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SCHINZEL-TYPE BOUNDS FOR THE MAHLER MEASURE ON LEMNISCATES

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Dedicated to the memory of Professor Andrzej Schinzel

Abstract. We study the generalized Mahler measure on lemniscates, and prove a sharp lower bound for the measure of totally real integer polynomials that includes the classical result of Schinzel expressed in terms of the golden ratio. Moreover, we completely characterize many cases when this lower bound is attained. For example, we explicitly describe all lemniscates and the corresponding quadratic polynomials that achieve our lower bound for the generalized Mahler measure. It turns out that the extremal polynomials attaining the bound must have even degree. The main computational part of this work is related to finding many extremals of degree four and higher, which is a new feature compared to the original Schinzel's theorem where only quadratic irreducible extremals are possible.

1. Mahler measure and its counterpart for lemniscates

For any nonzero polynomial

$$P(z) = a_n z^n + \cdots + a_1 z + a_0 = a_n \prod_{k=1}^{n} (z - \alpha_k) \in \mathbb{C}[z],$$

the Mahler measure of P is defined as the geometric mean of $|P|$ over the unit circle:

$$M(P) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left|P(e^{it})\right|\, dt\right) = |a_n| \prod_{k=1}^{n} \max\{1, |\alpha_k|\}, \tag{1.1}$$

with an empty product assumed to be 1. The second equality in (1.1) is an easy consequence of Jensen's formula. Additionally, if α is an algebraic integer, with P being the minimal polynomial of α, then the Mahler measure of α is $M(\alpha) := M(P)$. The Mahler measure is closely related to the Weil height, and it has been studied in number theory for many decades, see the surveys and books by Borwein [Bor02], Boyd [Boy81], Brunault and Zudilin [BZ20], Everest and Ward [EW99], McKee and Smyth [MS21], and Smyth [Smy08].

It is clear from the definition that $M(P) \ge 1$ for all $P(z) \in \mathbb{Z}[z],\ P \not\equiv 0$. Since $M(PQ) = M(P)M(Q)$ for arbitrary polynomials $P, Q \in \mathbb{Z}[z]$ by (1.1), one can restrict attention to only monic irreducible polynomials when finding lower bounds for the Mahler measure. The case of the smallest Mahler measure for integer polynomials is completely described by a classical result of Kronecker [Kro57]:

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Theorem 1.1 (Kronecker)**.** *Let $P(z) \in \mathbb{Z}[z], \deg(P) \geq 1$, be irreducible and monic. Then $M(P) = 1$ if and only if P is a cyclotomic polynomial or $P(z) = z$.*

In 1933, Lehmer [Leh33] asked if monic polynomials $P \in \mathbb{Z}[z]$ with $M(P) > 1$ could be found with $M(P)$ arbitrarily close to 1:

Conjecture 1.2 (Lehmer's Conjecture)**.**

$$\inf \{M(P) : P(z) \in \mathbb{Z}[z], M(P) > 1\} > 1.$$

Lehmer also noted that for $P \in \mathbb{Z}[z]$, the smallest value of $M(P) > 1$ which he could find was

$$M(\alpha_{\mathcal{L}}) = \alpha_{\mathcal{L}} = 1.1762\ldots,$$

where $\alpha_{\mathcal{L}}$ is the only root lying outside the closed unit disc of the polynomial

$$\mathcal{L}(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1. \tag{1.2}$$

To date, no polynomial $P \in \mathbb{Z}[z]$ has ever been found such that $1 < M(P) < \alpha_{\mathcal{L}}$. However, Lehmer's question has been answered for many subclasses of integer polynomials. We mention a couple of well known results from this area. A polynomial $P \in \mathbb{C}[z]$ with degree equal to n is said to be reciprocal if it satisfies $z^n P(1/z) = \pm P(z)$. Smyth [Smy71] found the sharp lower bound for the Mahler measure of integer polynomials that are not reciprocal:

Theorem 1.3 (Smyth)**.** *If $P \in \mathbb{Z}[z]$ is not reciprocal and $M(P) > 1$, then*

$$M(P) \geq M(z^3 - z - 1) = 1.3247\ldots.$$

As the polynomial $z^3 - z - 1$ is nonreciprocal, this lower bound is clearly attained. Another important result of this kind was proved by Schinzel [Sch73] for totally real polynomials, i.e., polynomials with all roots being real numbers:

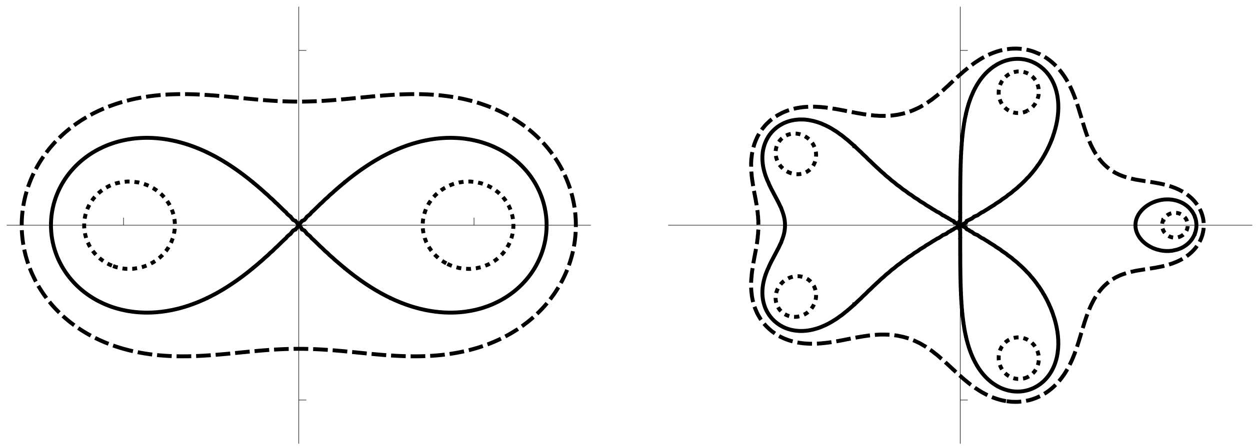
Theorem 1.4 (Schinzel)**.** *Let $P \in \mathbb{Z}[z]$ be a monic totally real polynomial of degree n that does not vanish at $0, 1, -1$. Then*

$$M(P) \geq \phi^{n/2}, \quad \textit{where } \phi = \frac{1 + \sqrt{5}}{2}.$$

By considering $P(z) = z^2 \pm z - 1$, we observe that this bound is sharp. Moreover, those are the only monic irreducible polynomials that attain the lower bound in Schinzel's result, see Corollary 4.3 below.

Lehmer's question has also been studied extensively by searching for integer polynomials, or equivalently algebraic integers α, such that $1 < M(P) = M(\alpha) < \alpha_{\mathcal{L}}$. In a series of two papers, Boyd [Boy80, Boy89] computed all algebraic integers α with degree $n \leq 20$ such that $1 < M(\alpha) < 1.3$, and found no α with $1 < M(\alpha) < \alpha_{\mathcal{L}}$. Building on this result, Mossinghoff was able to verify the same fact for $n \leq 24$. In 2007, Mossinghoff also published a database [Mos07] containing all irreducible polynomials $P \in \mathbb{Z}[z]$ having $\deg P \leq 180$ and $M(P) < 1.3$ known at the time. Improving on these computations, Flammang, Rhin, and Sac-Épée [FRS06] computed all monic irreducible polynomials $P \in \mathbb{Z}[z]$ with $1 < M(P) < 1.3247\ldots$ having degree $n \leq 36$. Following this, Mossinghoff, Rhin, and Wu [MRW08] showed that:

Theorem 1.5 (Mossinghoff, Rhin and Wu)**.** *If a polynomial $P \in \mathbb{Z}[z]$ satisfies $1 < M(P) < \alpha_{\mathcal{L}}$, then $\deg P \geq 56$.*



a) $|z^2-1| = r$ b) $|z^5-z^3-1| = r$

FIGURE 1. Examples of lemniscates with $r=1$, $r=\frac{1}{2}$ and $r=\frac{3}{2}$ graphed as solid, dotted and dashed curves, respectively.

In this paper, we study a generalized version of the Mahler measure introduced by the second author in [Pri21]. It is naturally associated with polynomial lemniscates, i.e., with level curves of polynomials. For a polynomial $V(z) = c_m \prod_{k=1}^m (z-z_k) \in \mathbb{C}[z]$ and for any $r>0$, the corresponding *lemniscate* and *filled-in* lemniscate are given respectively as

$$L := \{z \in \mathbb{C} : |V(z)| = r\} \quad \text{and} \quad E := \{z \in \mathbb{C} : |V(z)| \le r\}\,. \tag{1.3}$$

Lemniscates have been extensively studied in many areas of mathematics, beginning with such an early example as Bernoulli's lemniscate $\{z : |z^2-1| = 1\}$ depicted alongside several other lemniscates in Figure 1.

We define the *generalized Mahler measure* (over the lemniscate L) for any nonzero $P(z) = a_n \prod_{k=1}^n (z-\alpha_k)$ as follows:

$$M_L(P) := \exp\left(\int_L \log|P(z)|\,\frac{|V'(z)|}{2\pi m r}\,|dz|\right). \tag{1.4}$$

For α an algebraic integer with the minimal polynomial P, $M_L(\alpha)$ is defined as $M_L(P)$. Note that if $V(z) = z$ and $r=1$, then L is the unit circle $\mathbb{T} := \{|z|=1\}$, and $M_{\mathbb{T}}(P) \equiv M(P)$ for all P. Thus we deal with a natural generalization of the Mahler measure that inherits essentially all nice properties of its classical version. In particular, a closed form expression for $M_L(P)$ is stated below.

Proposition 1.6 ([Pri21])**.** *Let L be as in* (1.3)*, and let $P(z) = a_n \prod_{k=1}^n (z-\alpha_k) \in \mathbb{C}[z]$. Then*

$$M_L(P) = |a_n|\,|c_m|^{-n/m} \left(\textstyle\prod_{k=1}^n \max\{r, |V(\alpha_k)|\}\right)^{1/m}. \tag{1.5}$$

Moreover, a complete generalization of Kronecker's Theorem 1.1 holds.

Theorem 1.7 ([Pri21])**.** *Let $V(z) = z^m + \cdots + c_0 \in \mathbb{Z}[z]$ be monic, and let $r=1$. Define L and E as in* (1.3)*. Then for all nonzero $P(z) \in \mathbb{Z}[z]$, we have that $M_L(P) \ge 1$, with equality attained if and only if P has the leading coefficient equal to ± 1, and all the roots of P are located in E.*

More precisely, $M_L(P) = 1$ for a monic irreducible $P(z) \in \mathbb{Z}[z]$ if and only if either $P \mid V$ or $P \mid \Phi \circ V$ for some cyclotomic polynomial Φ. Equivalently, for α an algebraic integer, $M_L(\alpha) = 0$ if and only if $V(\alpha) = 0$ or $(\Phi \circ V)(\alpha) = 0$.

Setting $V(z) = z$, we arrive at the classical case $M_{\mathbb{T}}(P) = M(P)$, and recover the original Kronecker's Theorem 1.1. Theorem 1.7 shows that lemniscates of this type naturally carry infinitely many complete sets of conjugate algebraic integers.

Define $B_L := \inf \{M_L(P) : P \in \mathbb{Z}[z], M_L(P) > 1\}$ for any lemniscate L as in Theorem 1.7. It is immediate that the classical Lehmer's conjecture is equivalent to the statement $B_{\mathbb{T}} > 1$, where $\mathbb{T} = \{|z| = 1\}$, while its generalization should then state $B_L > 1$. A relationship between the classical Lehmer's conjecture and its generalization is given below.

Theorem 1.8 ([Pri21]). *For $V(z) = z^m + \cdots + c_1 z + c_0 \in \mathbb{Z}[z]$ and $r = 1$, define L as in (1.3). Then*

$$(B_{\mathbb{T}})^{1/m} \le B_L \le B_{\mathbb{T}}.$$

Thus the classical Lehmer's conjecture is true if and only if the generalized conjecture is true for a single lemniscate of the above type.

There are many interesting open questions related to the generalized Mahler measure on lemniscates. The goal of this paper is to address the questions on the lower bounds for the generalized Mahler measure of totally real polynomials, establishing results analogous to Theorem 1.4. Our main theoretic results are presented in the Section 2, while the numerical results and further theoretic explanations are included in Section 3. All proofs are given in Section 4.

2. Schinzel-type bounds on lemniscates

The main results of our work generalize Schinzel's Theorem 1.4 to the case of the Mahler measure on lemniscates.

Theorem 2.1. *Let $V(z) = z^m + \cdots + c_1 z + c_0 \in \mathbb{Z}[z]$, $r = 1$, and let L be defined as in (1.3). If $P(z) = a_n z^n + \cdots + a_1 z + a_0 \in \mathbb{Z}[z]$ is totally real and irreducible, with $M_L(P) > 1$ and $\gcd(a_n, \ldots, a_0) = 1$, then*

$$M_L(P) \ge \phi^{\frac{n}{2m}}, \qquad \text{where } \phi = \frac{1+\sqrt{5}}{2}. \tag{2.1}$$

Moreover, equality holds in (2.1) for an irreducible P if and only if $P \mid (V^2 \pm V - 1)$.

Hence for a totally real algebraic integer α of degree n such that $V(\alpha) \ne 0, \pm 1$, we have that $M_L(\alpha) \ge \phi^{\frac{n}{2m}}$, with equality holding if and only if α is a root of $V^2 \pm V - 1$.

Recall that all polynomials P satisfying $M_L(P) = 1$ are completely described in Theorem 1.7. In particular, condition $M_L(P) > 1$ in Theorem 2.1 is equivalent to $P \nmid V$ and $P \nmid \Phi \circ V$ for any cyclotomic polynomial Φ. We also note that the condition $P \mid (V^2 \pm V - 1)$ indicates that all irreducible polynomials P attaining equality in (2.1) must have leading coefficients ± 1, because V is monic.

For $V(z) = z$ and $r = 1$, we have that L is the unit circle $\mathbb{T}$, and Theorem 2.1 implies Theorem 1.4. McKee and Smith [MS21, pp. 262–263] proved several results for the classical Mahler measure of totally positive and totally real algebraic numbers, see Theorems 14.14 and 14.16, as well as Corollaries 14.15 and 14.17 in their book.

It is certainly of interest to give an even more explicit description when (2.1) provides a sharp lower bound. We first show that the bound is always sharp for any linear $V(z)$, which essentially represents a shifted version of the classical Schinzel's Theorem.

Proposition 2.2. *Let $V(z) = z + a_0 \in \mathbb{Z}[z]$, $r = 1$, and L be as defined in* (1.3). *Then equality in* (2.1) *is attained for monic integer quadratic polynomials if and only if*

$$P_1(z) = z^2 + (2a_0 - 1)z + a_0^2 - a_0 - 1 \quad \text{or} \quad P_2(z) = z^2 + (2a_0 + 1)z + a_0^2 + a_0 - 1.$$

We now consider the case where $V(z)$ is quadratic, and describe all *quadratic* extremal polynomials P.

Theorem 2.3. *Let $V(z) = z^2 + \cdots \in \mathbb{Z}[z]$, $r = 1$, and L be as defined in* (1.3). *Then equality in* (2.1) *is attained for a monic totally real $P \in \mathbb{Z}[z]$ of degree 2 if and only if either $V(z) = z^2 + 2\ell z + \ell^2 - 1$ or $V(z) = z^2 + 2\ell z + \ell^2 - 2$ for some $\ell \in \mathbb{Z}$. In particular, for both cases of V, the only monic irreducible quadratic polynomials that attain equality in* (2.1) *are*

$$P_1(z) = z^2 + (2\ell - 1)z + \ell^2 - \ell - 1 \quad \text{and} \quad P_2(z) = z^2 + (2\ell + 1)z + \ell^2 + \ell - 1.$$

Note that the above result does not provide all possible extremal polynomials for a given quadratic V, as it is restricted to quadratic P only. For example, if we set $\ell = -1$ in Theorem 2.3, and consider $V(z) = z^2 - 2z - 1$, then we obtain that $P_1(z) = z^2 - 3z + 1$ and $P_2(z) = z^2 - z - 1$ are the only quadratic extremals. Relating this to Theorem 2.1, we observe that $P_1(z)P_2(z) = V^2(z) + V(z) - 1$. However, using Theorem 2.1 again, we find another extremal polynomial $P(z) = z^4 - 4z^3 + z^2 + 6z + 1 = V^2(z) - V(z) - 1$. It follows from Proposition 3.1 below that the only possible degrees of the extremal polynomials P are 2 and 4, when V is quadratic. In fact, our computations showed that the extremal polynomials P of degree 4 represent the overwhelming majority of extremals for all quadratic polynomials V. This is not surprising as Theorem 2.3 explains that quadratic extremals occur for quadratic V of a very special form.

Our next result gives a complete description of possibilities when equality holds in (2.1) for a quadratic polynomial P, and the degree of V is at least 2.

Theorem 2.4. *Let $V(z) = z^m + \cdots + c_1 z + c_0 \in \mathbb{Z}[z]$, $m \geq 2$, $r = 1$, and L be as defined in* (1.3). *Equality in* (2.1) *is attained for a monic totally real $P \in \mathbb{Z}[z]$ of degree 2 if and only if there exist $Q(z) \in \mathbb{Z}[z]$ and $\ell \in \mathbb{Z}$ such that*

$$P(z) = z^2 \pm (2\ell - 1)z + \ell^2 - \ell - 1 \quad \text{or} \quad P(z) = z^2 \pm (2\ell + 1)z + \ell^2 + \ell - 1,$$

and

$$V(z) = P(z)Q(z) \pm z + \ell,$$

where $\pm$ is the same in all instances.

3. Computational results

Our numerical results exhibit many interesting patterns and properties of extremal polynomials that attain equality in (2.1). The following propositions explain some of them.

Proposition 3.1. *Under the assumptions of Theorem 2.1, if P is a monic irreducible totally real polynomial that attains equality in* (2.1), *then* $\deg(P)$ *is even, and* $\deg(P) \leq 2m$.

$V(z)$ coefficients	$P(z)$ coefficients
-3 -3 1	5 15 4 -6 1
-1 -2 1	1 6 1 -4 1
1 -3 1	-1 -3 10 -6 1
-3 -3 2 1	5 0 -6 1 1
-1 -1 2 1	-1 -4 0 3 1
1 0 -3 1	1 3 -1 -3 1
3 0 -3 1	-5 5 5 -5 1
-3 -2 3 1	5 10 -11 -17 5 6 1
-1 -2 3 1	1 6 -5 -15 5 6 1
1 -3 -3 1	1 -9 0 21 3 -6 1
3 -1 -3 1	5 -5 -14 11 7 -6 1
-1 -2 -1 2 1	1 -3 -2 2 1
1 3 -2 -2 1	1 1 -3 -1 1
1 3 -3 -2 1	-1 0 7 1 -6 -1 1
1 3 -2 -3 1	-1 3 7 -15 -13 18 5 -6 1
1 1 0 0 -3 1	1 3 -1 -3 1
-1 3 3 -1 0 -3 1	1 3 -1 -3 1
1 0 -2 3 -3 -3 1	-1 -1 7 2 -7 -2 1
-1 1 2 -3 3 -2 -3 1	1 2 -2 -3 1

TABLE 1. Examples of polynomials $V(z)$ with extremal polynomials $P(z)$. These polynomials were selected to show that all even degrees for $P(z)$ satisfying $\deg(P) \le 2\deg(V)$ can occur. Coefficients are listed in ascending order and the table is grouped by $\deg V$.

The ranges of degrees of the extremal polynomials, and of their coefficients, are illustrated in both Tables 1 and 2, where we list polynomials V that define the lemniscates together with the corresponding extremal polynomials P. All polynomials are represented by their coefficients listed in the order of increasing powers.

Proposition 3.2. *Under the assumptions of Theorem 2.1, suppose that P is a monic irreducible totally real polynomial that attains equality in* (2.1)*. There are two possibilities:*

(i) $P \mid (V^2 - V - 1)$*, which implies that P is also extremal for the lemniscate $L = \{z : |V(z) - 1| = 1\}$, in the sense that* (2.1) *also holds for the measure on the latter lemniscate,*
(ii) $P \mid (V^2 + V - 1)$*, which implies that P is also extremal for the lemniscate $L = \{z : |V(z) + 1| = 1\}$.*

Table 2 lists some examples of polynomials P that are extremal for two lemniscates simultaneously as described in Proposition 3.2.

Both Tables 1 and 2 were computed using Mathematica by considering $V(z) = c_0 + c_1 z + \cdots + z^m \in \mathbb{Z}[z]$ such that $|c_k| \le 3$ for all $k = 1, \ldots, m-1$. Moreover, all polynomials presented in Tables 1 and 2 were double checked with Maple. In Table 1, the further assumption was made that $|c_0| = 1$. Even with this restriction, we were able to find a

$V(z)$ coefficients	$P(z)$ coefficients
$-1\ -3\ \ 1$	$-1\ 3\ \ 8\ -6\ 1$
$0\ -3\ \ 1$	$-1\ 3\ \ 8\ -6\ 1$
$-2\ -1\ \ 1$	$1\ \ 3\ -2\ -2\ 1$
$-1\ -1\ \ 1$	$1\ \ 3\ -2\ -2\ 1$
$1\ -3\ -1\ \ 1$	$1\ -9\ \ 6\ \ 9\ -5\ -2\ \ 1$
$2\ -3\ -1\ \ 1$	$1\ -9\ \ 6\ \ 9\ -5\ -2\ \ 1$
$0\ -3\ -2\ \ 2\ \ 1$	$1\ -1\ -3\ \ 1\ \ 1$
$1\ -3\ -2\ \ 2\ \ 1$	$1\ -1\ -3\ \ 1\ \ 1$
$1\ -3\ -3\ \ 2\ \ 1$	$1\ -9\ \ 0\ \ 24\ \ 0\ -18\ -2\ \ 4\ \ 1$
$2\ -3\ -3\ \ 2\ \ 1$	$1\ -9\ \ 0\ \ 24\ \ 0\ -18\ -2\ \ 4\ \ 1$

TABLE 2. Examples of polynomials $V(z)$ and $V(z)+1$ sharing extremal polynomials. Coefficients are listed in ascending order.

wide range of polynomials V with the corresponding extremal polynomials P. Under the assumptions of Theorem 2.1, P attains equality in (2.1) if and only if $P \mid V^2 \pm V - 1$. With this in mind, we computed the extremal polynomials P by factoring $V^2 \pm V - 1$ completely into irreducible polynomials $P_1, \ldots, P_\ell \in \mathbb{Z}[z]$, using the standard Mathematica procedure. In the following steps, each factor P_i was checked to have $\deg P_i > 2$ and all real roots. The quadratic P_i were excluded because all quadratic extremal polynomials are well understood by Theorems 2.3 and 2.4. We checked that P_i are totally real applying the standard root finding method provided by Mathematica. For the sake of compactness, only a small portion of our computations are featured in Tables 1 and 2, but we are ready to share more computational results upon request.

Our further computations suggest some interesting conjectures. If the degree of polynomials V is kept fixed, then the proportion of lemniscates for which the bound (2.1) is attained grows with the naïve height of V (maximum modulus of all the coefficients of V). This is well illustrated in Table 3. Moreover, we found convincing numerical evidence that this proportion approaches 1 for quadratic polynomials V, see Table 3 (A). On the other hand, if $\deg(V) = 3$ then the proportion of lemniscates for which (2.1) is sharp seems to be much smaller, see Table 3 (B). However, when we add an additional assumption that the constant term of V satisfies $|c_0| = 1$, the proportion seems to approach 1 again, as evidenced in Table 3 (C).

If the naïve heights of V remain bounded, while the degrees of V increase to infinity, then the proportion of lemniscates for which the bound (2.1) is attained seems to rapidly decrease to 0. Table 4 contain the results of our computations that generate a random sample of polynomials V of size 10000, with the naïve height bounded by 1000, in order to estimate the proportion. These and many other observations will be addressed in our future papers.

4. PROOFS

We start with an auxiliary result borrowed from the paper of Höhn and Skoruppa [HS93], who provided a short proof of Schinzel's Theorem 1.4. We emphasize the complete description of equality cases in Schinzel's Theorem.

(A) $m = 2$

H	p
1	0.3333
5	0.6694
10	0.7755
50	0.9044
100	0.9328
200	0.9527
300	0.9614
500	0.9701

(B) $m = 3$

H	p
1	0.2222
5	0.2983
10	0.3922
20	0.4649
30	0.4964
40	0.5149
50	0.5274
100	0.5577

(C) $m = 3$, $|c_0| = 1$

H	p
1	0.2222
5	0.3829
10	0.5291
50	0.7780
100	0.8490
200	0.8931
300	0.9243
500	0.9323

TABLE 3. Proportion p of polynomials $V(z) = z^m + \cdots + c_1 z + c_0 \in \mathbb{Z}[z]$ with $|c_k| \leq H,\ k = 0, \ldots, m-1$, that have an extremal polynomial $P(z)$.

m	2	3	4	5	6	7	8
p	0.9788	0.6054	0.204	0.0355	0.0028	0.0001	0.0

TABLE 4. Proportion p of 10000 random polynomials $V(z) = z^m + \cdots + c_1 z + c_0 \in \mathbb{Z}[z]$ with $|c_k| \leq 1000,\ k = 0, \ldots, m-1$, that have an extremal polynomial $P(z)$.

Lemma 4.1. *For $x > 0$ and $x \neq 1$, we have that*

$$\log_+ x - \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\log x - \frac{1}{\sqrt{5}}\log|x-1| \geq \log\phi, \quad \text{where } \phi = \frac{1+\sqrt{5}}{2}.$$

Moreover, equality is attained if and only if $x = \phi^2$ or $x = \phi^{-2}$.

Proof. Note that for $x > 0$, $\log_+ x = \max\{0, \log x\}$. Let

$$f(x) := \log_+ x - \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\log x - \frac{1}{\sqrt{5}}\log|x-1|, \quad x > 0.$$

Then $f(\phi^2) = f(\phi^{-2}) = \log\phi$. For $x > 1$ and for $0 < x < 1$, we have that

$$f'(x) = \frac{(\sqrt{5}-1)\,x - \sqrt{5} - 1}{2x\,(x-1)\sqrt{5}} \qquad \text{and} \qquad f'(x) = \frac{(\sqrt{5}+1)\,x - \sqrt{5} + 1}{2x\,(1-x)\sqrt{5}},$$

respectively. Then ϕ^2 and ϕ^{-2} are the only critical points of $f(x)$. Since

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to\infty} f(x) = \infty,$$

it follows that the absolute minimum of $f(x)$ occurs only at the critical points ϕ^2 and ϕ^{-2}. ☐

As an immediate corollary of Lemma 4.1, we obtain the following sharp lower bound for the Mahler measure of totally positive irreducible integer polynomials.

Theorem 4.2. *If $P(z) = a_n z^n + \cdots + a_0 \in \mathbb{Z}[z]$ is totally positive and irreducible, and $P(1) \neq 0$, then*

$$M(P) \geq \phi^n.$$

Moreover, for P also irreducible, we have that $M(P) = \phi^n$ holds if and only if $P(z) = \pm(z^2 - 3z + 1)$.

Proof. Write $P(z) = a_n \prod_{k=1}^{n}(z - \alpha_k)$ with $\alpha_k > 0$ and $\alpha_k \neq 1$ for each k. By Lemma 4.1, we have that

$$\begin{aligned}
\log M(P) &= \log|a_n| + \sum_{k=1}^{n} \log_+ \alpha_k \\
&\geq \log|a_n| + \sum_{k=1}^{n} \left(\log\phi + \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\log\alpha_k + \frac{1}{\sqrt{5}}\log|\alpha_k - 1| \right) \\
&= \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\log|a_n| + n\log\phi + \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\log|P(0)| + \frac{1}{\sqrt{5}}\log|P(1)| \\
&\geq n\log\phi,
\end{aligned}$$

where the last inequality follows from the fact that a_n, $P(0)$ and $P(1)$ are nonzero integers. Hence, $M(P) \geq \phi^n$. Since equality in Lemma 4.1 is attained if and only if $\alpha = \phi^2$ or $\alpha = \phi^{-2}$, we deduce from the above inequalities that $M(P) = \phi^n$ holds if and only if $P(\phi^2) = P(\phi^{-2}) = 0$ and $|a_n| = 1$. The latter is equivalent to $P(z) = \pm(z^2 - 3z + 1)$. □

We now recover Schinzel's Theorem 1.4 together with the complete description of the extremal cases.

Corollary 4.3. *If $P(z) = a_n z^n + \cdots + a_0 \in \mathbb{Z}[z]$ is irreducible and totally real, with $P(0), P(\pm1) \neq 0$, then*

$$M(P) \geq \phi^{n/2}.$$

Furthermore, $M(P) = \phi^{n/2}$ if and only if $P(z) = \pm(z^2 \pm z - 1)$.

Proof. Write $P(z) = a_n \prod_{k=1}^{n}(z - \alpha_k)$ with $\alpha_k \in \mathbb{R}$ and $\alpha_k \neq 0, \pm1$ for each k. Then with $Q(z) = a_n^2 \prod_{k=1}^{n}(z - \alpha_k^2)$, we have $Q(z) \in \mathbb{Z}[z]$ is totally positive with $Q(0), Q(1) \neq 0$ and $M(P) = M(Q)^{1/2}$, so that by Theorem 4.2,

$$M(P) = M(Q)^{1/2} \geq \phi^{n/2}.$$

Assuming that $M(P) = \phi^{n/2}$, we obtain that $M(Q) = \phi^n$. Theorem 4.2 now implies that the roots of Q are ϕ^2 and ϕ^{-2}. Hence, the possible roots of P are $\pm\phi, \pm\phi^{-1}$. By the irreducibility of P, it follows that $P(z) = \pm(z^2 \pm z - 1)$. On the other hand, if $P(z) = \pm(z^2 \pm z - 1)$ then $M(P) = \phi$ by (1.1). □

Proof of Theorem 2.1. Let $P(z) = a_n \prod_{k=1}^{n}(z - \alpha_k)$, and define $Q(z) := a_n^m \prod_{k=1}^{n}(z - V(\alpha_k))$, so that

$$M_L(P) = |a_n| \left(\prod_{k=1}^{n} \max\{1, |V(\alpha_k)|\}\right)^{1/m} = M(Q)^{1/m}. \tag{4.1}$$

It is easy to see that Q is totally real since each $V(\alpha_k)$ is real. Further, the coefficients of

$$Q(z)/a_n^m = \prod_{k=1}^{n}(z - V(\alpha_k))$$

are elementary symmetric polynomials in $\{V(\alpha_k)\}_{k=1}^{n}$, so they are also symmetric polynomials in $\{\alpha_k\}_{k=1}^{n}$ with integer coefficients. It follows that the degree of α_k is at most m for each of these symmetric polynomials. By the Fundamental Theorem of Symmetric Polynomials, we then have that the coefficients of $Q(z)/a_n^m$ are integer polynomials, of degree at most m, in the elementary symmetric functions of $\{\alpha_k\}_{k=1}^n$, and when multiplied by a_n^m, these coefficients become integers. Hence, $Q(z) \in \mathbb{Z}[z]$.

Write $Q = Q_1 \dots Q_l$, where each factor $Q_i(z) = b_{n_i} \prod_{k=1}^{n_i}(z - V(\alpha_{j_k})) \in \mathbb{Z}[z]$ is totally real and irreducible, and $\prod_{i=1}^{l} b_{n_i} = a_n^m$. Suppose that $M(Q_i) = 1$ for some i. Then $|b_{n_i}| = 1$ and since Q_i is totally real and irreducible, Kronecker's Theorem 1.1 implies that $Q_i(z) = \pm z$ or $Q_i(z) = \pm(z \pm 1)$, because $z - 1$ and $z + 1$ are the only totally real cyclotomic polynomials. If $Q_i(z) = \pm z$ then $V(\alpha_k) = 0$ for some k, so that $P \mid V$ by the irreducibility of P. On the other hand, if $Q_i(z) = \pm(z \pm 1)$, then $V(\alpha_k) \pm 1 = 0$ for some k, so that $P \mid \Phi \circ V$ for $\Phi(z) = z \pm 1$. Since V and Φ are monic polynomials, we obtain that $|a_n| = 1$ in either of the considered cases. Moreover, Theorem 1.7 now gives that $M_L(P) = 1$, which contradicts our assumption that $M_L(P) > 1$. Hence, $Q_i(0), Q_i(\pm 1) \neq 0$ and Corollary 4.3 implies that $M(Q_i) \geq \phi^{\frac{n_i}{2}}$, because Q_i is irreducible, for $i = 1, \dots, l$. It follows that

$$M_L(P) = M(Q)^{1/m} = \prod_{i=1}^{l} M(Q_i)^{1/m} \geq \prod_{i=1}^{l} \phi^{\frac{n_i}{2m}} = \phi^{\frac{n}{2m}}.$$

If we assume that $M_L(P) = \phi^{\frac{n}{2m}}$, then $M(Q_i) = \phi^{\frac{n_i}{2}}$ for each $i = 1, \dots, l$, so that $Q_i = \pm(z^2 \pm z - 1)$ for each i by Corollary 4.3. Thus, $V(\alpha_k) \in \{\pm\phi, \pm 1/\phi\}$ for all $k = 1, \dots, n$. If $V(\alpha_k) = \phi$ or $V(\alpha_k) = -1/\phi$ for some $k = 1, \dots, n$, then $V(\alpha_k)^2 - V(\alpha_k) - 1 = 0$, so that $P \mid (V^2 - V - 1)$ by irreducibility. By similar reasoning, if $V(\alpha_k) = -\phi$ or $V(\alpha_k) = 1/\phi$ for some $k = 1, \dots, n$, then $P \mid (V^2 + V - 1)$. □

Proof of Proposition 2.2. It is known for totally real $P(z) = (z - \alpha_1)(z - \alpha_2) \in \mathbb{Z}[z]$ that $M(P) = \frac{1+\sqrt{5}}{2}$ if and only if $P(z) = z^2 - z - 1$ or $P(z) = z^2 + z - 1$, see Corollary 4.3. Suppose $M_L(P) = \frac{1+\sqrt{5}}{2}$, so that

$$M(P(z - a_0)) = \prod_{k=1}^{2} \max\{1, |\alpha_k + a_0|\} = M_L(P) = \frac{1 + \sqrt{5}}{2}.$$

This is equivalent by the classical case to either

$$P(z - a_0) = z^2 - z - 1 \quad \text{or} \quad P(z - a_0) = z^2 + z - 1.$$

Respectively, we then have that $M_L(P) = \frac{1+\sqrt{5}}{2}$ if and only if either

$$P(z) = z^2 + (2a_0 - 1)z + a_0^2 - a_0 - 1 \quad \text{or} \quad P(z) = z^2 + (2a_0 + 1)z + a_0^2 + a_0 - 1. \qquad \square$$

Proof of Theorem 2.3. Write $V(z) = z^2 + bz + c$, and suppose that $P(z) = (z - \alpha_1)(z - \alpha_2) \in \mathbb{Z}[z]$ is totally real with $M_L(P) = \left(\frac{1+\sqrt{5}}{2}\right)^{1/2}$. Let $Q(z) = (z - V(\alpha_1))(z - V(\alpha_2)) \in \mathbb{Z}[z]$, so that

$$M_L^2(P) = M(Q) = \max\{|V(\alpha_1)|, |V(\alpha_2)|\} = \frac{1 + \sqrt{5}}{2}.$$

It follows that $\min\{|V(\alpha_1)|, |V(\alpha_2)|\} = \frac{-1+\sqrt{5}}{2}$ by Corollary 4.3. Permuting α_1 and α_2, we may assume without loss of generality that

$$|V(\alpha_1)| = \frac{-1+\sqrt{5}}{2} \quad \text{and} \quad |V(\alpha_2)| = \frac{1+\sqrt{5}}{2}.$$

Hence, either $V(\alpha_2) = \frac{1+\sqrt{5}}{2}$ or $V(\alpha_2) = -\frac{1+\sqrt{5}}{2}$.

Case 1. If $V(\alpha_2) = \frac{1+\sqrt{5}}{2}$ then $V(\alpha_1) = \frac{1-\sqrt{5}}{2}$ and

$$V(\alpha_2) - V(\alpha_1) = \alpha_2^2 - \alpha_1^2 + b(\alpha_2 - \alpha_1) = \sqrt{5},$$

so that $\alpha_2 + \alpha_1 + b = \frac{\sqrt{5}}{\alpha_2 - \alpha_1}$. Now, as $\alpha_2 + \alpha_1$ is a coefficient of $-P(z) \in \mathbb{Z}[z]$, then $k = \alpha_2 + \alpha_1 + b \in \mathbb{Z}$, so that $\frac{\sqrt{5}}{\alpha_2 - \alpha_1} = k \in \mathbb{Z}$. Since $P(0) = \alpha_2\alpha_1 \in \mathbb{Z}$, we have that

$$\frac{5}{k^2} = \alpha_2^2 + \alpha_1^2 - 2\alpha_2\alpha_1 = (\alpha_2 + \alpha_1)^2 - 4\alpha_2\alpha_1 \in \mathbb{Z}.$$

Thus, $k^2 \mid 5$, so that $k = \pm 1$ since 5 is a prime. It follows that $\alpha_2 - \alpha_1 = \pm\sqrt{5}$ and

$$b = \frac{\sqrt{5}}{\alpha_2 - \alpha_1} - (\alpha_2 + \alpha_1) = \pm 1 - (\alpha_2 + \alpha_1). \tag{4.2}$$

Additionally, we have that

$$V(\alpha_2) + V(\alpha_1) = \alpha_2^2 + \alpha_1^2 + b(\alpha_2 + \alpha_1) + 2c = 1.$$

Using (4.2), we obtain from the above equation that

$$\pm(\alpha_2 + \alpha_1) = 2\alpha_2\alpha_1 - 2c + 1. \tag{4.3}$$

Hence, $\alpha_2 + \alpha_1$ is odd, so that b is even and $b = 2\ell$ for some $\ell \in \mathbb{Z}$. Therefore, we have that $\alpha_2 + \alpha_1 = \pm 1 - 2\ell$. Since $\alpha_2 - \alpha_1 = \pm\sqrt{5}$, we obtain that

$$\alpha_2 = \pm\frac{1+\sqrt{5}}{2} - \ell \quad \text{and} \quad \alpha_1 = \pm\frac{1-\sqrt{5}}{2} - \ell,$$

where $\pm$ takes the same values in both equations. It now follows by (4.3) and (4.2) that

$$\begin{aligned} 2c = 2\alpha_2\alpha_1 \mp (\alpha_2 + \alpha_1) + 1 &= 2\alpha_2\alpha_1 \pm 2\ell \\ &= 2\left(\pm\frac{1+\sqrt{5}}{2} - \ell\right)\left(\pm\frac{1-\sqrt{5}}{2} - \ell\right) \pm 2\ell \\ &= 2(-1 \mp \ell + \ell^2) \pm 2\ell = 2\ell^2 - 2, \end{aligned} \tag{4.4}$$

where $\pm$ in (4.4) is the same in all occurrences. We conclude that $c = \ell^2 - 1$, and that $V(z) = z^2 + 2\ell z + \ell^2 - 1$. Moreover, we have that

$$P_1(z) = \left(z - \left(\tfrac{1+\sqrt{5}}{2} - \ell\right)\right)\left(z - \left(\tfrac{1-\sqrt{5}}{2} - \ell\right)\right) = z^2 + (2\ell - 1)z + \ell^2 - \ell - 1$$

and

$$P_2(z) = \left(z + \left(\tfrac{1+\sqrt{5}}{2} + \ell\right)\right)\left(z + \left(\tfrac{1-\sqrt{5}}{2} + \ell\right)\right) = z^2 + (2\ell + 1)z + \ell^2 + \ell - 1$$

are the extremal polynomials attaining equality in (2.1).

Case 2. If $V(\alpha_2) = -\frac{1+\sqrt{5}}{2}$, then $V(\alpha_1) = -\frac{1-\sqrt{5}}{2}$ and

$$V(\alpha_1) - V(\alpha_2) = \alpha_1^2 - \alpha_2^2 + b(\alpha_1 - \alpha_2) = \sqrt{5}.$$

It follows that $\alpha_1 - \alpha_2 = \pm\sqrt{5}$ and $b = \pm 1 - (\alpha_1 + \alpha_2)$ by similar reasoning to the previous case. Additionally, we have

$$V(\alpha_1) + V(\alpha_2) = \alpha_1^2 + \alpha_2^2 + b(\alpha_1 + \alpha_2) + 2c = -1.$$

Inserting $b = \pm 1 - (\alpha_1 + \alpha_2)$ in the above equation, we find that $\pm(\alpha_1 + \alpha_2) = 2\alpha_1\alpha_2 - 2c - 1$, which means that $\alpha_1 + \alpha_2$ is odd and b is even. Let $b = 2\ell$ for some $\ell \in \mathbb{Z}$, so that $\alpha_1 + \alpha_2 = \pm 1 - 2\ell$. Since $\alpha_1 - \alpha_2 = \pm\sqrt{5}$, we obtain that

$$\alpha_1 = \pm\frac{1+\sqrt{5}}{2} - \ell \quad \text{and} \quad \alpha_2 = \pm\frac{1-\sqrt{5}}{2} - \ell,$$

where $\pm$ is the same in both equations. These roots of P now help us to compute c as follows:

$$\begin{aligned} 2c = 2\alpha_1\alpha_2 \mp (\alpha_1 + \alpha_2) - 1 &= 2\alpha_1\alpha_2 \pm 2\ell - 2 \\ &= 2\left(\pm\tfrac{1+\sqrt{5}}{2} - \ell\right)\left(\pm\tfrac{1-\sqrt{5}}{2} - \ell\right) \pm 2\ell - 2 \\ &= 2(-1 \mp \ell + \ell^2) \pm 2\ell - 2 = 2\ell^2 - 4, \end{aligned}$$

where $\pm$ is the same in all occurrences. Thus, $c = \ell^2 - 2$, so that $V(z) = z^2 + 2\ell z + \ell^2 - 2$. Finally, we again obtain the same extremal polynomials

$$P_1(z) = z^2 + (2\ell - 1)z + \ell^2 - \ell - 1 \quad \text{and} \quad P_2(z) = z^2 + (2\ell + 1)z + \ell^2 + \ell - 1.$$

For the converse, we first assume that $V(z) = z^2 + 2\ell z + \ell^2 - 1$, and check directly that $M_L(P_1) = M_L(P_2) = \left(\frac{1+\sqrt{5}}{2}\right)^{1/2}$. Note that $P_1(z) = z^2 + (2\ell - 1)z + \ell^2 - \ell - 1$ has the real roots

$$\alpha_2 = \frac{1+\sqrt{5}}{2} - \ell \quad \text{and} \quad \alpha_1 = \frac{1-\sqrt{5}}{2} - \ell.$$

Since $V(z) = P_1(z) + z + \ell$, we have that

$$M_L^2(P_1) = \max\{|V(\alpha_1)|, |V(\alpha_2)|\} = \max\{|\alpha_1 + \ell|, |\alpha_2 + \ell|\} = \frac{1+\sqrt{5}}{2}.$$

Likewise, $P_2(z) = z^2 + (2\ell + 1)z + \ell^2 + \ell - 1$ has the real roots

$$\alpha_2 = -\frac{1+\sqrt{5}}{2} - \ell \quad \text{and} \quad \alpha_1 = -\frac{1-\sqrt{5}}{2} - \ell,$$

and using $V(z) = P_2(z) - z - \ell$, we obtain that

$$M_L^2(P_2) = \max\{|V(\alpha_1)|, |V(\alpha_2)|\} = \max\{|-\alpha_1 - \ell|, |-\alpha_2 - \ell|\} = \frac{1+\sqrt{5}}{2}.$$

If $V(z) = z^2 + 2\ell z + \ell^2 - 2$ then we have that $V(z) = P_1(z) + z + \ell - 1 = P_2(z) - z - \ell - 1$, which implies in the same way that

$$M_L^2(P_1) = M_L^2(P_2) = \frac{1+\sqrt{5}}{2}.$$

An alternative proof of the converse follows from Theorem 2.1 by observing that for $V(z) = z^2 + 2\ell z + \ell^2 - 1$ we have $V^2 - V - 1 = P_1P_2$, while for $V(z) = z^2 + 2\ell z + \ell^2 - 2$ we instead have that $V^2 + V - 1 = P_1P_2$.

□

Proof of Theorem 2.4. Assume that $P(z) = (z-\alpha_2)(z-\alpha_1) \in \mathbb{Z}[z]$ is totally real with $M_L(P) = \left(\frac{1+\sqrt{5}}{2}\right)^{1/m}$. Arguing as in the proof of Theorem 2.3, we note that

$$M_L^m(P) = \max\{|V(\alpha_1)|, |V(\alpha_2)|\} = \frac{1+\sqrt{5}}{2},$$

and assume without loss of generality that

$$|V(\alpha_2)| = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad |V(\alpha_1)| = \frac{-1+\sqrt{5}}{2}.$$

Case 1. Suppose that $V(\alpha_2) = \frac{1+\sqrt{5}}{2}$, then $V(\alpha_1) = \frac{1-\sqrt{5}}{2}$. By the Division Algorithm, we may write

$$V(z) = (z-\alpha_2)(z-\alpha_1)Q(z) + kz + \ell$$

for some $Q(z) = b_0 + b_1 z + \cdots + z^{m-2} \in \mathbb{Z}[z]$ and $k, \ell \in \mathbb{Z}$. It follows that

$$V(\alpha_2) - V(\alpha_1) = k(\alpha_2 - \alpha_1) = \sqrt{5}.$$

Repeating the same argument as in the proof of Theorem 2.3, we obtain that

$$\frac{5}{k^2} = \alpha_2^2 + \alpha_1^2 - 2\alpha_2\alpha_1 \in \mathbb{Z},$$

which implies that $k = \pm 1$ and $\alpha_2 - \alpha_1 = \pm\sqrt{5}$. On the other hand, we have that

$$V(\alpha_2) + V(\alpha_1) = k(\alpha_2 + \alpha_1) + 2\ell = 1,$$

so that $\alpha_2 + \alpha_1 = \pm(1 - 2\ell)$. It now follows that

$$\alpha_2 = \pm\left(\frac{1+\sqrt{5}}{2} - \ell\right) \quad \text{and} \quad \alpha_1 = \pm\left(\frac{1-\sqrt{5}}{2} - \ell\right),$$

and we have

$$P(z) = (z-\alpha_2)(z-\alpha_1) = z^2 \pm (2\ell - 1)z + \ell^2 - \ell - 1.$$

Case 2. Suppose that $V(\alpha_2) = -\frac{1+\sqrt{5}}{2}$, so that $V(\alpha_1) = -\frac{1-\sqrt{5}}{2}$. As before, we write

$$V(z) = (z-\alpha_2)(z-\alpha_1)Q(z) + kz + \ell$$

for some $Q(z) = b_0 + b_1 z + \cdots + z^{m-2} \in \mathbb{Z}[z]$ and $k, \ell \in \mathbb{Z}$. If follows that

$$V(\alpha_1) - V(\alpha_2) = k(\alpha_1 - \alpha_2) = \sqrt{5},$$

so that $k = \pm 1$ and $\alpha_1 - \alpha_2 = \pm\sqrt{5}$. Additionally,

$$V(\alpha_2) + V(\alpha_1) = k(\alpha_1 + \alpha_2) + 2\ell = -1,$$

so that $-(\alpha_1 + \alpha_2) = \pm(1 + 2\ell)$. Thus,

$$\alpha_2 = \pm\left(-\frac{1+\sqrt{5}}{2} - \ell\right) \quad \text{and} \quad \alpha_1 = \pm\left(-\frac{1-\sqrt{5}}{2} - \ell\right),$$

which immediately gives that

$$P(z) = (z-\alpha_2)(z-\alpha_1) = z^2 \pm (2\ell + 1) + \ell^2 + \ell - 1.$$

For the converse, assume first that for some $Q(z) \in \mathbb{Z}[z]$ and $\ell \in \mathbb{Z}$, we have

$$P(z) = z^2 \pm (2\ell - 1)z + \ell^2 - \ell - 1 \quad \text{and} \quad V(z) = P(z)Q(z) \pm z + \ell,$$

where $\pm$ is the same in all occurrences. Then the roots of P are

$$\alpha_2 = \pm\left(\frac{1+\sqrt{5}}{2} - \ell\right) \quad \text{and} \quad \alpha_1 = \pm\left(\frac{1-\sqrt{5}}{2} - \ell\right).$$

Thus, $V(\alpha_2) = \frac{1+\sqrt{5}}{2}$ and $V(\alpha_1) = \frac{1-\sqrt{5}}{2}$, so that $M_L(P) = \left(\frac{1+\sqrt{5}}{2}\right)^{1/m}$. Similarly, if instead we assume that

$$P(z) = z^2 \pm (2\ell+1)z + \ell^2 + \ell - 1 \quad \text{and} \quad V(z) = P(z)Q(z) \pm z + \ell,$$

where $\pm$ is the same in all occurrences, then the roots of P are

$$\alpha_2 = \pm\left(-\frac{1+\sqrt{5}}{2} - \ell\right) \quad \text{and} \quad \alpha_1 = \pm\left(-\frac{1-\sqrt{5}}{2} - \ell\right).$$

Hence, $V(\alpha_2) = -\frac{1+\sqrt{5}}{2}$ and $V(\alpha_1) = -\frac{1-\sqrt{5}}{2}$, so that $M_L(P) = \left(\frac{1+\sqrt{5}}{2}\right)^{1/m}$ again. $\square$

Proof of Proposition 3.1. As in the proof of Theorem 2.1, let $P(z) = a_n \prod_{k=1}^{n}(z-\alpha_k)$ and $Q(z) = a_n^m \prod_{k=1}^{n}(z - V(\alpha_k))$. We also write $Q = Q_1 \dots Q_l$, where each factor $Q_i \in \mathbb{Z}[z]$ is totally real and irreducible. The proof of Theorem 2.1 showed that $M_L(P) = \phi^{\frac{n}{2m}}$ holds if and only if $Q_i = \pm(z^2 \pm z - 1)$ for each i. Hence $\deg(P) = \deg(Q)$ is an even number. Moreover, since $P \mid (V^2 \pm V - 1)$ by Theorem 2.1, we conclude that $\deg(P) \leq 2\deg(V) = 2m$. $\square$

Proof of Proposition 3.2. By Theorem 2.1, P attains equality in (2.1) if and only if $P \mid (V^2 - V - 1)$ or $P \mid (V^2 + V - 1)$. Note that

$$(V-1)^2 + (V-1) - 1 = V^2 - V - 1 \quad \text{and} \quad (V+1)^2 - (V+1) - 1 = V^2 + V - 1.$$

Thus, $P \mid (V^2 - V - 1)$ if and only if $P \mid ((V-1)^2 + (V-1) - 1)$, and $P \mid (V^2 + V - 1)$ if and only if $P \mid ((V+1)^2 - (V+1) - 1)$. $\square$

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