ON THE SPECTRUM OF THE HEIGHT FOR TOTALLY REAL NUMBERS

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ABSTRACT. C.J. Smyth and later Flammang studied the spectrum of the Weil height in the field of all totally real numbers, establishing both lower and upper bounds for the limit infimum of the height of all totally real integers and determining isolated values of the height. We remove the hypothesis that we consider only integers and establish an lower bound on the limit infimum of the height for all totally real numbers. Our proof relies on a quantitative equidistribution theorem for numbers of small height.

Note: *The main theorem of this article relies on a result of Favre and Rivera-Letelier which was incorrect as stated. Specifically, in Théorèm 3 of* [5] *a factor of 2 is missing before the term $h_\rho(F)$. When this note was first written up the authors were unaware of this error. Since discovering the error, the bound in the main theorem of this paper was reduced by a factor of 2, and thus the bound in this paper is no longer an improvement on existing results.*

Recall that an algebraic number is said to be *totally real* if all of its Galois conjugates lie in the field $\mathbb{R}$ under any choice of embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. The totally real numbers form a field which we denote $\mathbb{Q}^{\mathrm{tr}}$. Schinzel [12] established a lower bound on the infimum of the nonzero values of the absolute logarithmic Weil height h for all totally real numbers:[1]

Theorem (Schinzel 1973)**.** *Let $\alpha \in \mathbb{Q}^{\mathrm{tr}}$, $\alpha \neq 0, \pm 1$ be a totally real number. Then*

$$h(\alpha) \geq h\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1}{2}\log\frac{1+\sqrt{5}}{2} = 0.2406059\ldots$$

where h denotes the absolute logarithmic Weil height.

Work of Smyth [13, 14], later improved by Flammang [7], established bounds on the limit infimum of the height for the ring of totally real integers:

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[1]In fact Schinzel's result originally implied this result for integers, however, it is easy to see that it generalizes to totally real nonintegers as well; cf. e.g., [8, 9].

Theorem (Smyth 1980, Flammang 1996)**.** *Let $\mathbb{Z}^{\mathrm{tr}}$ denote the ring of all totally real integers. Then*

$$\liminf_{\alpha\in\mathbb{Z}^{\mathrm{tr}}} h(\alpha) \geq \frac{1}{2}\log 1.720566 = 0.271327\ldots$$

and there are precisely six isolated values of the Weil height of a totally real integer in the interval $(0, 0.271327\ldots)$*. Further,*

$$\liminf_{\alpha\in\mathbb{Z}^{\mathrm{tr}}} h(\alpha) \leq \log 1.31427\ldots = 0.27328\ldots$$

In this paper we establish the following bound for the limit infimum of the height which removes the hypothesis that we only consider integers and is instead a bound over all totally real numbers, and establishes a 'gap' between the infimum of the positive values of height, set by Schinzel, and the limit infimum. Specifically, we prove the following:

Theorem. *Let $\mathbb{Q}^{\mathrm{tr}}$ denote the field of all totally real numbers. Then*

$$\liminf_{\alpha\in\mathbb{Q}^{\mathrm{tr}}} h(\alpha) \geq \frac{140}{3}\cdot\left(\frac{1}{8}-\frac{1}{6\pi}\right)^2 = 0.120786\ldots$$

Our result may be thought of as similar in spirit to the result of Zagier on the spectrum of the height on the curve $y = x+1$ over $\overline{\mathbb{Q}}$, that is, of $h(\alpha)+h(1-\alpha)$ (see [15, §3.A]), and the bounds of Bombieri and Zannier for the limit infimum of the height of totally p-adic numbers [2].

Our proof relies on results about the equidistribution of points of small Weil height. Early work on the distribution of numbers by height included results like the uniform distribution of the Farey fractions, culminating in quantitative generalization of the distribution of rational points at all places by Choi [3, 4]. Shortly after Choi's work, Bilu [1] formulated an equidistribution for algebraic points of small height at the archimedean place, proving that the Galois conjugates of a sequence of numbers with Weil height tending to zero must equidistribute along the unit circle in $\mathbb{C}$ in the sense of weak convergence of measures. Later this theorem was made quantitative by Petsche [10], and then vastly generalized by Favre and Rivera-Letelier [5] (see also the corrigendum [6]). We will use the result of Favre and Rivera-Letelier in our proof.

We note that of course the upper bound for the limit infimum set by Smyth remains valid for all totally real numbers. It is interesting to note that Smyth established this upper bound by constructing an explicit sequence of totally real algebraic integers with height limiting to the value $0.27328\ldots$ While Smyth's work precedes the development of arithmetic dynamics, his construction can naturally be thought of as dynamical in spirit, as the sequence he constructs is a certain sequence of preperiodic points for the rational map

$$H(x) = x - 1/x$$

which has a totally real archimedean Julia set. As these points have H-canonical height 0, it follows by work of Petsche, Szpiro, and Tucker [11, Theorem 1 and Prop. 16] that their standard Weil height must tend to a limit which is given by the the value of the Arakelov-Zhang pairing of H with the usual squaring map:

$$\langle H(z), z^2\rangle_{\mathrm{AZ}} = \int_{\mathbb{R}} \log^+|x|\, d\mu_H(x) = 0.27328\ldots$$

where μ_H is the canonical measure associated to iteration of $H(x)$. Smyth computed this value by determining a remarkable iterative formula for the distribution of μ_H (see [13, Theorem 3]).

Before beginning our proof, we set some notation. Suppose that $f \in C^1(\mathbb{P}^1(\mathbb{C}))$. We denote the Lipschitz constant of f with respect to the chordal metric by

$$\mathrm{Lip}(f) = \sup_{x,y\in\mathbb{P}^1(\mathbb{C})} \frac{|f(x)-f(y)|}{\mathsf{d}(x,y)},$$

where

$$\mathsf{d}(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\cdot\sqrt{1+|y|^2}}$$

denotes the archimedean chordal metric on $\mathbb{P}^1(\mathbb{C})$. Write $df = \partial f + \bar\partial f$ where ∂f is a form of type $(1,0)$ on $\mathbb{C} = \mathbb{R}\oplus i\mathbb{R}$ and $\bar\partial f$ is of type $(0,1)$, and define $d^c f = \frac{1}{2\pi i}(\partial f - \bar\partial f)$. (This normalization is to ensure that the Laplacian $\Delta g = dd^c g$ satisfies $\Delta g_\rho = \rho$ where $g_\rho(x) = \int_{\mathbb{C}} \log|x-y|\,d\rho(y)$, for suitable Borel probability measures ρ on $\mathbb{C}$.) For $f,g \in C^1(\mathbb{P}^1(\mathbb{C}))$ real-valued functions, we define the Dirichlet form to be

$$\langle f,g\rangle = \int_{\mathbb{C}} df\wedge d^c g = \int_{\mathbb{C}} \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right)\frac{dx\wedge dy}{2\pi}.$$

This defines a bilinear form which satisfies the Cauchy-Schwartz inequality $\langle f,g\rangle \le \langle f,f\rangle^{1/2}\langle g,g\rangle^{1/2}$.

Let λ denote the Haar measure of the unit circle in $\mathbb{C}$ and G the absolute Galois group. Now [5, 6, Theorem 7] (corrected; see the note at the beginning of this article) tells us that, since our difference with the standard archimedean measure is 0, the Hölder constant κ in can be taken to be 1, and then there exists a constant $c > 0$ such that for all $f \in C^1(\mathbb{P}^1(\mathbb{C}))$ and $\alpha \in \overline{\mathbb{Q}}$,

$$\left|\frac{1}{d}\sum_{z\in G\alpha} f(z) - \int_{\mathbb{P}^1(\mathbb{C})} f(z)\,d\lambda\right| \le \frac{\mathrm{Lip}(f)}{d} + \left(2h(\alpha) + c\frac{\log d}{d}\right)^{1/2}\langle f,f\rangle^{1/2} \tag{1}$$

where $d = \#G\alpha$ is the degree of α and $\mathrm{Lip}(f)$ denotes the Lipschitz constant of f on $\mathbb{P}^1(\mathbb{C})$ as defined above. We are now ready to prove our theorem.

Proof of Theorem. We will obtain our lower bound by integrating a test function $f(z)$ and computing the exact values of the integral of f against λ and the average of f over the Galois conjugates of any totally real number. Specifically, let

$$f(z) = \mathsf{d}(z,i)^3\cdot\mathsf{d}(z,-i)^3 = \left(\frac{\sqrt{x^2+(y-1)^2}\sqrt{x^2+(y+1)^2}}{2(1+x^2+y^2)}\right)^3$$

where $z = x+iy$ and as defined above d denotes the chordal metric on $\mathbb{P}^1(\mathbb{C})$. Notice that f is C^1 on $\mathbb{C} = \mathbb{R}\oplus i\mathbb{R} \cong \mathbb{R}^2$ since both partial derivatives are continuous. Further, for $x \in \mathbb{R}$, we have $d(x,i) = d(x,-i) = 1/\sqrt{2}$, and so $f(x) = 1/8$. Thus, for *any* totally real α, the average

$$\frac{1}{d}\sum_{z\in G\alpha} f(z) = \frac{1}{8}.$$

Further, it is a simple matter to compute that

$$\int_{\mathbb{P}^1(\mathbb{C})} f(z)\,d\lambda(z) = \frac{1}{6\pi}.$$

Thus by equation (1) we have

$$\left|\frac{1}{8}-\frac{1}{6\pi}\right| \le \frac{\mathrm{Lip}(f)}{d} + \left(2h(\alpha)+c\frac{\log d}{d}\right)^{1/2} \langle f,f\rangle^{1/2}.$$

Notice that as we take the limit infimum as α ranges over all elements of $\mathbb{Q}^{\mathrm{tr}}$, we only care about numbers of a bounded height, so by Northcott's theorem the degree d must tend to infinity. Thus when we take the limit infimum above, we obtain the lower bound:

$$\left|\frac{1}{8}-\frac{1}{6\pi}\right| \le \liminf_{\alpha\in\mathbb{Q}^{\mathrm{tr}}} \left(2h(\alpha)\right)^{1/2}\cdot\langle f,f\rangle^{1/2}$$

With the aid of a computer it is easy to calculate the exact value $\langle f,f\rangle = 3/140$, and thus squaring both sides and rearranging terms, we arrive at

$$\frac{1}{2}\cdot\frac{140}{3}\cdot\left|\frac{1}{8}-\frac{1}{6\pi}\right|^2 = 0.120786\ldots \le \liminf_{\alpha\in\mathbb{Q}^{\mathrm{tr}}} h(\alpha)$$

which concludes the proof. □

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