmas. *Let $\{\mathcal{E}_n\}_{n=1}^\infty$ be a sequence of arbitrary events.*
(i) *If $\sum_{n=1}^\infty \mathbb{P}(\mathcal{E}_n) < \infty$ then $\mathbb{P}(\mathcal{E}_n$ occurs infinitely often$) = 0$.*
(ii) *If events $\{\mathcal{E}_n\}_{n=1}^\infty$ are independent and $\sum_{n=1}^\infty \mathbb{P}(\mathcal{E}_n) = \infty$, then $\mathbb{P}(\mathcal{E}_n$* i.o.$) = 1$.

Proof of Lemma 3.1. For any fixed $\varepsilon > 0$, define events $\mathcal{E}_n = \{|A_n| \ge e^{\varepsilon n}\},\ n \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{n=1}^\infty \mathbb{P}(\mathcal{E}_n) &= \sum_{n=1}^\infty \mathbb{P}(\{\log^+ |A_n| \ge \varepsilon n\}) = \sum_{n=1}^\infty \mathbb{P}\left(\left\{\frac{1}{\varepsilon}\log^+ |A_0| \ge n\right\}\right) \\ &\le \frac{1}{\varepsilon}\mathbb{E}[\log^+ |A_0|] < \infty.\end{aligned}$$

Hence $\mathbb{P}(\mathcal{E}_n$ occurs infinitely often$) = 0$ by the first Borel-Cantelli Lemma, so that the complementary event $\mathcal{E}_n^c$ must happen for all large n with probability 1. This means that $|A_n|^{1/n} \le e^\varepsilon$ for all sufficiently large $n \in \mathbb{N}$ almost surely. We obtain that

$$\limsup_{n\to\infty} |A_n|^{1/n} \le e^\varepsilon \quad \text{a.s.},$$

and since $\varepsilon > 0$ was arbitary, this shows that

$$\limsup_{n\to\infty} |A_n|^{1/n} \le 1 \quad \text{a.s.} \tag{3.3}$$

On the other hand, A_0 is non-trivial, so there exists $c > 0$ such that $\mathbb{P}(|A_0| > c) > 0$. Therefore

$$\sum_{n=1}^{\infty} \mathbb{P}(|A_n| > c) = \infty,$$

since A_n are i.i.d. random variables. Using the second Borel-Cantelli Lemma, it follows that with probability one, there exist infinitely many n such that $|A_n| > c$. Combining this with (3.3), we obtain (3.1). An elementary argument shows that (3.2) is a consequence of (3.1). □

The next lemma shows that the random coefficients cannot be "too small too often" under our assumptions. The specific form given below appears in [23], but its roots are found in [13]. Again, we provide a short proof for the sake of completeness.

Lemma 3.2. *If $\{A_k\}_{k=0}^{\infty}$ are non-trivial i.i.d. complex random variables, then there is $b > 0$ such that*

$$\liminf_{n\to\infty} \left(\max_{n-b\log n<k\le n} |A_k| \right)^{1/n} \ge 1 \quad a.s. \tag{3.4}$$

Proof. We use a modified idea of Fernández [13] in this proof. Let $\alpha_n \le n$, $n \in \mathbb{N}$, be a sequence of natural numbers that will be specified later. Consider

$$M_n := \max_{n-\alpha_n<k\le n} |A_k|.$$

The statement

$$\liminf_{n\to\infty} (M_n)^{1/n} \ge 1 \quad \text{a.s.}$$

is equivalent to

$$\mathbb{P}(\{M_n \le \lambda^n \text{ i.o.}\}) = 0$$

for all positive $\lambda < 1$. The latter would follow from the first Borel-Cantelli Lemma if we show that

$$\sum_{n=1}^{\infty} \mathbb{P}(\{M_n \le \lambda^n\}) < \infty$$

for all positive $\lambda < 1$. Since our variables $\{A_k\}_{k=0}^{\infty}$ are i.i.d., we have

$$\mathbb{P}(\{M_n \le \lambda^n\}) = \mathbb{P}(\{|A_0| \le \lambda^n\})^{\alpha_n}.$$

As A_0 is non-trivial, we can find $c > 0$ and $p \in (0,1)$ such that $\mathbb{P}(\{|A_0| \le c\}) \le p$. Hence for any $\lambda < 1$ there is $N = N(p, \lambda) \in \mathbb{N}$ such that $\mathbb{P}(\{|A_0| \le \lambda^n\}) \le p$ for all $n \ge N$. This gives

$$\sum_{n=1}^{\infty} \mathbb{P}(\{M_n \le \lambda^n\}) \le \sum_{n=1}^{\infty} p^{\alpha_n} < \infty,$$

provided $p^{\alpha_n} \le 1/n^2$ for large n. It suffices to take $\alpha_n \ge (-2/\log p) \log n$ to satisfy the latter condition. $\square$

Deterministic results for the asymptotic zero distribution of Laurent rational functions go back to Edrei [11], and they were generalized in [19] and [20]. The coefficient conditions employed in [19] are not applicable in the setting of Theorem 2.1, but they work well under assumptions (2.5) and (2.6), see the proof of Theorem 2.4 below. Thus our proof of Theorem 2.1 rely on [20], where we combined and generalized ideas from the papers of Edrei [11] and Grothmann [14]. It suffices for our current purposes to consider the special case of Theorem 1.1 from [20] given below.

Theorem 3.3. *Consider an infinite sequence of Laurent rational functions*

$$L_{m,n}(z) = \sum_{k=0}^{m} s_k z^k + \sum_{k=1}^{n} t_k z^{-k}, \quad m, n \in \mathbb{N},$$

such that both $m, n \to \infty$ *and* $\lim_{m,n\to\infty} \dfrac{m}{m+n} = \alpha$ *along this sequence. Given* $0 < r < R < \infty$, *let* $A := \{z : r < |z| < R\}$, $G := \{z : |z| < r\}$ *and* $\Omega := \{z : |z| > R\}$. *Suppose that the sequence* $L_{m,n}$ *converges to* $f \not\equiv 0$ *locally uniformly in* A. *If there is a compact set* $S_1 \subset \Omega$ *such that*

$$\liminf_{m,n\to\infty} \max_{z \in S_1} \frac{|L_{m,n}(z)|^{1/m}}{|z|} \ge \frac{1}{R}, \tag{3.5}$$

and a compact set $S_2 \subset G$ *such that*

$$\liminf_{m,n\to\infty} \max_{z \in S_2} |z|\, |L_{m,n}(z)|^{1/n} \ge r, \tag{3.6}$$

then the normalized zero counting measures of $L_{m,n}$ *satisfy* $\tau_{m,n} \xrightarrow{w} \alpha\mu_R + (1-\alpha)\mu_r$ *as* $m, n \to \infty$.

Proof of Theorem 2.1. The strategy of this proof is to apply Theorem 3.3. We first show that the random Laurent rational functions (2.3) converge uniformly on compact subsets of the annulus A, with probability one, under our assumptions. Define

$$P_m(z) := \sum_{k=0}^{m} A_k a_k z^k \quad \text{and} \quad Q_n(z) := \sum_{k=1}^{n} B_k b_k z^{-k}, \tag{3.7}$$

so that $L_{m,n}(z) = P_m(z) + Q_n(z)$. It follows from (2.1) and (3.1) that

$$\limsup_{k\to\infty} |a_k A_k|^{1/k} = 1/R \quad \text{and} \quad \limsup_{k\to\infty} |b_k B_k|^{1/k} = r \tag{3.8}$$

hold with probability one. Hence P_m represent the partial sums of a random series with radius of convergence R, i.e., P_m converge almost surely to an analytic function $g \not\equiv 0$ in the disk $\{z : |z| < R\}$. Similarly, we obtain after the change of variable $z \to 1/z$ that Q_n converge almost surely to an analytic function $h \not\equiv 0$ in the domain $\{z : |z| > r\}$. It follows that $L_{m,n}$ converge almost surely to an analytic function $f = g + h \not\equiv 0$ in the annulus A. Clearly, this convergence is locally uniform in A. We now show that (3.5) and (3.6) hold for our random Laurent rational functions (2.3) with probability one. Assume for (3.5) that $S_1 = \{z : |z| = R'\}$ with $R' > R$, and use the standard integral representation for the coefficients

$$a_k A_k = \frac{1}{2\pi i} \int_{|z|=R'} \frac{L_{m,n}(z)\, dz}{z^{k+1}}, \quad k = 0, 1, 2, \ldots.$$

It follows that

$$|a_k A_k| \le \frac{\max_{|z|=R'} |L_{m,n}(z)|}{(R')^k}$$

and

$$\max_{|z|=R'} \frac{|L_{m,n}(z)|}{|z|^m} \ge (R')^{k-m} |a_k A_k|, \quad k = 0, 1, 2, \ldots.$$

We choose $b > 0$ as in Lemma 3.2, and use the above inequality together with (3.4) and (2.1) to obtain that

$$\begin{aligned}
\liminf_{m\to\infty} \max_{|z|=R'} \frac{|L_{m,n}(z)|^{1/m}}{|z|} &\ge \liminf_{m\to\infty} \max_{m-b\log m<k\le m} (R')^{k/m-1} |a_k A_k|^{1/m} \\
&= \frac{1}{R} \liminf_{m\to\infty} \max_{m-b\log m<k\le m} |A_k|^{1/m} \ge \frac{1}{R}
\end{aligned}$$

holds with probability one. A similar argument shows that (3.6) holds with probability one for any set $S_2 = \{z : |z| = r'\} \subset G$ with $r' < r$. Indeed, we have

$$|b_k B_k| \le (r')^k \max_{|z|=r'} |L_{m,n}(z)|, \quad k \in \mathbb{N},$$

which gives as before that

$$\begin{aligned}
\liminf_{n\to\infty} \max_{|z|=r'} |z| |L_{m,n}(z)|^{1/n} &\ge \liminf_{n\to\infty} \max_{n-b\log n<k\le n} (r')^{1-k/n} |b_k B_k|^{1/n} \\
&= r \liminf_{n\to\infty} \max_{n-b\log n<k\le n} |B_k|^{1/n} \ge r.
\end{aligned}$$

Hence $\tau_{m,n} \xrightarrow{w} \alpha\mu_R + (1-\alpha)\mu_r$ with probability one by Theorem 3.3. □

We also need a companion of Theorem 3.3 for the case when $r = R$, i.e., when the annulus A degenerates to a circumference. Let $\|\cdot\|_E$ denote the supremum norm on a set E. Assumption (3.9) below replaces the corresponding assumption in Theorem 3.3 that $L_{m,n} \to f \not\equiv 0$ locally uniformly in A.

Theorem 3.4. *Consider an infinite sequence of Laurent rational functions*

$$L_{m,n}(z) = \sum_{k=0}^{m} s_k z^k + \sum_{k=1}^{n} t_k z^{-k}, \quad m,n \in \mathbb{N},$$

such that both $m,n \to \infty$ and $\lim_{m,n\to\infty} \dfrac{m}{m+n} = \alpha$ along this sequence. For $\rho \in (0,\infty)$, define $E := \{z : |z| = \rho\}$, $G := \{z : |z| < \rho\}$ and $\Omega := \{z : |z| > \rho\}$. Suppose that

$$\limsup_{m,n\to\infty} \|L_{m,n}\|_E^{1/(m+n)} \le 1. \tag{3.9}$$

If there is a compact set $S_1 \subset \Omega$ such that

$$\liminf_{m,n\to\infty} \max_{z\in S_1} \frac{|L_{m,n}(z)|^{1/m}}{|z|} \ge \frac{1}{\rho}, \tag{3.10}$$

and a compact set $S_2 \subset G$ such that

$$\liminf_{m,n\to\infty} \max_{z\in S_2} |z|\, |L_{m,n}(z)|^{1/n} \ge \rho, \tag{3.11}$$

then the normalized zero counting measures of $L_{m,n}$ satisfy $\tau_{m,n} \xrightarrow{w} \mu_\rho$ as $m,n \to \infty$, where $d\mu_\rho(\rho e^{i\theta}) = d\theta/(2\pi)$.

We note that the coefficients s_k and t_k in $L_{m,n}$ may depend on m and n, i.e., they may be different in different Laurent rational functions. A proof of Theorem 3.4 is implicitly contained in the proof of Theorem 1.1 from [20]. Since this result is not explicitly mentioned in the literature, we sketch a proof based on Theorem 1.1 of [1, p. 50] for polynomials.

Proof. We first note that the change of variable $z \to z/\rho$ reduces this result to the case $\rho = 1$. We provide the proof for the latter case, i.e., for $E = \mathbb{T} = \{z : |z| = 1\}$. Let d_m be the exact degree of the polynomial part for $L_{m,n}$, so that s_{d_m} is the highest non-zero coefficient. Denote the exact degree of the Laurent part for $L_{m,n}$ by e_n, with $t_{e_n} \ne 0$ being the lowest coefficient. Consider the polynomials $P_{m,n}(z) := z^{e_n} L_{m,n}(z)$ of the exact degree $d_m + e_n \le m + n$, whose zeros coincide with the zeros of $L_{m,n}(z)$. We apply Theorem 1.1 of [1, p. 50] to this sequence of polynomials. Assumption (1.2) of Theorem 1.1 in [1] is identical to our assumption (3.9). It follows from Lemma 4.1 of [20] and (3.9)-(3.11) that

$$\lim_{m,n\to\infty} \frac{d_m}{m} = \lim_{m,n\to\infty} \frac{e_n}{n} = 1.$$

Hence assumption (1.4) of Theorem 1.1 [1] is satisfied by our assumption (3.10) because

$$\liminf_{m,n\to\infty} \max_{z\in S_1} \left(\frac{1}{m+n} \log |P_{m,n}(z)| - \log |z| \right) = \liminf_{m,n\to\infty} \frac{m}{m+n} \log \max_{z\in S_1} \frac{|L_{m,n}(z)|^{1/m}}{|z|} \geq 0.$$

It remains to show that assumption (1.2) of Theorem 1.1 [1] holds, which translates into

$$\lim_{m,n\to\infty} \tau_{m,n}(M) = 0$$

for any compact set $M \subset G = \{z : |z| < 1\}$. This follows from Lemma 1.6 of [1, p. 52] applied to the polynomials $Q_{m,n}(z) := z^{d_m} L_{m,n}(1/z)$ by using our assumptions (3.9) and (3.11). □

Proof of Theorem 2.2. We apply Theorem 3.4 to the random Laurent rational functions (2.3) with $\rho = R$. Observe that

$$\begin{aligned} \|L_{m,n}\|_E &\leq \sum_{k=0}^{m} |A_k a_k| R^k + \sum_{k=1}^{n} |B_k b_k| R^{-k} \\ &\leq (m+1) \max_{0\leq k\leq m} \left(|a_k| R^k\right) \max_{0\leq k\leq m} |A_k| + n \max_{1\leq k\leq n} \left(|b_k| R^{-k}\right) \max_{1\leq k\leq n} |B_k|. \end{aligned} \tag{3.12}$$

Since $R \lim_{m\to\infty} |a_m|^{1/m} = R^{-1} \lim_{n\to\infty} |b_n|^{1/n} = 1$, we have that

$$\lim_{m\to\infty} \left(\max_{0\leq k\leq m} |a_k| R^k \right)^{1/m} = \lim_{n\to\infty} \left(\max_{1\leq k\leq n} |b_k| R^{-k} \right)^{1/n} = 1.$$

Applying these limits together with (3.2) and (3.12), we verify that (3.9) holds with probability one. We now need to show that (3.10) and (3.11) hold with probability one. This is done in essentially the same way as in the proof of Theorem 2.1. Let $S_1 = \{z : |z| = R'\} \subset \Omega$ with $R' > R$, and let $S_2 = \{z : |z| = r'\} \subset G$ with $r' < R$. Then we have

$$\max_{|z|=R'} \frac{|L_{m,n}(z)|}{|z|^m} \geq (R')^{k-m} |a_k A_k|, \quad k = 0, 1, 2, \ldots,$$

and

$$\max_{|z|=r'} |L_{m,n}(z)| \geq (r')^{-k} |b_k B_k|, \quad k \in \mathbb{N}.$$

Selecting $b > 0$ as in Lemma 3.2, and combining the above estimates with (3.4), we obtain that

$$\begin{aligned} \liminf_{m,n\to\infty} \max_{|z|=R'} \frac{|L_{m,n}(z)|^{1/m}}{|z|} &\geq \liminf_{m\to\infty} \max_{m-b\log m<k\leq m} (R')^{k/m-1} |a_k A_k|^{1/m} \\ &= \frac{1}{R} \liminf_{m\to\infty} \max_{m-b\log m<k\leq m} |A_k|^{1/m} \geq \frac{1}{R} \end{aligned}$$

and

$$\begin{aligned}\liminf_{m,n\to\infty} \max_{|z|=r'} |z||L_{m,n}(z)|^{1/n} &\geq \liminf_{n\to\infty} \max_{n-b\log n<k\leq n} (r')^{1-k/n}|b_k B_k|^{1/n} \\ &= R \liminf_{n\to\infty} \max_{n-b\log n<k\leq n} |B_k|^{1/n} \geq R\end{aligned}$$

both hold with probability one. Hence $\tau_{m,n} \xrightarrow{w} \mu_R$ with probability one by Theorem 3.4. $\square$

Proof of Theorem 2.3. We first assume that $\alpha \in (0,1)$, and apply Theorem 3.4 to the random Laurent rational functions

$$\tilde{L}_{m,n}(z) = \left(\frac{r}{R}\right)^{\alpha(\alpha-1)(m+n)} L_{m,n}(z),$$

with $E := \{z : |z| = \rho = R^\alpha r^{1-\alpha}\}$. Note that both the polynomial and the Laurent parts of $L_{m,n}(z)$ diverge in the annulus $\{z : R < |z| < r\}$ as $m,n \to \infty$. The value of ρ is essentially determined by matching the growth rates of the polynomial and the Laurent parts of $L_{m,n}(z)$ on E. It is clear that

$$\begin{aligned}\|L_{m,n}\|_E &\leq \sum_{k=0}^{m} |A_k a_k|\rho^k + \sum_{k=1}^{n} |B_k b_k|\rho^{-k} \qquad (3.13)\\ &\leq (m+1) \max_{0\leq k\leq m} \left(|a_k|\rho^k\right) \max_{0\leq k\leq m} |A_k| + n \max_{1\leq k\leq n} \left(|b_k|\rho^{-k}\right) \max_{1\leq k\leq n} |B_k| \\ &=: p_{m,n} + q_{m,n}.\end{aligned}$$

We use the following elementary inequality

$$\limsup_{k\to\infty} \left(\max_{1\leq j\leq k} |c_j|\right)^{1/l_k} \leq \max\left(\limsup_{k\to\infty} |c_k|^{1/l_k}, 1\right)$$

that holds for any sequence of complex numbers $\{c_k\}_{k=1}^\infty$ and any sequence of natural numbers $\{l_k\}_{k=1}^\infty$ increasing to infinity. Applying this inequality with $c_k = |a_k|\rho^k$ and $l_m = 1/(m+n)$, we obtain by (2.1) that

$$\begin{aligned}\limsup_{m,n\to\infty} \left(\max_{0\leq k\leq m} \left(|a_k|\rho^k\right)\right)^{1/(m+n)} &\leq \max\left(\limsup_{m,n\to\infty} \left(|a_m|\rho^m\right)^{1/(m+n)}, 1\right) \\ &= \max\left(R^{-\alpha}\rho^\alpha, 1\right) = \max\left((R/r)^{\alpha^2-\alpha}, 1\right) \\ &= (R/r)^{\alpha^2-\alpha}.\end{aligned}$$

It follows from the above inequality and (3.2) that

$$\limsup_{m,n\to\infty} \left((r/R)^{\alpha(\alpha-1)(m+n)}\, p_{m,n}\right)^{1/(m+n)} \leq (r/R)^{\alpha(\alpha-1)}(R/r)^{\alpha^2-\alpha} = 1$$

with probability one. A similar argument shows that

$$\limsup_{m,n\to\infty}\left(\max_{0\le k\le n}\left(|b_k|\rho^{-k}\right)\right)^{1/(m+n)} \le \max\left(r^{1-\alpha}\rho^{\alpha-1},1\right) = \max\left((R/r)^{\alpha^2-\alpha},1\right) = (R/r)^{\alpha^2-\alpha},$$

and that

$$\limsup_{m,n\to\infty}\left((r/R)^{\alpha(\alpha-1)(m+n)}\, q_{m,n}\right)^{1/(m+n)} \le 1 \quad \text{a.s.}$$

Hence we verified by (3.13) that (3.9) holds for $\tilde{L}_{m,n}(z)$ with probability one.

We next show that (3.10) and (3.11) also hold for $\tilde{L}_{m,n}(z)$ with probability one. The idea is the same as in the proofs of Theorems 2.1 and 2.2. Consider $S_1 = \{z : |z| = R'\} \subset \Omega$ with $R' > \rho$, and $S_2 = \{z : |z| = r'\} \subset G$ with $r' < \rho$. We again have

$$\max_{|z|=R'} \frac{|L_{m,n}(z)|}{|z|^m} \ge (R')^{k-m}|a_k A_k|, \quad k = 0,1,2,\ldots,$$

and

$$\max_{|z|=r'} |L_{m,n}(z)| \ge (r')^{-k}|b_k B_k|, \quad k \in \mathbb{N}.$$

Choose $b > 0$ as in Lemma 3.2, and use the above estimates with (3.4) to obtain that

$$\begin{aligned}\liminf_{m,n\to\infty}\max_{|z|=R'}\frac{|\tilde{L}_{m,n}(z)|^{1/m}}{|z|} &\ge \liminf_{m,n\to\infty}\left(\frac{r}{R}\right)^{\alpha(\alpha-1)(m+n)/m}\max_{m-b\log m<k\le m}(R')^{k/m-1}|a_k A_k|^{1/m}\\ &= \left(\frac{r}{R}\right)^{\alpha-1}\frac{1}{R}\liminf_{m\to\infty}\max_{m-b\log m<k\le m}|A_k|^{1/m} \ge \frac{1}{R^\alpha r^{1-\alpha}} = \frac{1}{\rho}\end{aligned}$$

and

$$\begin{aligned}\liminf_{m,n\to\infty}\max_{|z|=r'}|z||\tilde{L}_{m,n}(z)|^{1/n} &\ge \liminf_{m,n\to\infty}\left(\frac{r}{R}\right)^{\alpha(\alpha-1)(m+n)/n}\max_{n-b\log n<k\le n}(r')^{1-k/n}|b_k B_k|^{1/n}\\ &= \left(\frac{r}{R}\right)^{-\alpha} r\liminf_{n\to\infty}\max_{n-b\log n<k\le n}|B_k|^{1/n} \ge R^\alpha r^{1-\alpha} = \rho\end{aligned}$$

both hold with probability one. Hence $\tau_{m,n} \xrightarrow{w} \mu_\rho$ with probability one by Theorem 3.4.

The proof is completed by considering the remaining cases $\alpha = 0$ and $\alpha = 1$. If $\alpha = 0$ then we repeat the above proof by applying Theorem 3.4 to the random Laurent rational functions

$$\tilde{L}_{m,n}(z) = \left(\frac{R}{r}\right)^m L_{m,n}(z).$$

For $\alpha = 1$, one needs to apply Theorem 3.4 to the random Laurent rational functions

$$\tilde{L}_{m,n}(z) = \left(\frac{R}{r}\right)^n L_{m,n}(z).$$

□

Proof of Theorem 2.4. The proofs of analogs for Theorems 2.1, 2.2 and 2.3 under assumptions (2.5) and (2.6) are even more direct, because the random coefficients now satisfy

$$\lim_{m\to\infty} |A_m|^{1/m} = \lim_{m\to\infty} \left(\max_{0\le k\le m} |A_k|\right)^{1/m} = \lim_{n\to\infty} |B_n|^{1/n} = \lim_{n\to\infty} \left(\max_{0\le k\le n} |B_k|\right)^{1/n} = 1 \quad (3.14)$$

with probability one, see Lemma 4.2 in [21] or [22]. We immediately obtain that

$$\lim_{m\to\infty} |a_m A_m|^{1/m} = 1/R \quad \text{and} \quad \lim_{n\to\infty} |b_n B_n|^{1/n} = r \quad \text{a.s.} \quad (3.15)$$

Since $0 < r < R < \infty$ in the context of Theorem 2.1, (3.15) implies that the sequence $L_{m,n}$ converges to $f \not\equiv 0$ locally uniformly in $A = \{z : r < |z| < R\}$ with probability one. Furthermore, the conclusion of our Theorem 2.1 now follows from Theorem 2.1 of [19] and (3.15).

Theorems 2.2 and 2.3 follow from Theorem 2.6 of [19] and (3.14)-(3.15) in a routine way. Indeed, one only needs to verify that (2.20) in Theorem 2.6 of [19] holds almost surely. But that condition is identical to (3.9) in this paper, which is verified in the same fashion as in our proofs of Theorems 2.2 and 2.3 by using (3.14)-(3.15). □

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