# Asymptotic zero distribution of random polynomials spanned by general bases

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*Dedicated to Professor E. B. Saff on his 70th Birthday*

ABSTRACT. Zeros of Kac polynomials spanned by monomials with i.i.d. random coefficients are asymptotically uniformly distributed near the unit circumference. We give estimates of the expected discrepancy between the zero counting measure and the normalized arclength on the unit circle. Similar results are established for polynomials with random coefficients spanned by different bases, e.g., by orthogonal polynomials. We show almost sure convergence of the zero counting measures to the corresponding equilibrium measures, and quantify this convergence, relying on the potential theoretic methods developed for deterministic polynomials. Applications include estimates of the expected number of zeros in various sets. Random coefficients may be dependent and need not have identical distributions in our results.

## 1. Introduction

Zeros of polynomials with random coefficients have been intensively studied since 1930s. The early work concentrated on the expected number of real zeros $\mathbb{E}[N_n(\mathbb{R})]$ for polynomials of the form $P_n(z) = \sum_{k=0}^n A_k z^k$, where $\{A_n\}_{k=0}^n$ are independent and identically distributed random variables. Apparently the first paper that initiated the study is due to Bloch and Pólya [**7**]. They gave an upper bound $\mathbb{E}[N_n(\mathbb{R})] = O(\sqrt{n})$ for polynomials with coefficients selected from the set $\{-1, 0, 1\}$ with equal probabilities. Further results generalizing and improving that estimate were obtained by Littlewood and Offord [**32**]-[**33**], Erdős and Offord [**15**] and others. Kac [**29**] established the important asymptotic result

$$\mathbb{E}[N_n(\mathbb{R})] = (2/\pi + o(1)) \log n \quad \text{as } n \to \infty,$$

for polynomials with independent real Gaussian coefficients. Refined forms of this asymptotic were developed by Kac [**30**], Hammersley [**21**], Wang [**54**], Edelman and Kostlan [**14**], and others. It appears that the sharpest known version is given by the asymptotic series of Wilkins [**55**]. Many additional references and further

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2010 *Mathematics Subject Classification.* Primary: 30C15; Secondary: 30B20, 60B10.

*Key words and phrases.* Polynomials, random coefficients, expected number of zeros, uniform distribution, random orthogonal polynomials.

This research was partially supported by the National Security Agency (grant H98230-12-1-0227) and by the AT&T Foundation.

directions of work on the expected number of real zeros may be found in the books of Bharucha-Reid and Sambandham [**4**], and of Farahmand [**17**]. The book [**4**] remains the only comprehensive reference devoted to random polynomials, despite being somewhat outdated. Zeros of random polynomials are continuously gaining popularity, with numerous papers published every year, so that our brief overview of the area is necessarily incomplete.

While the number of real zeros is rather small, Shparo and Shur [**46**], Arnold [**2**], and many other authors showed that most of zeros of random polynomials are accumulating near the unit circumference, being equidistributed in the angular sense, under mild conditions on the probability distribution of the coefficients. Introducing modern terminology, we define the *zero counting measure*

$$\tau_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{Z_k},$$

where $\{Z_k\}_{k=1}^n$ are the zeros of a polynomial $P_n$ of degree $n$, and $\delta_{Z_k}$ is the unit point mass at $Z_k$. The fact of equidistribution for the zeros of random polynomials is expressed via the weak convergence of $\tau_n$ to the normalized arclength measure $\mu_{\mathbb{T}}$ on the unit circumference, where $d\mu_{\mathbb{T}}(e^{it}) := dt/(2\pi)$. Namely, we have that $\tau_n \stackrel{w}{\to} \mu_{\mathbb{T}}$ with probability 1 (abbreviated as a.s. or almost surely). More recent work on the global limiting distribution of zeros of random polynomials include papers of Hughes and Nikeghbali [**23**], Ibragimov and Zeitouni [**24**], Ibragimov and Zaporozhets [**25**], Kabluchko and Zaporozhets [**26, 27**], etc. In particular, Ibragimov and Zaporozhets [**25**] proved that if the coefficients are independent and identically distributed, then the condition $\mathbb{E}[\log^+|A_0|] < \infty$ is necessary and sufficient for $\tau_n \stackrel{w}{\to} \mu_{\mathbb{T}}$ almost surely. As usual, $\mathbb{E}[X]$ denotes the expectation of a random variable $X$. The results of Shepp and Vanderbei [**41**] provide asymptotics for the expected number of complex zeros, when random polynomials have Gaussian coefficients. Ibragimov and Zeitouni [**24**] obtained generalizations of those results for random coefficients from the domain of attraction of the stable law. A Java program that computes and plots the complex roots of random polynomials may be found on the web page of Vanderbei [**53**].

Another interesting direction is related to the study of zeros of random polynomials spanned by various bases, e.g., by orthogonal polynomials. These questions were considered by Shiffman and Zelditch [**42**]-[**44**], Bloom [**8**] and [**9**], Bloom and Shiffman [**11**], Bloom and Levenberg [**10**], Bayraktar [**3**] and others. Many of the mentioned papers used potential theoretic approach to study the limiting zero distribution, which is well developed for deterministic polynomials, see Blatt, Saff and Simkani [**5**], and Andrievskii and Blatt [**1**]. We also rely on the potential theoretic techniques of [**5**] and [**1**] in our study of zeros of random polynomials spanned by various bases, with random coefficients from quite general classes.

Thus the study of global zero distribution for random polynomials may be naturally divided into two related directions on counting the expected number of zeros in various sets, and on the almost sure limits of zero counting measures. We address both groups of problems in this paper. The majority of available results require the coefficients $\{A_k\}_{k=0}^\infty$ be independent and identically distributed (i.i.d.) random variables. Asymptotic results on the expected number of zeros require further stringent assumption on the distributions of coefficients. One of our main goals is to remove unnecessary restrictions, and prove results on zeros of polynomials whose coefficients need not have identical distributions and may be dependent. We continue the line of research from the papers [**37**], [**38**] and [**36**].

We do not discuss the local scaling limit results on the zeros of random polynomials in this paper, but refer to Bleher and Di [**6**], Tao and Vu [**50**], and Sinclair and Yattselev [**47**].

Section 2 deals with almost sure convergence of the zero counting measures for polynomials with random coefficients that satisfy only weak log-integrability assumptions. Section 3 is devoted to the discrepancy results, and establishes expected rates of convergence of the zero counting measures to the equilibrium measures. Again, the random coefficients in Section 3 are neither independent nor identically distributed, and their distributions only satisfy certain uniform bounds for the fractional and logarithmic moments. We also consider random polynomials spanned by general bases in Sections 2 and 3, which includes random orthogonal polynomials and random Faber polynomials on various sets in the plane. Section 3 also mentions asymptotic results for the expected number of real zeros of random orthogonal polynomials. All proofs are given in Section 4.

## 2. Asymptotic Equidistribution of Zeros

We first review recent results from [**36**] on the equidistribution of zeros for sequences of polynomials of the form

$$P_n(z) = \sum_{k=0}^{n} A_k z^k, \quad n \in \mathbb{N}.$$

Let $A_k,\ k = 0, 1, 2, \ldots$, be complex valued random variables that are not necessarily independent, nor they are required to be identically distributed. Denoting the distribution function of $|A_k|$ by $F_k$, we introduce the following assumptions.

**Assumption 1** There is $N \in \mathbb{N}$ and a decreasing function $f : [a, \infty) \to [0, 1],\ a > 1$, such that

$$\int_a^\infty \frac{f(x)}{x}\, dx < \infty \text{ and } 1 - F_k(x) \le f(x),\ \forall\, x \in [a, \infty), \tag{2.1}$$

holds for all $k \ge N$.

**Assumption 2** There is $N \in \mathbb{N}$ and an increasing function $g : [0, b] \to [0, 1],\ 0 < b < 1$, such that

$$\int_0^b \frac{g(x)}{x}\, dx < \infty \text{ and } F_k(x) \le g(x),\ \forall\, x \in [0, b], \tag{2.2}$$

holds for all $k \ge N$.

If the random variables $|A_k|,\ k = 0, 1, \ldots$, are identically distributed, then assumptions (2.1)-(2.2) are equivalent to $\mathbb{E}[|\log |A_0||] < \infty$. Assumption (2.2) clearly implies that $\mathbb{P}(\{A_k = 0\}) = 0$ for all $k$. The work of Schehr and Majumdar [**40**] shows that equidistribution of zeros near the unit circumference *requires* certain uniform assumptions on coefficients.

We proved results on almost sure limits for the zero counting measures of random polynomials (see [**36**]) by using potential theoretic techniques of Blatt, Saff and Simkani [**5**] combined with the following facts about the random coefficients:

$$\lim_{n\to\infty} |A_0|^{1/n} = \lim_{n\to\infty} |A_n|^{1/n} = \lim_{n\to\infty} \max_{0\le k\le n} |A_k|^{1/n} = 1 \quad \text{a.s.}$$

The simplest result from [**36**] is as follows.

THEOREM 2.1. *If the coefficients of* $P_n(z) = \sum_{k=0}^n A_k z^k,\ n \in \mathbb{N}$, *are complex random variables that satisfy assumptions* (2.1) *and* (2.2), *then the zero counting measures* $\tau_n$ *for this sequence converge almost surely to* $\mu_{\mathbb{T}}$ *as* $n \to \infty$.

We also considered [**36**] more general ensembles of random polynomials

$$P_n(z) = \sum_{k=0}^{n} A_k B_k(z)$$

spanned by the bases $\{B_k\}_{k=0}^\infty$. Let $B_k(z) = \sum_{j=0}^k b_{j,k} z^j$, where $b_{j,k} \in \mathbb{C}$ for all $j$ and $k$, and $b_{k,k} \neq 0$ for all $k$, be a polynomial basis. Note that deg $B_k = k$ for all $k \in \mathbb{N} \cup \{0\}$. Given a compact set $E \subset \mathbb{C}$ of positive logarithmic capacity $\text{cap}(E)$ (cf. Ransford [**39**]), we assume that

$$\limsup_{k\to\infty} \|B_k\|_E^{1/k} \le 1 \quad \text{and} \quad \lim_{k\to\infty} |b_{k,k}|^{1/k} = 1/\text{cap}(E), \tag{2.3}$$

where $\|B_k\|_E := \sup_E |B_k|$. Condition (2.3) holds for many standard bases used for representing analytic functions on $E$, e.g., for various sequences of orthogonal polynomials (cf. Stahl and Totik [**48**]) and for Faber polynomials (see Suetin [**49**]). Random orthogonal polynomials and their asymptotic zero distribution was recently studied in a series of papers by Shiffman and Zelditch [**43**], Bloom [**8**] and [**9**], Bloom and Shiffman [**11**], Bloom and Levenberg [**10**] and Bayraktar [**3**]. In particular, it was shown that the counting measures of zeros converge weakly to the equilibrium measure of $E$ denoted by $\mu_E$, which is a positive unit Borel measure supported on the outer boundary of $E$ [**39**]. Most of mentioned papers also considered multivariate polynomials. They assumed that the basis polynomials are orthonormal with respect to a measure satisfying the Bernstein-Markov property, and that the coefficients are complex i.i.d. random variables with uniformly bounded distribution density function with respect to the area measure, and with proper decay at infinity. We also used the results of Blatt, Saff and Simkani [**5**] for deterministic polynomials in [**36**], in a similar way as some of the above papers, but were able to relax conditions on the random coefficients and to consider more general bases.

THEOREM 2.2. *Suppose that a compact set* $E \subset \mathbb{C}$, $\text{cap}(E) > 0$, *has empty interior and connected complement. If the coefficients* $\{A_k\}_{k=0}^\infty$ *satisfy* (2.1)-(2.2), *and the basis polynomials* $\{B_k\}_{k=0}^\infty$ *satisfy* (2.3), *then the zero counting measures of* $P_n(z) = \sum_{k=0}^n A_k B_k(z)$ *converge almost surely to* $\mu_E$ *as* $n \to \infty$.

For sets with interior points, we introduced an extra assumption on the constant term $A_0$.

THEOREM 2.3. *Let* $E \subset \mathbb{C}$ *be any compact set of positive capacity. If* (2.1)-(2.3) *hold,* $A_0$ *is independent from* $\{A_n\}_{n=1}^\infty$, *and there is* $t > 1$ *such that*

$$\sup_{z\in\mathbb{C}} \mathbb{E}\left[(\log^- |A_0 - z|)^t\right] < \infty, \tag{2.4}$$

*then the zero counting measures of* $P_n(z) = \sum_{k=0}^n A_k B_k(z)$ *converge almost surely to* $\mu_E$ *as* $n \to \infty$.

Assumption (2.4) rules out the possibility that $A_0$ takes any specific value with positive probability. On the other hand, if $A_0$ is a continuous random variable satisfying (2.4), its density need not be bounded. For example, if the probability measure $\nu$ of $A_0$ is absolutely continuous with respect to the area measure $dA$ and has density $d\nu/dA(w)$ uniformly bounded by $C/|w-z|^s,\ s<2$, near every $z\in\mathbb{C}$, then (2.4) holds.

Several applications of Theorems 2.2-2.3 to random orthogonal and random Faber polynomials are given in [**36**]. We provide a generalization of Theorem 2.2 from Kabluchko and Zaporozhets [**27**] as another application.

THEOREM 2.4. *Let $\{w_k\}_{k=0}^\infty$ be a sequence of complex numbers such that*

$$\lim_{k\to\infty} |w_k|^{1/k} = 1/R, \quad R>0.$$

*If (2.1)-(2.2) hold, then the zero counting measures of $P_n(z)=\sum_{k=0}^n A_k w_k z^k$ converge almost surely to the uniform distribution $d\theta/(2\pi)$ on the circle $|z|=R$.*

When $\lim_{k\to\infty}|w_k|^{1/k}=0$, we essentially deal with the partial sums of a random entire function. This interesting case is considered in detail in [**27**], see also [**50**] for local scaling limit results. We do not pursue this case here, as it requires separate treatment.

We now extend the results of [**36**] to more general sequences of random polynomials of the form

$$P_n(z)=\sum_{k=0}^n A_{k,n}B_k(z).$$

Thus we deal with a triangular array of complex random coefficients $A_{k,n},\ k=0,1,\dots,n,\ n\in\mathbb{N}$, instead of a sequence $\{A_k\}_{k=0}^\infty$ considered before. It is necessary to introduce slightly stronger conditions on this array, in order to prove results on the zero distribution.

**Assumption 1*** There is $N\in\mathbb{N}$ such that $\{|A_{k,n}|\}_{k=0}^n$ are jointly independent for each $n\ge N$. Furthermore, there is a function $f:[a,\infty)\to[0,1],\ a>1$, such that $f(x)\log x$ is decreasing, and

$$\int_a^\infty f(x)\,\frac{\log x}{x}\,dx<\infty \text{ and } 1-F_{k,n}(x)\le f(x),\ \forall\, x\in[a,\infty), \tag{2.5}$$

holds for all $k=0,1,\dots,n$, and all $n\ge N$.

**Assumption 2*** There is $N\in\mathbb{N}$ and an increasing function $g:[0,b]\to[0,1],\ 0<b<1$, such that

$$\int_0^b \frac{g(x)}{x}\,dx<\infty \text{ and } F_{k,n}(x)\le g(x),\ \forall\, x\in[0,b], \tag{2.6}$$

holds for all $k=0,1,\dots,n$, and all $n\ge N$.

Theorems 2.1-2.4 have natural generalizations if we replace (2.1) and (2.2) by (2.5) and (2.6). We state two of them below.

THEOREM 2.5. *Suppose that a compact set $E\subset\mathbb{C}$, $\operatorname{cap}(E)>0$, has empty interior and connected complement. If the coefficients $A_{k,n}$ satisfy (2.5)-(2.6), and the basis polynomials $B_k$ satisfy (2.3), then the zero counting measures of $P_n(z)=\sum_{k=0}^n A_{k,n}B_k(z)$ converge almost surely to $\mu_E$ as $n\to\infty$.*

There are many application of this general result. Perhaps most interesting cases are related to random orthogonal and random Faber polynomials. Orthogonality below is considered with respect to the weighted arclength measure $w(s)\,ds$ defined on the rectifiable set $E$.

Corollary 2.6. *Assume that conditions* (2.5)-(2.6) *hold for the coefficients.*
(i) *Suppose that $E$ is a finite union of rectifiable Jordan arcs with connected complement. If the basis polynomials $B_k$ are orthonormal with respect to a positive Borel measure $\mu$ supported on $E$ such that the Radon-Nikodym derivative $w(s) = d\mu/ds > 0$ for almost every $s$, then* (2.3) *is satisfied and $\tau_n$ converge almost surely to $\mu_E$ as $n \to \infty$.*
(ii) *Suppose that $E$ is a compact connected set with empty interior and connected complement, and that $E$ is not a single point. If the basis polynomials $B_k$ are the Faber polynomials of $E$, then* (2.3) *holds true and $\tau_n$ converge almost surely to $\mu_E$ as $n \to \infty$.*

If $E$ has interior, then we again need to prevent accumulation of zeros there by imposing an additional assumption.

Theorem 2.7. *Let $E \subset \mathbb{C}$ be any compact set of positive capacity. If* (2.3), (2.5) *and* (2.6) *hold, and there is $t > 1$ such that*

$$\limsup_{n\to\infty} \sup_{z\in\mathbb{C}} \mathbb{E}\left[(\log^- |A_{0,n} - z|)^t\right] < \infty, \tag{2.7}$$

*then the zero counting measures of $P_n(z) = \sum_{k=0}^n A_{k,n}B_k(z)$ converge almost surely to $\mu_E$ as $n \to \infty$.*

One can give applications of this theorem to random orthogonal polynomials with respect to the arclength and the area measures, as well as to random Faber polynomials, similar to Corollary 2.6.

## 3. Expected Number of Zeros

We now discuss problems on bounds and asymptotic results for the expected number of zeros in various sets. The first group of results provide quantitative estimates for the weak convergence of the zero counting measures of random polynomials to the corresponding equilibrium measures. In particular, we study the deviation of $\tau_n$ from $\mu_E$ on certain sets, which is often referred to as discrepancy between those measures. We again assume that the complex valued random variables $A_k,\ k = 0, 1, 2, \ldots$, are not necessarily independent nor identically distributed. It is convenient to first discuss the simplest case of the unit circle, which originated in [**37**]. A standard way to study the deviation of $\tau_n$ from $\mu_{\mathbb{T}}$ is to consider the discrepancy of these measures in the annular sectors of the form

$$A_r(\alpha, \beta) = \{z \in \mathbb{C} : r < |z| < 1/r,\ \alpha \le \arg z < \beta\}, \quad 0 < r < 1.$$

The recent paper [**38**] contains the following estimate of the discrepancy.

Theorem 3.1. *Suppose that the coefficients of $P_n(z) = \sum_{k=0}^n A_k z^k$ are complex random variables that satisfy:*

(1) $\mathbb{E}[|A_k|^t] < \infty,\ k = 0, \ldots, n,$ *for a fixed* $t \in (0, 1]$
(2) $\mathbb{E}[\log |A_0|] > -\infty$ *and* $\mathbb{E}[\log |A_n|] > -\infty$.

*Then we have for all large $n \in \mathbb{N}$ that*

(3.1)
$$\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right] \le C_r \left[\frac{1}{n}\left(\frac{1}{t}\log\sum_{k=0}^{n}\mathbb{E}[|A_k|^t] - \frac{1}{2}\mathbb{E}[\log|A_0 A_n|]\right)\right]^{1/2},$$

*where*

$$C_r := \sqrt{\frac{2\pi}{\mathbf{k}}} + \frac{2}{1-r} \quad \textit{with} \quad \mathbf{k} := \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$$

*being Catalan's constant.*

Introducing uniform bounds, [**38**] also provides the rates of convergence for the expected discrepancy as $n \to \infty$.

Corollary 3.2. *Let $P_n(z) = \sum_{k=0}^{n} A_{k,n} z^k,\ n \in \mathbb{N}$, be a sequence of random polynomials. If*

$$M := \sup\{\mathbb{E}[|A_{k,n}|^t] \mid k = 0,\ldots,n,\ n \in \mathbb{N}\} < \infty$$

*and*

$$L := \inf\{\mathbb{E}[\log|A_{k,n}|] \mid k = 0 \,\&\, n,\ n \in \mathbb{N}\} > -\infty,$$

*then*

$$\begin{aligned}\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right] &\le C_r\left[\frac{1}{n}\left(\frac{\log(n+1)+\log M}{t} - L\right)\right]^{1/2} \\ &= O\left(\sqrt{\frac{\log n}{n}}\right) \quad \textit{as } n \to \infty.\end{aligned}$$

It is well known from the work of Erdős and Turán [**16**] that the order $\sqrt{\log n/n}$ is optimal in the deterministic case. Papers [**37**] and [**38**] explain how one can obtain quantitative results about the expected number of zeros of random polynomials in various sets, see Propositions 2.3-2.5 of [**38**]. The basic observation here is that the number of zeros of $P_n$ in a set $S \subset \mathbb{C}$ denoted by $N_n(S)$ is equal to $n\tau_n(S)$, and the estimates for $\mathbb{E}[N_n(S)]$ readily follow from Theorem 3.1 and Corollary 3.2.

We now turn to random polynomials spanned by the general bases $B_k(z) = \sum_{j=0}^{k} b_{j,k} z^j,\ k = 0,1,\ldots,$ where $b_{j,k} \in \mathbb{C}$ for all $j$ and $k$, and $b_{k,k} \neq 0$ for all $k$. These bases are considered in connection with an arbitrary compact set $E$ of positive capacity in the plane, whose equilibrium measure is denoted by $\mu_E$. In [**36**], we obtained several expected discrepancy results for the pair $\tau_n$ and $\mu_E$ on smooth closed domains and arcs by using the corresponding results for deterministic polynomials due to Andrievskii and Blatt [**1**]. We continue here with similar estimates for quasiconformal domains that may have infinitely many corners at the boundary. A closed Jordan curve $L$ is called quasiconformal (or quasicircle) if there is a constant $a > 0$ such that

$$\min(\operatorname{diam}\gamma_1(z,t), \operatorname{diam}\gamma_2(z,t)) \le a|z-t| \quad \forall\, z,t \in L,$$

where $\gamma_1(z,t)$ and $\gamma_2(z,t)$ are the two subarcs of $L$ with endpoints $z$ and $t$. It is well known that quasicircles need not be rectifiable and may have corners even at a dense subset of itself, see Appendix B of [**1**] for background and further references.

In order to obtain the rates of convergence as in Corollary 3.2, we assume that the basis satisfies

$$\|B_k\|_E = O(k^p) \quad \text{and} \quad |b_{k,k}|(\text{cap}(E))^k \geq c\, k^{-q} \quad \text{as } k \to \infty, \tag{3.2}$$

with fixed positive constants $c, p, q$. This condition holds for many important bases such as orthogonal polynomials and Faber polynomials.

Instead of the annular sectors $A_r(\alpha, \beta)$, we use the "generalized sectors" $A_r$ defined by conformal mappings. For any closed Jordan curve $L$, its complement $\overline{\mathbb{C}}\setminus L$ consists of the bounded domain $G$ and the unbounded domain $\Omega$. We introduce the standard conformal mappings $\phi : G \to \mathbb{D},\ \phi(z_0) = 0,\ \phi'(z_0) > 0$, where $z_0 \in G$, and $\Phi : \Omega \to \Delta := \{w : |w| > 1\},\ \Phi(\infty) = \infty,\ \Phi'(\infty) > 0$. It is well known that both mappings can be extended to $L$ so that $\phi$ becomes a homeomorphism between $\overline{G}$ and $\overline{\mathbb{D}}$, while $\Phi$ becomes a homeomorphism between $\overline{\Omega}$ and $\overline{\Delta}$. Denote the inverse mappings by $\psi := \phi^{-1}$ and $\Psi := \Phi^{-1}$. For any subarc $J \subset L$ and $r \in (0, 1)$, let

$$\begin{aligned} A_r = A_r(J) = \{z \in \overline{\Omega} : 1 \leq |\Phi(z)| \leq 1/r \text{ and } \Phi(z)/|\Phi(z)| \in \Phi(J)\} \\ \cup \{z \in \overline{G} : r \leq |\phi(z)| \leq 1 \text{ and } \phi(z)/|\phi(z)| \in \phi(J)\}. \end{aligned}$$

Thus $A_r$ is a curvilinear strip around $J$ that is bounded by the level curves $|\Phi(z)| = 1/r$ and $|\phi(z)| = r,\ 0 < r < 1$. It is known [**1**] that the conformal maps $\psi$ and $\Phi$ are Hölder continuous up to the boundary of their domains of definition when $L$ is a quasicirle. Hence there is $\alpha \in (0, 1]$ and $b > 0$ such that

$$|\Phi \circ \psi(w_1) - \Phi \circ \psi(w_2)| \leq b|w_1 - w_2|^\alpha \quad \forall\, w_1, w_2 \in \mathbb{T}.$$

This Hölder exponent $\alpha$ is crucial for the expected discrepancy estimate.

Theorem 3.3. *Suppose that $E \subset \mathbb{C}$ is a compact set bounded by a quasiconformal curve $L = \partial E$ with interior $G = E^\circ$, and that $w \in G$. For $P_n(z) = \sum_{k=0}^n A_k B_k(z)$, let $\{A_k\}_{k=0}^n$ satisfy $\mathbb{E}[|A_k|^t] < \infty,\ k = 0, \ldots, n$, for a fixed $t \in (0, 1]$. If $\mathbb{E}[\log|A_n P_n(w)|] > -\infty$ then for every arc $J \subset L$ and $A_r = A_r(J)$ we have*

$$\begin{aligned} &\mathbb{E}\left[|(\tau_n - \mu_E)(A_r)|\right] \\ &\leq C\left[\frac{1}{n}\left(\frac{2}{t}\log\left(\sum_{k=0}^n \mathbb{E}[|A_k|^t]\right) + \log\frac{\max_{0\leq k\leq n}\|B_k\|_E^2}{|b_{n,n}|(\text{cap}(E))^n} - \mathbb{E}[\log|A_n P_n(w)|]\right)\right]^{\alpha/(1+\alpha)}, \end{aligned} \tag{3.3}$$

*where $C > 0$ is independent of $n$, $P_n$ and $J$.*

*In particular, if $\mathbb{E}[\log|A_n|] > -\infty$, $A_0$ is independent from $A_1, A_2, \ldots, A_n$, and $\mathbb{E}[\log|A_0 + z|] \geq T > -\infty$ for all $z \in \mathbb{C}$, then*

$$\mathbb{E}[\log|A_n P_n(w)|] \geq \log|b_{0,0}| + \mathbb{E}[\log|A_n|] + T > -\infty, \tag{3.4}$$

*and* (3.3) *holds.*

The Hölder exponent $\alpha$ depends on the geometric properties of $L$, mainly on the angles. Thus if $L$ is Dini-smooth (cf. [**1**, p. 72]), we have that $\alpha = 1$ and the above result essentially reduces to Theorem 3.6 of [**36**]. Furthermore, if $L$ consists of $m$ Dini-smooth arcs that form interior (in $G$) corners of magnitudes $\beta_j\pi,\ j = 1, \ldots, m$, at the junction points, then (see [**1**, p. 72])

$$\alpha = \min\left(\min_{1\leq j\leq m}\frac{\beta_j}{2-\beta_j}, 1\right).$$

Note also that if $\nu$ is the probability measure of $A_0$, then the assumption $\mathbb{E}[\log|A_0+z|] \ge T > -\infty$ for all $z \in \mathbb{C}$ may be interpreted in terms of the logarithmic potential of $\nu$ as $U^\nu(z) = -\int \log|t-z|\,d\nu(t) \le -T < \infty$ for all $z \in \mathbb{C}$. Measures with uniformly bounded above potentials are well understood in potential theory, and they do not have large local concentration of mass, e.g., point masses.

Corollary 3.4. *Let $P_n(z) = \sum_{k=0}^n A_{k,n}B_k(z),\ n \in \mathbb{N}$, be a sequence of random polynomials, and let $L$ satisfy the assumptions of Theorem 3.3. Suppose that for $t \in (0,1]$ we have*

$$\limsup_{n\to\infty} \max_{k=0,\dots,n} \mathbb{E}[|A_{k,n}|^t] < \infty. \tag{3.5}$$

*Assume further that*

$$\liminf_{n\to\infty} \mathbb{E}[\log|A_{n,n}|] > -\infty, \tag{3.6}$$

*$A_{0,n}$ is independent from $\{A_{k,n}\}_{k=1}^n$ for all large $n$, and*

$$\liminf_{n\to\infty} \inf_{z\in\mathbb{C}} \mathbb{E}[\log|A_{0,n}+z|] > -\infty. \tag{3.7}$$

*If the basis polynomials $B_k$ satisfy* (3.2)*, then*

$$\mathbb{E}\left[|(\tau_n-\mu_E)(A_r)|\right] = O\left[\left(\frac{\log n}{n}\right)^{\alpha/(1+\alpha)}\right] \quad \text{as } n\to\infty. \tag{3.8}$$

We give examples of typical bases satisfying (3.2) below.

Corollary 3.5. *Assume that conditions* (3.5)*,* (3.6) *and* (3.7) *hold for the coefficients.*
(i) *Suppose that $L$ is a rectifiable quasiconformal curve. If the basis polynomials $B_k$ are orthonormal with respect to a positive Borel measure $\mu$ supported on $L$ such that $d\mu(s) = w(s)\,ds$, where $w(s) \ge c > 0$ for almost every point of $L$ in $ds$-sense, then* (3.2) *is satisfied and* (3.8) *holds true.*
(ii) *Suppose that $L$ is an arbitrary Jordan curve. If the basis polynomials $B_k$ are the Faber polynomials of $E$, then* (3.2) *holds true. Hence* (3.8) *is valid provided $L$ is a quasiconformal curve.*
(iii) *Suppose that $L$ is an arbitrary quasiconformal curve with interior $G$. If the basis polynomials $B_k$ are orthonormal with respect to $d\mu(z) = w(z)\,dA(z)$, where $dA$ is the area measure on $G$ and $w(z) \ge c > 0$ a.e. in $dA$-sense, then* (3.2) *is satisfied and* (3.8) *holds true.*

Similar results can be proved for quasiconformal arcs by using Theorem 2.4 of [**1**, p. 69]. The case of smooth arcs was already considered in [**36**], see Theorem 3.3 and Corollaries 3.4 and 3.5 in that paper. We do not attempt to generalize along this line, but instead consider the illuminating case of random orthogonal polynomials on a real line segment. Let $\mu$ be a positive Borel measure with finite moments of all orders, supported on $[-1,1]$. Consider the orthonormal polynomials $\{B_k\}_{k=0}^\infty$ with respect to this measure, and the corresponding random polynomials

$$P_n(x) = \sum_{k=0}^n A_k B_k(z).$$

If $d\mu(x) = w(x)dx$, where $w(x) > 0$ a.e. on $[-1,1]$, then

$$\tau_n \xrightarrow{w} \mu_{[-1,1]} = \frac{dx}{\pi\sqrt{1-x^2}} \quad a.s.$$

under very general assumptions on random coefficients. This holds for the coefficients satisfying (2.1)-(2.2) by Corollary 2.3 of [**36**], and for the coefficients satisfying (2.5)-(2.6) by Corollary 2.6 of this paper.

Assume for simplicity that the coefficients $\{A_k\}_{k=0}^\infty$ are i.i.d. random variables such that $\mathbb{E}[|A_0|^t] < \infty$ for a fixed $t \in (0,1]$ and $\mathbb{E}[\log |A_0|] > -\infty$. Corollary 3.4 of [**36**] implies that

$$\mathbb{E}[N_n(A_r)] = \mu_{[-1,1]}([a,b])\, n + o(n) = \frac{\arcsin b - \arcsin a}{\pi}\, n + o(n),$$

for any interval $[a,b] \subset [-1,1]$ and its neighborhood $A_r$. Thus we clearly see that most of the zeros are asymptotically distributed near $[-1,1]$ according to the measure $\mu_{[-1,1]}$, and we can even provide estimates of the expected number of zeros near every subinterval of $[-1,1]$. It is very interesting, and is not obvious, that a large fraction of zeros of such random orthogonal polynomials are actually *real*.

If the coefficients are i.i.d. with standard real Gaussian distribution, Das [**12**] considered random Lengendre polynomials, and found that $\mathbb{E}[N_n(-1,1)]$ is asymptotically equal to $n/\sqrt{3}$. Wilkins [**56**] improved the error term in this asymptotic relation by showing that $\mathbb{E}[N_n(-1,1)] = n/\sqrt{3} + o(n^\varepsilon)$ for any $\varepsilon > 0$. Later, Das and Bhatt [**13**] concluded that $\mathbb{E}[N_n(-1,1)]$ is asymptotically equal to $n/\sqrt{3}$ for random Jacobi polynomials with the standard Gaussian coefficients. Zeros of a random Legendre polynomial are pictured in Figure 1. One may find more interesting pictures and computations of zeros of random orthogonal polynomials by Trefethen on his CHEBFUN page [**52**]. We conjecture that the asymptotic relation

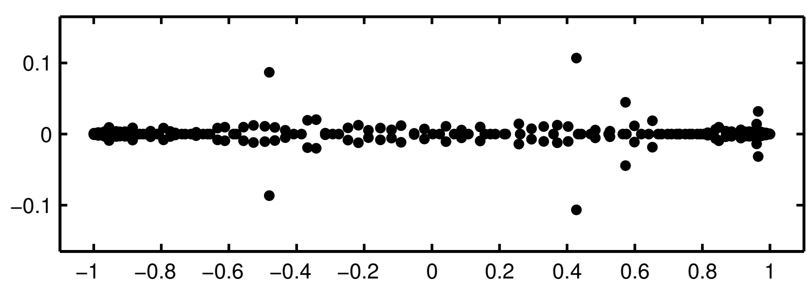


FIGURE 1. Zeros of a random Legendre polynomial of degree 200

$\mathbb{E}[N_n(-1,1)] \sim n/\sqrt{3}$ holds for large classes of random orthogonal polynomials with Gaussian coefficients, under weak assumptions on the orthogonality measure $\mu$.

It is worth mentioning that random trigonometric polynomials were also studied by many authors, see [**4**] and [**17**]. In fact, random trigonometric polynomials are related to random Chebyshev polynomials by a change of variable. There are many other interesting directions of research in the general area of random analytic functions that are not even touched here, see [**22**] for example.

## 4. Proofs

**4.1. Proofs for Section 2.** We state a slightly modified version of the result due to Blatt, Saff and Simkani [**5**], which is used to prove all equidistribution theorems of Section 2.

**Theorem BSS.** *Let $E \subset \mathbb{C}$ be a compact set,* $\mathrm{cap}(E) > 0$. *If a sequence of polynomials $P_n(z) = \sum_{k=0}^n c_{k,n} z^k$ satisfy*

$$\limsup_{n\to\infty} \|P_n\|_E^{1/n} \le 1 \quad \text{and} \quad \lim_{n\to\infty} |c_{n,n}|^{1/n} = 1/\mathrm{cap}(E), \tag{4.1}$$

*and for any closed set $A$ in the bounded components of $\mathbb{C} \setminus \operatorname{supp} \mu_E$ we have*

$$\lim_{n\to\infty} \tau_n(A) = 0, \tag{4.2}$$

*then the zero counting measures $\tau_n$ converge weakly to $\mu_E$ as $n \to \infty$.*

It is known that (4.2) holds if every bounded component of $\mathbb{C}\setminus\operatorname{supp}\mu_E$ contains a compact set $K$ such that

$$\liminf_{n\to\infty} \|P_n\|_K^{1/n} \ge 1, \tag{4.3}$$

see Grothmann [**19**] (and also [**1**]) for the case of unbounded component of $\mathbb{C} \setminus \operatorname{supp}\mu_E$, and see Bloom [**8, 9**]. In applications, this compact set $K$ is often selected as a single point.

One of the main ingredients in the applications of this result is the $n$-th root limiting behavior of coefficients. We provide the following probabilistic versions of such limits. Let $\{X_n\}_{n=1}^\infty$ be a sequence of complex valued random variables, and let $F_n$ be the distribution function of $|X_n|$, $n \in \mathbb{N}$. We use the assumptions on random variables $X_n$ that match those of (2.1) and (2.2) in Section 2.

Lemma 4.1. *If there is $N \in \mathbb{N}$ and a decreasing function $f : [a,\infty) \to [0,1]$, $a > 1$, such that*

$$\int_a^\infty \frac{f(x)}{x}\,dx < \infty \text{ and } 1 - F_n(x) \le f(x),\ \forall\, x \in [a,\infty),$$

*holds for all $n \ge N$, then*

$$\limsup_{n\to\infty} |X_n|^{1/n} \le 1 \qquad a.s. \tag{4.4}$$

*Further, if there is $N \in \mathbb{N}$ and an increasing function $g : [0,b] \to [0,1]$, $0 < b < 1$, such that*

$$\int_0^b \frac{g(x)}{x}\,dx < \infty \text{ and } F_n(x) \le g(x),\ \forall\, x \in [0,b],$$

*holds for all $n \ge N$, then*

$$\liminf_{n\to\infty} |X_n|^{1/n} \ge 1 \qquad a.s. \tag{4.5}$$

*Hence if both assumptions are satisfied, then*

$$\lim_{n\to\infty} |X_n|^{1/n} = 1 \qquad a.s. \tag{4.6}$$

We use a standard method for finding the almost sure limits of (4.4)-(4.6) via the first Borel-Cantelli lemma stated below (see, e.g., [**20**, p. 96]).

**Borel-Cantelli Lemma** *Let $\{\mathcal{E}_n\}_{n=1}^\infty$ be a sequence of arbitrary events. If $\sum_{n=1}^\infty \mathbb{P}(\mathcal{E}_n) < \infty$ then $\mathbb{P}(\mathcal{E}_n$ occurs infinitely often$) = 0$.*

Proof of Lemma 4.1. We first prove (4.4). For any fixed $\varepsilon > 0$, define events $\mathcal{E}_n = \{|X_n| > e^{\varepsilon n}\},\ n \in \mathbb{N}$. Using the first assumption and letting $m := \max(N, \lfloor \frac{1}{\varepsilon} \log a \rfloor) + 2$, we obtain

$$\sum_{n=m}^{\infty} \mathbb{P}(\mathcal{E}_n) = \sum_{n=m}^{\infty} (1 - \mathbb{P}(\{|X_n| \le e^{\varepsilon n}\})) = \sum_{n=m}^{\infty} (1 - F_n(e^{\varepsilon n})) \le \sum_{n=m}^{\infty} f(e^{\varepsilon n})$$
$$\le \int_{m-1}^{\infty} f(e^{\varepsilon t})\,dt \le \frac{1}{\varepsilon} \int_a^{\infty} \frac{f(x)}{x}\,dx < \infty.$$

Hence $\mathbb{P}(\mathcal{E}_n$ occurs infinitely often$) = 0$ by the first Borel-Cantelli lemma, so that the complementary event $\mathcal{E}_n^c$ must happen for all large $n$ with probability 1. This means that $|X_n|^{1/n} \le e^{\varepsilon}$ for all sufficiently large $n \in \mathbb{N}$ almost surely. Thus

$$\limsup_{n\to\infty} |X_n|^{1/n} \le e^{\varepsilon} \quad \text{a.s.},$$

and (4.4) follows because $\varepsilon > 0$ may be arbitrarily small.

The proof of (4.5) proceeds in a similar way. For any given $\varepsilon > 0$, we set $\mathcal{E}_n = \{|X_n| \le e^{-\varepsilon n}\},\ n \in \mathbb{N}$. Using the second assumption and letting $m := \max(N, \lfloor -\frac{1}{\varepsilon} \log b \rfloor) + 2$, we have

$$\sum_{n=m}^{\infty} \mathbb{P}(\mathcal{E}_n) = \sum_{n=m}^{\infty} F_n(e^{-\varepsilon n}) \le \sum_{n=m}^{\infty} g(e^{-\varepsilon n})$$
$$\le \int_{m-1}^{\infty} g(e^{-\varepsilon t})\,dt \le \frac{1}{\varepsilon} \int_0^{b} \frac{g(x)}{x}\,dx < \infty.$$

Hence $\mathbb{P}(\mathcal{E}_n$ i.o.$) = 0$, and $|X_n|^{1/n} > e^{-\varepsilon}$ holds for all sufficiently large $n \in \mathbb{N}$ almost surely. We obtain that

$$\liminf_{n\to\infty} |X_n|^{1/n} \ge e^{-\varepsilon} \quad \text{a.s.},$$

and (4.5) follows by letting $\varepsilon \to 0$. □

Lemma 4.1 implies that any infinite sequence of coefficients satisfying Assumptions 1 and 2 of Section 2 must also satisfy (4.6). We state this as follows.

Lemma 4.2. *Suppose that* (2.1) *and* (2.2) *hold for the coefficients $A_n$ of random polynomials. Then the following limits exist almost surely:*

$$\lim_{n\to\infty} |A_n|^{1/n} = 1 \quad a.s., \tag{4.7}$$

$$\lim_{n\to\infty} |A_k|^{1/n} = 1 \quad a.s.,\ k = 0, 1, 2, \ldots, \tag{4.8}$$

*and*

$$\lim_{n\to\infty} \max_{0\le k\le n} |A_k|^{1/n} = 1 \quad a.s. \tag{4.9}$$

Proof of Lemma 4.2. Limit (4.7) follows from Lemma 4.1 by letting $X_n = A_n,\ n \in \mathbb{N}$. Similarly, if we set for a fixed $k \in \mathbb{N} \cup \{0\}$ that $X_n = A_k,\ n \in \mathbb{N}$, then (4.8) is immediate.

We deduce (4.9) from (4.7). Let $\omega$ be any elementary event such that

$$\lim_{n\to\infty} |A_n(\omega)|^{1/n} = 1,$$

which holds with probability one. We immediately obtain that

$$\liminf_{n\to\infty} \max_{0\le k\le n} |A_k(\omega)|^{1/n} \ge \liminf_{n\to\infty} |A_n(\omega)|^{1/n} = 1.$$

On the other hand, elementary properties of limits imply that

$$\limsup_{n\to\infty} \max_{0\le k\le n} |A_k(\omega)|^{1/n} \le 1.$$

Indeed, for any $\varepsilon > 0$ there $n_\varepsilon \in \mathbb{N}$ such that $|A_n(\omega)|^{1/n} \le 1+\varepsilon$ for all $n \ge n_\varepsilon$ by (4.7). Hence

$$\max_{0\le k\le n} |A_k(\omega)|^{1/n} \le \max\left(\max_{0\le k\le n_\varepsilon} |A_k(\omega)|^{1/n}, 1+\varepsilon\right) = 1+\varepsilon \quad \text{for all large } n,$$

and the result follows by letting $\varepsilon \to 0$. □

The following lemma replaces Lemma 4.2 under Assumptions 1* and 2*.

Lemma 4.3. *Suppose that* (2.5) *and* (2.6) *hold for the coefficients* $A_{k,n}$ *of random polynomials. Then the following limits exist almost surely:*

$$\lim_{n\to\infty} |A_{n,n}|^{1/n} = 1 \quad a.s., \tag{4.10}$$

$$\lim_{n\to\infty} |A_{k,n}|^{1/n} = 1 \quad a.s.,\ k \in \mathbb{N}\cup\{0\}, \tag{4.11}$$

*and*

$$\lim_{n\to\infty} \max_{0\le k\le n} |A_{k,n}|^{1/n} = 1 \quad a.s. \tag{4.12}$$

Proof of Lemma 4.3. Limits (4.10) and (4.11) follow from Lemma 4.1 by correspondingly letting $X_n = A_{n,n},\ n \in \mathbb{N}$, and $X_n = A_{k,n},\ n\in\mathbb{N}$, for a fixed $k \in \mathbb{N}\cup\{0\}$. In fact, this argument holds under weaker assumptions such as (2.1) and (2.2), and does not require independence of coefficients.

In order to prove (4.12), we introduce the random variable $Y_n = \max_{0\le k\le n} |A_{k,n}|$, and denote its distribution function by $F_n(x),\ n\in\mathbb{N}$. Note that

$$\liminf_{n\to\infty} |Y_n|^{1/n} \ge \liminf_{n\to\infty} |A_{n,n}|^{1/n} = 1 \quad \text{a.s.}$$

Using independence of $|A_{k,n}|,\ k=0,\dots,n$, for each $n\ge N$, and applying (2.5), we estimate

$$F_n(x) = \prod_{k=0}^{n} F_{k,n}(x) \ge (1-f(x))^{n+1} \ge 1-(n+1)f(x), \quad x \ge a.$$

For any fixed $\varepsilon > 0$, define events $\mathcal{E}_n = \{|Y_n| > e^{\varepsilon n}\},\ n\in\mathbb{N}$. Letting $m := \max(N, \lfloor \frac{1}{\varepsilon}\log a\rfloor) + 2$, we obtain from the above estimate and (2.5) that

$$\begin{aligned}\sum_{n=m}^{\infty} \mathbb{P}(\mathcal{E}_n) &= \sum_{n=m}^{\infty} (1-\mathbb{P}(\{|Y_n| \le e^{\varepsilon n}\})) = \sum_{n=m}^{\infty} (1-F_n(e^{\varepsilon n})) \le \sum_{n=m}^{\infty} (n+1) f(e^{\varepsilon n}) \\ &\le 2\int_{m-1}^{\infty} t\, f(e^{\varepsilon t})\, dt \le \frac{2}{\varepsilon^2}\int_a^\infty \frac{f(x)\log x}{x}\, dx < \infty.\end{aligned}$$

Hence $\mathbb{P}(\mathcal{E}_n \text{ i.o.}) = 0$ by the first Borel-Cantelli lemma, and $|Y_n|^{1/n} \le e^{\varepsilon}$ for all sufficiently large $n\in\mathbb{N}$ almost surely. We obtain that

$$\limsup_{n\to\infty} |Y_n|^{1/n} \le e^{\varepsilon} \quad \text{a.s.},$$

and (4.12) follows after letting $\varepsilon \to 0$. $\square$

The proofs of Theorems 2.1, 2.2 and 2.3 may be found in [**36**], so that we omit them here.

Proof of Theorem 2.4. We assume that $w_0 \neq 0$, for otherwise we can replace $P_n(z)$ with $P_n(z)/z^m$, where $m = \min(k \in \mathbb{N} : w_k \neq 0)$. The result is deduced from Theorem BSS with $E = \{z : |z| = R\}$. Recall that $\mathrm{cap}(E) = R$, and the equilibrium measure of $E$ is $d\mu_E(Re^{it}) = dt/(2\pi)$, see [**39**]. Note that

$$\|P_n\|_E \le \sum_{k=0}^{n} |A_k w_k z^k| \le (n+1) \max_{0\le k\le n} |A_k| \max_{0\le k\le n} |w_k| R^k.$$

It follows from an elementary argument as in the proof of (4.9) that

$$\lim_{k\to\infty} |w_k|^{1/k} R = 1 \quad \Rightarrow \quad \lim_{n\to\infty} \left( \max_{0\le k\le n} |w_k| R^k \right)^{1/n} = 1.$$

Indeed, we immediately obtain that

$$\liminf_{n\to\infty} \left( \max_{0\le k\le n} |w_k| R^k \right)^{1/n} \ge \liminf_{n\to\infty} |w_n|^{1/n} R = 1.$$

For any $\varepsilon > 0$ there is $k_\varepsilon \in \mathbb{N}$ such that $|w_k| R^k \le (1+\varepsilon)^k$ for all $k \ge k_\varepsilon$. This gives

$$\left( \max_{0\le k\le n} |w_k| R^k \right)^{1/n} \le \max \left( \max_{0\le k\le k_\varepsilon} |w_k|^{1/n} R^{k/n}, 1+\varepsilon \right) = 1+\varepsilon \quad \text{for all large } n.$$

Thus we have that

$$\limsup_{n\to\infty} \left( \max_{0\le k\le n} |w_k| R^k \right)^{1/n} \le 1+\varepsilon,$$

and the claim follows by letting $\varepsilon \to 0$.

Using (4.7) and (4.9) of Lemma 4.2, we conclude that (4.1) holds almost surely. On the other hand, (4.8) with $k = 0$ also gives that

$$\lim_{n\to\infty} |P_n(0)|^{1/n} = \lim_{n\to\infty} |w_0 A_0|^{1/n} = \lim_{n\to\infty} |A_0|^{1/n} = 1 \quad \text{a.s.},$$

meaning that (4.3) is satisfied for $K = \{0\}$ almost surely. Hence (4.2) holds a.s. for any compact subset $A$ of the unit disk, which completes the proof. $\square$

Proof of Theorem 2.5. Since $\mathrm{supp}\,\mu_E \subset E$, we have that $\mathbb{C} \setminus \mathrm{supp}\,\mu_E$ has no bounded components in this case, and (4.2) of Theorem BSS holds trivially. Thus we only need to prove (4.1) for polynomials

$$P_n(z) = \sum_{k=0}^{n} A_{k,n} B_k(z) = A_{n,n} b_{n,n} z^n + \ldots, \quad n \in \mathbb{N}.$$

Applying (4.10) of Lemma 4.3 and (2.3), we obtain for their leading coefficients that

$$\lim_{n\to\infty} |A_{n,n} b_{n,n}|^{1/n} = 1/\mathrm{cap}(E) \quad \text{a.s.}$$

Furthermore,

$$\|P_n\|_E \le \sum_{k=0}^{n} |A_{k,n}| \|B_k\|_E \le (n+1) \max_{0\le k\le n} |A_{k,n}| \max_{0\le k\le n} \|B_k\|_E.$$

Note that (2.3) implies by a simple argument (already used in the proof of Lemma 4.2) that

$$\limsup_{n\to\infty} \max_{0\le k\le n} \|B_k\|_E^{1/n} \le 1.$$

Combining this fact with (4.12) of Lemma 4.3, we obtain that

$$\limsup_{n\to\infty} \|P_n\|_E^{1/n} \le 1 \quad \text{a.s.}$$

□

PROOF OF COROLLARY 2.6. Since the coefficient conditions (2.5)-(2.6) hold by our assumptions, we only need to verify that the bases satisfy (2.3) in both cases (i) and (ii). Then almost sure convergence of $\tau_n$ to $\mu_E$ will follow from Theorem 2.5.

(i) Our assumptions on the orthogonality measure $\mu$ and set $E$ imply that the orthogonal polynomials have regular asymptotic behavior expressed by (2.3) according to Theorem 4.1.1 and Corollary 4.1.2 of [**48**, pp. 101-102]. Corollary 4.1.2 is stated for a set $E$ consisting of smooth arcs and curves, but its proof holds for arbitrary rectifiable case, because $\mu$ and $\mu_E$ are both absolutely continuous with respect to the arclength $ds$. In fact, it is known that the density of the equilibrium measure is expressed via normal derivatives of the Green function $g_E$ for the complement of $E$ from both sides of the arcs:

$$d\mu_E = \frac{1}{2\pi}\left(\frac{\partial g_E}{\partial n_+} + \frac{\partial g_E}{\partial n_-}\right) ds,$$

see Theorem 1.1 and Example 1.2 of [**35**]. Furthermore, $d\mu_E/ds > 0$ almost everywhere in the sense of arclength on $E$, see Garnett and Marshall [**18**].

(ii) Assumptions imposed on $E$ imply that $\text{cap}(E) > 0$, and that Faber polynomials are well defined. In particular, the Faber polynomials of $E$ satisfy $B_n(z) = z^n/(\text{cap}(E))^n + \dots,\ n = 0, 1, \dots$, by definition, see [**49**]. Furthermore, Kövari and Pommerenke [**28**] showed that the Faber polynomials of any compact connected set do not grow fast:

$$\|B_n\|_E = O(n^s) \quad \text{as } n \to \infty,$$

where $s < 1/2$. Hence (2.3) holds true in this case. □

PROOF OF THEOREM 2.7. We use Theorem BSS again. Since (4.1) is verified exactly as in the proof of Theorem 2.5, we do not repeat that argument.

It remains to show that (4.2) holds almost surely as a consequence of (2.7), which is again done via (4.3). In particular, we prove that

$$\liminf_{n\to\infty} |P_n(w)|^{1/n} \ge 1 \tag{4.13}$$

holds almost surely for every given $w \in \mathbb{C}$. Define the events

$$\mathcal{E}_n = \{|P_n(w)| \le e^{-\varepsilon n}\} = \left\{\frac{1}{\varepsilon}\log^- |P_n(w)| \ge n\right\}, \quad n \in \mathbb{N}.$$

For any fixed $t > 1$, Chebyshev's inequality gives

$$\mathbb{P}(\mathcal{E}_n) \le \frac{1}{n^t}\mathbb{E}\left[\left(\frac{1}{\varepsilon}\log^- |P_n(w)|\right)^t\right], \quad n \in \mathbb{N}.$$

Note that

$$\left(\log^- |P_n(w)|\right)^t \le \left(\log^- |b_{0,0}| + \log^- \left|A_{0,n} + \sum_{k=1}^n \frac{A_{k,n}}{b_{0,0}} B_k(w)\right|\right)^t$$
$$\le 2^t \left(\left(\log^- |b_{0,0}|\right)^t + \left(\log^- \left|A_{0,n} + \sum_{k=1}^n \frac{A_{k,n}}{b_{0,0}} B_k(w)\right|\right)^t\right).$$

Denoting the value of limsup in (2.7) by $C$, we obtain that

$$\mathbb{E}\left[\left(\log^- |P_n(w)|\right)^t\right] \le 2^t \left(\left(\log^- |b_{0,0}|\right)^t + C + 1\right)$$

holds for all sufficiently large $n \in \mathbb{N}$. It follows that

$$\sum_{n=1}^\infty \mathbb{P}(\mathcal{E}_n) \le \frac{2^t}{\varepsilon^t} \left(\left(\log^- |b_{0,0}|\right)^t + C + 1\right) \sum_{n=1}^\infty \frac{1}{n^t} < \infty.$$

Hence $\mathbb{P}(\mathcal{E}_n \text{ i.o.}) = 0$ by the first Borel-Cantelli lemma, and $|P_n(w)|^{1/n} > e^{-\varepsilon}$ holds for all sufficiently large $n \in \mathbb{N}$ with probability one. We obtain that

$$\liminf_{n\to\infty} |P_n(w)|^{1/n} \ge e^{-\varepsilon} \quad \text{a.s.,}$$

and (4.13) follows by letting $\varepsilon \to 0$. $\square$

**4.2. Proofs for Section 3.** A proof of Theorem 3.1 may be found in [**38**], while that of Corollary 3.2 is immediate from (3.1) and the bounds $M$ and $L$.

If $A_k,\ k = 0, \ldots, n$, are complex random variables satisfying $\mathbb{E}[|A_k|^t] < \infty,\ k = 0, \ldots, n$, for a fixed $t \in (0, 1]$, then we have by Jensen's inequality that

$$\mathbb{E}\left[\log \sum_{k=0}^n |A_k|\right] \le \frac{1}{t} \log \left(\sum_{k=0}^n \mathbb{E}[|A_k|^t]\right). \tag{4.14}$$

A proof of this elementary fact is contained in [**36**], see Lemma 4.4.

Proof of Theorem 3.3. Observe that the leading coefficient of $P_n$ is $A_n b_{n,n}$. Let $A_r$ be a "strip" around a subarc $J \subset L$. We use Theorem 2.1 from Chapter 2 of [**1**, p. 59] for the needed discrepancy estimate:

$$|(\tau_n - \mu_E)(A_r)| \le C \left(\frac{1}{n} \log \frac{\|P_n\|_E}{|A_n b_{n,n}|(\mathrm{cap}(E))^n} + \frac{1}{n} \log \frac{\|P_n\|_E}{|P_n(w)|}\right)^{\alpha/(1+\alpha)}, \tag{4.15}$$

where $w \in G$ and the constant $C > 0$ is independent of $n$, $P_n$ and $J$. Jensen's inequality implies that

$$\mathbb{E}\left[|(\tau_n - \mu_E)(A_r)|\right] \le C \left(\frac{1}{n} \mathbb{E}\left[\log \frac{\|P_n\|_E}{|A_n b_{n,n}|(\mathrm{cap}(E))^n}\right] + \frac{1}{n} \mathbb{E}\left[\log \frac{\|P_n\|_E}{|P_n(w)|}\right]\right)^{\alpha/(1+\alpha)}.$$

It is clear that

$$\|P_n\|_E \le \sum_{k=0}^n |A_k| \|B_k\|_E \le \max_{0\le k\le n} \|B_k\|_E \sum_{k=0}^n |A_k|.$$

Hence (4.14) yields

$$\begin{aligned}\mathbb{E}\left[\log\|P_n\|_E\right] &\le \mathbb{E}\left[\log\sum_{k=0}^{n}|A_k|\right] + \log\max_{0\le k\le n}\|B_k\|_E \\ &\le \frac{1}{t}\log\left(\sum_{k=0}^{n}\mathbb{E}[|A_k|^t]\right) + \log\max_{0\le k\le n}\|B_k\|_E.\end{aligned}$$

and

$$\mathbb{E}\left[\log\frac{\|P_n\|_E}{|A_n b_{n,n}|(\operatorname{cap}(E))^n}\right] \le \frac{1}{t}\log\left(\sum_{k=0}^{n}\mathbb{E}[|A_k|^t]\right) + \log\frac{\max_{0\le k\le n}\|B_k\|_E}{|b_{n,n}|(\operatorname{cap}(E))^n} - \mathbb{E}[\log|A_n|].$$

Thus (3.3) follows as combination of the above estimates.

We now proceed to the lower bound for the expectation of $\log|A_nP_n(w)|$ in (3.4) by estimating that

$$\begin{aligned}\mathbb{E}[\log|A_nP_n(w)|] &= \mathbb{E}\left[\log\left|A_n\sum_{k=0}^{n}A_kB_k(w)\right|\right] \\ &= \mathbb{E}[\log|A_n|] + \log|b_{0,0}| + \mathbb{E}\left[\log\left|A_0+\sum_{k=1}^{n}A_k\frac{B_k(w)}{b_{0,0}}\right|\right].\end{aligned}$$

Let $\nu_k$ be the probability measure of $A_k,\ k=0,1,\ldots,n$. Since $A_0$ is independent from $A_1,A_2,\ldots,A_n$, and $\mathbb{E}[\log|A_0+z|]\ge T>-\infty$ for all $z\in\mathbb{C}$, we obtain that

$$\begin{aligned}&\mathbb{E}\left[\log\left|A_0+\sum_{k=1}^{n}A_k\frac{B_k(w)}{b_{0,0}}\right|\right] \\ &= \int\ldots\int\left(\int\log\left|A_0+\sum_{k=1}^{n}A_k\frac{B_k(w)}{b_{0,0}}\right|\,d\nu_0(A_0)\right)d\nu_1(A_1)\ldots d\nu_n(A_n) \\ &\ge T>-\infty.\end{aligned}$$

□

Proof of Corollary 3.4. We use (3.3). Thus (3.5) implies that

$$\frac{2}{tn}\log\left(\sum_{k=0}^{n}\mathbb{E}[|A_{k,n}|^t]\right)\le O\left(\frac{\log n}{n}\right)\quad\text{as } n\to\infty,$$

and (3.6) implies that

$$-\frac{1}{n}\,\mathbb{E}[\log|A_{n,n}|]\le O\left(\frac{1}{n}\right)\quad\text{as } n\to\infty.$$

Moreover, our assumption (3.2) about the basis gives

$$\frac{1}{n}\log\frac{\max_{0\le k\le n}\|B_k\|_E^2}{|b_{n,n}|(\operatorname{cap}(E))^n}\le O\left(\frac{\log n}{n}\right)\quad\text{as } n\to\infty.$$

Using (3.7), we further estimate as in the proof of Theorem 3.3 that

$$\begin{aligned} -\frac{1}{n}\,\mathbb{E}[\log|P_n(w)|] &= -\frac{1}{n}\,\mathbb{E}\left[\log\left|\sum_{k=0}^{n} A_{k,n}B_k(w)\right|\right] \\ &= -\frac{1}{n}\left(\log|b_{0,0}| + \mathbb{E}\left[\log\left|A_{0,n} + \sum_{k=1}^{n} A_{k,n}\frac{B_k(w)}{b_{0,0}}\right|\right]\right) \\ &\le O\left(\frac{1}{n}\right) \quad \text{as } n\to\infty. \end{aligned}$$

Hence (3.8) follows from (3.3) and the above estimates. □

Proof of Corollary 3.5. All parts of Corollary 3.5 follow from Corollary 3.4 provided we show that the corresponding bases satisfy (3.2). It is convenient to first consider part (ii).

(ii) In fact, (3.2) was already verified for the Faber polynomials of any compact connected set $E$ in the proof of Corollary 2.6. Recall that the Faber polynomials of $E$ have the form $F_n(z) = z^n/(\mathrm{cap}(E))^n + \dots,\ n = 0,1,\dots,$ by definition, see [**49**]. Furthermore, $\|F_n\|_E = O(n^s)$ as $n\to\infty$, where $s < 1/2$, by [**28**].

(i) The leading coefficient $b_{n,n}$ of the orthonormal polynomial $B_n$ (with respect to any measure $\mu$) provides the solution of the following extremal problem [**48**]:

$$|b_{n,n}|^{-2} = \inf\left\{\int |Q_n|^2\,d\mu \ :\ Q_n \text{ is a monic polynomial of degree } n\right\}.$$

We use the monic polynomial $Q_n(z) = (\mathrm{cap}(E))^n F_n(z)$ that satisfies $\|Q_n\|_E \le C_1 n^s(\mathrm{cap}(E))^n$, where $C_1 > 0$ depends only on $E$, to estimate that

$$\begin{aligned} |b_{n,n}| &\ge \left(\int |Q_n|^2\,d\mu\right)^{-1/2} \ge (\mu(E))^{-1/2}\,\|Q_n\|_E^{-1} \\ &\ge C_1^{-1}\,(\mu(E))^{-1/2}\,n^{-s}(\mathrm{cap}(E))^{-n}. \end{aligned}$$

Thus the second part of (3.2) is proved. For the proof of the first part, we apply the Nikolskii type inequality (see Theorem 1.1 of [**34**] and comments on page 689):

$$\|B_n\|_E \le C_2 n\left(\int_E |B_n|^2\,ds\right)^{1/2} \le \frac{C_2}{\sqrt{c}}\,n\left(\int_E |B_n|^2\,w(s)ds\right)^{1/2} = \frac{C_2}{\sqrt{c}}\,n.$$

We used that $B_n$ is orthonormal with respect to $d\mu(s) = w(s)ds$ on the last step.

(iii) The proof of this part is similar to that of part (i). The estimate of the leading coefficient $b_{n,n}$ for the second part of (3.2) proceeds in the same way. The first part of (3.2) follows from the area Nikolskii type inequality (see Theorem 1.3 of [**34**] and remark (i) on page 689):

$$\|B_n\|_E \le C_3 n^2\left(\int_E |B_n|^2\,dA\right)^{1/2} \le \frac{C_3}{\sqrt{c}}\,n^2\left(\int_E |B_n|^2\,w\,dA\right)^{1/2} = \frac{C_3}{\sqrt{c}}\,n^2,$$

where we used that the weighted area $L_2$ norm of $B_n$ is equal to 1 by definition. □

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