QUANTITATIVE HEIGHT BOUNDS UNDER SPLITTING CONDITIONS

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Abstract. In an earlier work, the first author and Petsche used potential theoretic techniques to establish a lower bound for the height of algebraic numbers that satisfy splitting conditions, such as being totally real or p-adic, improving on earlier work of Bombieri and Zannier in the totally p-adic case. These bounds applied as the degree of the algebraic number over the rationals tended towards infinity. In this paper, we use discrete energy approximation techniques on the Berkovich projective line to make the dependence on the degree in these bounds explicit, and we establish lower bounds for algebraic numbers which depend only on local properties of the numbers.

1. Introduction

In a previous work of the first author and Petsche [9], it was established that if S is a set of rational places and L_S denotes an extension of $\mathbb{Q}$ containing all algebraic numbers whose Galois conjugates all lie in the local fields $L_p \neq \mathbb{C}$ for each $p \in S$, then for all $\alpha \in L_S$,

$$h(\alpha) \geq \frac{1}{2} \sum_{p \in S} I(\mu_{L_p}) + o(1) \quad \text{as} \quad d = [\mathbb{Q}(\alpha) : \mathbb{Q}] \to \infty. \tag{1}$$

The local energies $I(\mu_{L_p})$ appearing above arise naturally as the solutions to a certain energy minimization problem for local fields, and the resulting bounds improved on earlier constants obtained by Bombieri and Zannier [4].

The goal of this note is to establish a bound in which the dependence on the degree is made explicit. We begin by fixing some notation to be used throughout this paper:

$S \subseteq M_{\mathbb{Q}}$	will be a given set of rational primes, possibly containing the archimedean prime.
$L_p/\mathbb{Q}_p$	will be a given finite normal extension for each $p \in S$.
L_S	will denote the field of all algebraic numbers all of whose Galois conjugates lie in L_p for each $p \in S$.

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Further, for each finite prime $p \in S$, we will denote by:

$e = e_p$ the ramification degree of $L_p/\mathbb{Q}_p$,
$f = f_p$ the inertial degree of $L_p/\mathbb{Q}_p$,
$q = p^f$ the order of the residue field of L_p, and
O_{L_p} the ring of integers of L_p.

To ease notation and avoid too many subscripts, we will leave tacit the dependence of e, f, q on the prime $p \in S$ below. Our first result is the following:

Theorem 1. *Let L_S be as above and $\alpha \in L_S$ with $d = [\mathbb{Q}(\alpha) : \mathbb{Q}] > 1$. Set*

$$V_\infty = \begin{cases} 0 & \text{if } \infty \notin S \\ \max\left\{ \frac{7\zeta(3)}{4\pi^2} - \frac{0.95d + 2}{2d^2} - \frac{(d-2)\log d}{2d(d-1)}, 0 \right\} & \text{if } \infty \in S \end{cases}$$

where $\zeta(3) = \sum_{n\geq 1} 1/n^3$. Then we have

$$h(\alpha) \geq -\frac{\log d}{2(d-1)} + V_\infty + \frac{1}{2} \sum_{\substack{p\in S \\ p\neq\infty,\ p^{1/e}<d}} \left((1 - \frac{1}{q^{n_p - 1}}) \frac{q \log p}{e(q^2-1)} - \frac{\log d}{d} \right) \tag{2}$$

where for each p,

$$n_p = \left\lfloor \frac{e \log d}{\log p} \right\rfloor.$$

If in addition α is an algebraic integer, then we have

$$h(\alpha) \geq -\frac{\log d}{2(d-1)} + V_\infty + \frac{1}{2} \sum_{\substack{p\in S \\ p\neq\infty,\ p^{1/e}<d}} \left((1 - \frac{1}{q^{n_p - 1}}) \frac{\log p}{e(q-1)} - \frac{\log d}{d} \right), \tag{3}$$

and if α is an algebraic unit, then

$$h(\alpha) \geq -\frac{\log d}{2(d-1)} + V_\infty + \frac{1}{2} \sum_{\substack{p\in S \\ p\neq\infty,\ p^{1/e}<d}} \left((1 - \frac{1}{q^{n_p - 1}}) \frac{q \log p}{e(q-1)^2} - \frac{\log d}{d} \right). \tag{4}$$

Theorem 1 should be compared to Bombieri and Zannier [4, Theorem 3]. Note that the result of Bombieri and Zannier does not cover the case where $\infty \in S$, but uses a form of Mahler's inequality much as our result does in (2). We note that in case $\infty \in S$, V_∞ is positive if and only if $d > 6$, i.e. only for algebraic numbers of degree > 6.

With trivial modifications, Theorem 1 can be stated over an arbitrary base number field $K/\mathbb{Q}$. This statement is given as Theorem 13 below.

Note that the height bounds from Theorem 1 are negative for algebraic numbers of small degree d. But as there are only finitely many algebraic numbers of bounded height and bounded degree, we recover the aforementioned result of the first author and Petsche:

$$\liminf_{\alpha \in L_S} h(\alpha) \geq \sum_{p \in S\setminus\{\infty\}} \frac{q \log p}{e(q^2-1)} + \begin{cases} \frac{7\zeta(3)}{4\pi^2} & \text{if } \infty \in S \\ 0 & \text{else} \end{cases}.$$

Although our bounds are trivial in some cases, we can use Theorem 1 to give absolute lower bounds for non roots of unity in $L_S^\times$. As such lower bounds depend mainly on the smallest prime in S, we will focus on the case $|S| = 1$. Note that we have trivially $L_S \subseteq L_{S'}$, whenever $S' \subseteq S$. In case $S = \{\infty\}$, Schinzel [15] gave the sharp lower bound $1/2 \log(1+\sqrt{5})/2$.

Theorem 2. *Let p be a rational prime and $\alpha \in L_{\{p\}}^{\times}$, not a root of unity. Then we have*

$$h(\alpha) \geq \begin{cases} \dfrac{\log p}{15(q-1)e\log\left(\frac{5(q-1)e}{\log p}\right)^4} & \text{if } e \geq 2 \\ \dfrac{\log(2)\log(p)}{10(q+1)\log\left(\frac{10(q+1)}{\log p}\right)} & \text{if } e = 1 \end{cases}$$

1.1. **Background.** We will now recall some of the notation and results regarding potential theory and Berkovich space which we will use. For background we refer the reader to [3, 7, 8]. Our notation largely follows that of Favre and Rivera-Letelier [7, 8]. For simplicity, we will state our results with $\mathbb{Q}$ as the ground field, however, any number field K can be substituted for the ground field with the usual renormalizations of absolute values.

We will denote by $\mathbb{A}^1, \mathbb{P}^1$ the usual affine and projective lines and by $\mathsf{A}^1, \mathsf{P}^1$ the Berkovich affine and projective lines, respectively. We refer the reader to [3, 7, 2] for some basic references on Berkovich space. We define the *standard measures* λ_p on $\mathsf{P}^1(\mathbb{C}_p)$ to be the probability measures which are either the Dirac measure on the Gauss point of $\mathsf{P}^1(\mathbb{C}_p)$ if $p \nmid \infty$ or the normalized Haar measure on the unit circle of $\mathbb{C}^\times$ if $p \mid \infty$. We let Δ denote the measure-valued Laplacian on P^1. We call $\rho = (\rho_p)_{p \in M_\mathbb{Q}}$ an *adelic measure* if for each $p \in M_\mathbb{Q}$, ρ_p is a Borel probability measure on $\mathsf{P}^1(\mathbb{C}_p)$ which is equal to λ_p for all but finitely many p and *admits a continuous potential* with respect to λ_p at the remaining places in the sense of [7], that is to say, for which $\rho_p - \lambda_p = \Delta g$ for some $g \in C(\mathsf{P}^1(\mathbb{C}_p))$. For any ρ_p, σ_p signed finite Borel measures on $\mathsf{P}^1(\mathbb{C}_p)$, we define, when it exists, the *local mutual energy pairing* to be

$$(\rho_p, \sigma_p)_p = \iint_{\mathsf{A}_p^1 \times \mathsf{A}_p^1 \setminus \mathrm{Diag}_p} -\log|x-y|_p \, d\rho_p(x)\, d\sigma_p(y) \tag{5}$$

where $\mathrm{Diag}_p = \{(x,x) : x \in \mathbb{C}_p\}$ denotes the diagonal of classical (or 'type I') points and $|\cdot|_p$ denotes the usual p-adic (or archimedean if $p \mid \infty$) absolute value, normalized so as to agree with the usual Euclidean absolute value when $p = \infty$ and for finite primes to satisfy the usual normalization for the p-adic absolute value where $|p|_p = 1/p$. Note that the notation in our integral here is loose in the p-adic setting, where for non-classical points x or y, the distance $|x-y|_p$ should be read as the natural extension of $|x-y|_p$ to the Berkovich projective line, denoted by $\sup\{x,y\}$ in the article of Favre and Rivera-Letelier [7, §3.3] and as the *Hsia kernel* $\delta(x,y)_\infty$ in the book of Baker and Rumely [3, §4].

When $\rho = (\rho_p), \sigma = (\sigma_p)$ are adelic measures we will sometimes write $(\rho, \sigma)_p$ instead of $(\rho_p, \sigma_p)_p$ to ease notation. When well-defined it is easy to see that the local mutual energy is symmetric. The local mutual energy exists in particular when ρ_p and σ_p are either Borel probability measures with continuous potentials with respect to the standard measure or are probability measures supported on a finite subset of $\mathbb{P}^1(\overline{\mathbb{Q}})$. In particular this applies for our adelic measures, and extends naturally by bilinearity to the vector space of signed measures arising from these measures. We refer the reader to [7] for proofs of these results.

For a local field $L_p/\mathbb{Q}_p$ with absolute value $|\cdot| = |\cdot|_p$, the first author and Petsche [9] defined the *energy integral* of a Borel probability measure ν on L_p to be

$$I(\nu) = \iint_{\mathbb{P}^1(L_p)\times\mathbb{P}^1(L_p)} -\log\delta(x,y)\,d\nu(x)\,d\nu(y),$$

where $\delta : \mathbb{P}^1(L_p) \times \mathbb{P}^1(L_p) \to \mathbb{R}$ is defined by

$$\delta(x,y) = \frac{|x_0y_1 - y_0x_1|}{\max\{|x_0|,|x_1|\}\max\{|y_0|,|y_1|\}}$$

for $x = (x_0 : x_1)$ and $y = (y_0 : y_1)$ in $\mathbb{P}^1(L_p)$. (We suppress the dependence on p in the above notation.) When L_p is non-archimedean, δ is precisely the spherical metric on $\mathbb{P}^1(L_p)$. In the case that ν admits a continuous potential with respect to λ_p, then by necessity the diagonal must be of $(\nu-\lambda_p)\otimes(\nu-\lambda_p)$-measure zero, and it is easy to see that the energy $I(\nu)$ corresponds exactly to the energy pairing of ν with the standard p-adic measure λ_p in (5):

$$I(\nu) = (\nu-\lambda_p, \nu-\lambda_p)_p$$

It follows from [9, Theorem 1] that for each L_p there exists a unique minimal Borel probability measure μ_{L_p} such that

$$I(\nu) \geq I(\mu_{L_p}) \tag{6}$$

for every Borel probability measure ν supported on $\mathbb{P}^1(L_p)$, with equality if and only if $\nu = \mu_{L_p}$. However, it is important to note that if ν has any point masses, then $I(\nu) = \infty$, as the diagonal cannot be excluded in the definition of I in order for the main theorems from [9] to apply.

Favre and Rivera-Letelier demonstrate in [7] that the Weil height of α can be written

$$h(\alpha) = \frac{1}{2}\sum_{p\in M_\mathbb{Q}} (\lambda_p - [\alpha], \lambda_p - [\alpha])_p \tag{7}$$

where λ_p is a standard measures described above on the Berkovich analytic line $\mathsf{P}^1(\mathbb{C}_p)$ with the usual p-adic absolute value (or achimedean absolute value when $p=\infty$) and the measure $[\alpha]$ is defined by

$$[\alpha] = \frac{1}{|G_\mathbb{Q}\alpha|}\sum_{z\in G_\mathbb{Q}\alpha}\delta_z$$

where $G_\mathbb{Q}$ denotes the usual absolute Galois group over $\mathbb{Q}$ and δ_z the Dirac measure with point mass at z. We regard this as a measure on $\mathsf{P}^1(\mathbb{C}_p)$ by fixing for all primes p an embedding from an algebraic closure of $\mathbb{Q}$ to $\mathbb{C}_p$.

The main idea behind the proof of Theorem 1 is to apply an inequality of the same type as (6) to the terms in (7) for which $p \in S$ in order to get lower bounds on the local energy pairings. As the measures $[\alpha]$ consist purely of a finite sum of point masses, however, in order to apply our bound, we must first approximate the measure $[\alpha]$ by an appropriate regularization of the measure which admits a continuous potential. We give these regularizations in Section 2. These regularizations are not typically supported in L_p, so in Section 3 we will prove several results akin to [9, Theorem 1] providing lower bounds for the energy of an ϵ-neighborhood around the line $\mathbb{P}^1(L_p)$ for each $p \in S$. We will then use these lower bounds to prove the main results in Section 4.

In the final Section of this paper we will combine our results from Theorem 1 with the general lower height bound of Dobrowolski [5] to achieve Theorem 2.

2. Regularized measures

Suppose our $\alpha \in L_S$ as in the formulation of Theorem 1 for $p \in S$. We wish to find a regularization of the measure

$$[\alpha] = \frac{1}{|G_{\mathbb{Q}}\alpha|} \sum_{z \in G_{\mathbb{Q}}\alpha} \delta_z$$

supported on $\mathbb{P}^1(L_p)$ which admits a continuous potential with respect to the standard measure λ_p. (This regularization may be supported on a larger space; for example, when $L_\infty = \mathbb{R}$, our regularization will be supported on $\mathbb{C}$.)

2.1. Archimedean regularization of measures.

We start with the real case $L_\infty = \mathbb{R}$. Our technique will be to replace the point masses in the probability measure $[\alpha]$ defined above by measures which are suitably regular. Our approach is quite similar to that of [7, §2] but our choice of regularization is slightly simpler and achieves the minimal logarithmic energy possible.

Specifically, for a given Dirac point mass δ_x for $x \in \mathbb{C}$, we will define $\delta_{x,\epsilon}$ to be the normalized unit Lebesgue measure of the circle $\{z \in \mathbb{C} : |z - x| = \epsilon\}$. For $F \subset \mathbb{C}$ a finite set and $[F] = {}^{1}/_{|F|} \sum_{x \in F} \delta_x$, we will define

$$[F]_\epsilon = \frac{1}{|F|} \sum_{x \in F} \delta_{x,\epsilon}.$$

It is immediate that these measures admit a continuous potential as defined above in Section 1.1.

We now prove a few easy lemmas regarding our regularized measures (cf. Lemmas 2.9 and 2.10 of [7]).

Lemma 3. *Let $F \subset \mathbb{C}$ be a finite set and $\epsilon > 0$. Then*

$$|([F], \lambda_\infty) - ([F]_\epsilon, \lambda_\infty)| \le \epsilon,$$

where $[F], [F]_\epsilon$ are the probability measures defined above.

Proof. Our proof is essentially the same as that of [7, Lemma 2.9]. It suffices to prove the bound for a singleton $z \in F$ with measures $\delta_z, \delta_{z,\epsilon}$. We use the standard notation $\log^+$ for the function $\max\{\log, 0\}$. Note that

$$(\delta_{z,\epsilon}, \lambda_\infty) - (\delta_z, \lambda_\infty) = \int_0^1 \log^+|z + \epsilon \cdot e^{2\pi i t}|\, dt - \log^+|z|,$$

but $|\log^+|z + \epsilon \cdot e^{2\pi i t}| - \log^+|z|| \le \epsilon$ for every real t. The result follows. □

Lemma 4. *Let $F \subset \mathbb{C}$ be a finite set and $\epsilon > 0$. Then*

$$([F]_\epsilon, [F]_\epsilon) \le ([F], [F]) - \frac{\log \epsilon}{|F|}.$$

Note that this lemma improves on [7, Lemma 2.10] as the term $C/|F|$ on the right hand side is removed.

Proof. We follow the same method as in the proof of [7, Lemma 2.10]. We note that for $\epsilon > 0$ and two points $z \neq z' \in \mathbb{C}$,

$$\begin{aligned}-(\delta_{z,\epsilon}, \delta_{z',\epsilon}) &= \int_0^1 \int_0^1 \log|z + \epsilon \cdot e^{2\pi i t} - (z' + \epsilon \cdot e^{2\pi i s})| \, dt \, ds \\ &= \int_0^1 \max\{\log|z - (z' + \epsilon \cdot e^{2\pi i s})|, \log \epsilon\} \, ds \\ &\geq \max\left\{ \int_0^1 \log|z - (z' + \epsilon \cdot e^{2\pi i s})| \, ds, \log \epsilon \right\} \\ &\geq \max\{\log|z - z'|, \log \epsilon\} \geq \log|z - z'| = -(\delta_z, \delta_{z'})\end{aligned}$$

so for each $z \neq z'$, we have $(\delta_z, \delta_{z'}) \geq (\delta_{z,\epsilon}, \delta_{z',\epsilon})$. On the other hand, we have what is essentially the logarithmic capacity:

$$(\delta_{z,\epsilon}, \delta_{z,\epsilon}) = -\log \epsilon.$$

Thus

$$\begin{aligned}([F]_\epsilon, [F]_\epsilon) &= \frac{1}{|F|^2} \sum_{\substack{z,z' \in F \\ z \neq z'}} (\delta_{z,\epsilon}, \delta_{z',\epsilon}) + \frac{1}{|F|^2} \sum_{z \in F} (\delta_{z,\epsilon}, \delta_{z,\epsilon}) \\ &\leq \frac{1}{|F|^2} \sum_{\substack{z,z' \in F \\ z \neq z'}} (\delta_z, \delta_{z'}) + \frac{1}{|F|^2} \cdot |F| \cdot (-\log \epsilon) \\ &= ([F], [F]) - \frac{\log \epsilon}{|F|}.\end{aligned}$$

□

We now prove our main result of this section:

Proposition 5. *Let $F \subset \mathbb{C}$ be a finite set and $\epsilon > 0$. Then*

$$([F] - \lambda_\infty,\ [F] - \lambda_\infty) \geq ([F]_\epsilon - \lambda_\infty,\ [F]_\epsilon - \lambda_\infty) - 2\epsilon + \frac{\log \epsilon}{|F|}.$$

Proof. We use the bilinearity of the energy pairing to write

$$([F] - \lambda_\infty, [F] - \lambda_\infty) = ([F], [F]) - 2([F], \lambda_\infty) + (\lambda_\infty, \lambda_\infty)$$

(noting that these individual energy pairings must be finite by [7, Lemma 4.3]), and from this it follows that

$$([F] - \lambda_\infty, [F] - \lambda_\infty) = ([F]_\epsilon - \lambda_\infty, [F]_\epsilon - \lambda_\infty) + E_1 + E_2$$

where

$$E_1 = -2([F], \lambda_\infty) + 2([F]_\epsilon, \lambda_\infty), \quad \text{and} \quad E_2 = ([F], [F]) - ([F]_\epsilon, [F]_\epsilon).$$

By applying Lemma 3 to E_1 and Lemma 4 to E_2, we get the desired bound. □

2.2. **Non-archimedean regularization of measures.** We fix a rational prime p and an associated local field L_p of residue characteristic p. We will follow the regularization technique introduced in [7, §4]: for $x \in \mathbb{C}_p$ and $\epsilon \in \mathbb{R}$, $\epsilon > 0$, we let $\zeta_{x,\epsilon}$ denote the type II or type III point of the Berkovich affine line $\mathsf{A}^1(\mathbb{C}_p)$, corresponding to the disc of radius ϵ around x. For our measure $[\alpha]$ on $\mathsf{P}^1(\mathbb{C}_p)$ for $\alpha \in L_S$ we define the *regularized measure* $[\alpha]_\epsilon$ to be

$$[\alpha]_\epsilon = \frac{1}{|G_{\mathbb{Q}}\alpha|} \sum_{z \in G_{\mathbb{Q}}\alpha} \delta_{z,\epsilon}. \tag{8}$$

where $\delta_{z,\epsilon}$ denotes the Dirac unit point mass supported on the point $\zeta_{z,\epsilon} \in \mathsf{A}^1(\mathbb{C}_p)$.

We will always assume that $0 < \epsilon < 1$ in our regularization throughout this paper. Let $e = e(L_p/\mathbb{Q}_p)$ and $f = f(L_p/\mathbb{Q}_p)$ denote the local ramification and inertial degrees of L_p, respectively. As we have fixed the prime p in this section, we will set $(\cdot,\cdot) = (\cdot,\cdot)_p$ for the energy pairing (5). The following result provides an estimate on how far the energy pairing of $[\alpha]_\epsilon$ with the standard measure λ_p is from the energy pairing of $[\alpha]$ with λ_p:

Proposition 6 (Favre, Rivera-Letelier [7, Prop. 9])**.** *For all* $0 < \epsilon < 1$*,*

$$([\alpha] - \lambda_p, [\alpha] - \lambda_p) \geq ([\alpha]_\epsilon - \lambda_p, [\alpha]_\epsilon - \lambda_p) + \frac{\log \epsilon}{|G_{\mathbb{Q}}\alpha|}. \tag{9}$$

3. Potential theoretic results

In both the non-archimedean and archimedean settings, our regularized measures lie in an ϵ-neighborhood of the line $\mathbb{P}^1(L_p)$ for $L_p/\mathbb{Q}_p$ our chosen local field at each place, and thus the energies (the δ-Robin constants) of the lines $\mathbb{P}^1(L_p)$ computed in [9] cannot be directly applied. In order to get a lower bound on the energy, we will compute in this section estimates on the energy of the ϵ-neighborhoods of the appropriate lines.

3.1. **Archimedean results.** Our goal in this section is to prove a potential theoretic result which is perhaps of independent interest for the $-\log \delta(x,y)$ kernel which will be used to prove our main results in the archimedean case. In [9] it is shown that the δ-Robin constant of $\mathbb{P}^1(\mathbb{R})$ is given by $I(\mu_{\mathbb{R}})$, where

$$\mu_{\mathbb{R}}(z) = \frac{1}{\pi^2 z} \log \left| \frac{z+1}{z-1} \right| \, dz.$$

As our regularized measures for the archimedean places are not supported entirely on the projective real line, we cannot directly apply this result even when $F \subset \mathbb{R}$ in order to obtain a lower bound on the local factors of the height. Instead, for each $0 < \epsilon < 1$, we will prove a bound on the δ-Robin constant of the set

$$E_\epsilon = \{z \in \mathbb{C} : |\mathrm{Im}\, z| \leq \epsilon\} \cup \{\infty\} \subset \mathbb{P}^1(\mathbb{C}),$$

on which our measures $[F]_\epsilon$ for a finite set $F \subset \mathbb{R}$ are supported, and use this lower bound on the δ-energy in an analogue of [9, Theorem 1]. Our result is the following:

Theorem 7. *Let E_ϵ be as above for $0 < \epsilon < 1$. Then the δ-Robin constant of E_ϵ satisfies*

$$V_\delta(E_\epsilon) \geq \frac{7\zeta(3)}{2\pi^2} - 0.95\sqrt{\epsilon}.$$

In particular, for any Borel probability measure μ supported on E_ϵ, we have

$$I(\mu) = \iint_{E_\epsilon \times E_\epsilon} -\log \delta(x,y) \, d\mu(x) \, d\mu(y) \geq \frac{7\zeta(3)}{2\pi^2} - 0.95\sqrt{\epsilon}.$$

Before proving this theorem, we will prove two technical lemmas which will be useful:

Lemma 8. *Let $y \leq 1$ be a positive real number. For all $z \in \mathbb{R} \setminus \{0\}$ one has*

$$\frac{4y^2 \log\left(1+\frac{y^2}{z^2}\right)}{z^2+y^2z^2} \geq \left(\log\left(1+\frac{y^2}{z^2}\right)\right)^2.$$

Proof of lemma. Since every z appears with an even power, we may assume that $z > 0$. After dividing both sides by the positive value $\log\left(1+\frac{y^2}{z^2}\right)$, it is enough to prove

$$\frac{4y^2}{z^2+y^2z^2} \geq \log\left(1+\frac{y^2}{z^2}\right). \tag{10}$$

We regard both sides of (10) as functions in z. The derivative of $\frac{4y^2}{z^2+y^2z^2}$ (with respect to z) is $\frac{-8y^2}{(1+y^2)z^3}$ and the derivative of $\log\left(1+\frac{y^2}{z^2}\right)$ (with respect to z) is $\frac{-2y^2}{z(y^2+z^2)}$. Using $0 < y \leq 1$ we see that

$$\frac{-8y^2}{(1+y^2)z^3} \leq \frac{-8y^2}{2z^3} \leq \frac{-2y^2}{zz^2} \leq \frac{-2y^2}{z(y^2+z^2)} \quad \text{for all } z > 0.$$

Since both functions in (10) tend to zero as z tends to infinity, this implies that the inequality (10), and hence the statement of the lemma, is true. □

Lemma 9. *Let $y \leq 1$ be a positive real number. For all $x \in \mathbb{R}$ one has*

$$\int_{\mathbb{R}} \log\left(1+\frac{y^2}{(x-z)^2}\right) \cdot \frac{1}{z\pi^2} \log\left|\frac{z+1}{z-1}\right| dz \leq \sqrt{y} \cdot 1.9.$$

Proof of lemma. We start by applying the Cauchy-Schwarz-Bunyakovsky inequality to get

$$\begin{aligned} &\int_{\mathbb{R}} \log\left(1+\frac{y^2}{(x-z)^2}\right) \cdot \frac{1}{z\pi^2} \log\left|\frac{z+1}{z-1}\right| dz \\ &\qquad \leq \sqrt{\int_{\mathbb{R}} \left(\log\left(1+\frac{y^2}{(x-z)^2}\right)\right)^2 dz} \cdot \sqrt{\int_{\mathbb{R}} \left(\frac{1}{z\pi^2} \log\left|\frac{z+1}{z-1}\right|\right)^2 dz} \end{aligned} \tag{11}$$

The second factor is a constant $c_{\mathbb{R}} = 0.450158\ldots$, so we are left to find an upper bound for the integral

$$\int_{\mathbb{R}} \left(\log\left(1+\frac{y^2}{(x-z)^2}\right)\right)^2 dz = \int_{\mathbb{R}} \left(\log\left(1+\frac{y^2}{z^2}\right)\right)^2 dz.$$

We want to regard this latter integral as a function in y. Therefore, we define

$$F(y) = \int_{\mathbb{R}} \left(\log\left(1+\frac{y^2}{z^2}\right)\right)^2 dz$$

for all $y \in (0,1)$ (note that the integral defining $F(y)$ converges for every y). We claim that $\frac{F(y)}{y}$ is strictly increasing on $(0,1)$. By the Leibniz rule we get

$$\begin{aligned} \frac{d}{dy}\frac{F(y)}{y} &= \frac{y\int_{\mathbb{R}} \frac{d}{dy}\left(\log\left(1+\frac{y^2}{z^2}\right)\right)^2 dz - \int_{\mathbb{R}}\left(\log\left(1+\frac{y^2}{z^2}\right)\right)^2 dz}{y^2} \\ &= \frac{\int_{\mathbb{R}} \frac{4y^2\log\left(1+\frac{y^2}{z^2}\right)}{z^2+y^2z^2} - \left(\log\left(1+\frac{y^2}{z^2}\right)\right)^2 dz}{y^2}. \end{aligned}$$

By Lemma 8 this is non-negative for all $0 < y \leq 1$, proving the claim. It follows that $F(y) \leq yF(1) \leq y \cdot 17.420688\ldots$. From (11) we get

$$\int_{\mathbb{R}} \log\left(1 + \frac{y^2}{(x-z)^2}\right) \cdot \frac{1}{\pi^2} \log\left|\frac{z+1}{z-1}\right| \, dz \leq \sqrt{y} \cdot \sqrt{F(1)} \cdot c_{\mathbb{R}} \leq 1.87887\ldots\sqrt{y},$$

which concludes the proof of this lemma. □

We are now ready to prove the theorem.

Proof of Theorem 7. Our proof relies on [9, Theorem 7], namely, that for any closed set $E \subset \mathbb{P}^1(\mathbb{C})$ of finite δ-Robin constant $V_\delta(E)$ and any Borel probability measure ν supported on E, we have

$$\inf_{z \in E} U_\delta^\nu(z) \leq V_\delta(E) \leq \sup_{z \in E} U_\delta^\nu(z) \tag{12}$$

where U_δ^ν is the δ-potential associated to the measure ν given by

$$U_\delta^\nu(z) = \int_E -\log \delta(w, z) \, d\nu(w).$$

The idea of the proof will be to study the decay of the potential associated to the minimal energy measure $\mu_{\mathbb{R}}$ given in [9, Theorem 1]. Since $\mu_{\mathbb{R}}$ is supported on $\mathbb{P}^1(\mathbb{R}) \subset E_\epsilon$, E_ϵ obviously has finite δ-energy, and the infimum of $U_\delta^{\mu_{\mathbb{R}}}$ on E_ϵ will determine a lower bound for the δ-Robin constant of E_ϵ.

Specifically, we note that, for $x + yi \in E_\epsilon \setminus \{\infty\}$, we have

$$\begin{aligned} U_\delta^{\mu_{\mathbb{R}}}(x+iy) &= \log^+|x+iy| + \int_{\mathbb{R}} \log^+|z| \, d\mu_{\mathbb{R}}(z) - \int_{\mathbb{R}} \log|x+iy-z| \, d\mu_{\mathbb{R}}(z) \\ &= \log^+|x+iy| + \frac{7\zeta(3)}{2\pi^2} - \int_{\mathbb{R}} \log|x+iy-z| \, d\mu_{\mathbb{R}}(z) \end{aligned}$$

where we have used the result $\int_{\mathbb{R}} \log^+|z| \, d\mu_{\mathbb{R}}(z) = {}^{7\zeta(3)}/_{2\pi^2}$ which follows from the computations in the proof of [9, Theorem 1]. Now,

$$\begin{aligned} \int_{\mathbb{R}} \log|x+iy-z| \, d\mu_{\mathbb{R}}(z) &= \frac{1}{2} \int_{\mathbb{R}} \log\left((x-z)^2 + y^2\right) \, d\mu_{\mathbb{R}}(z) \\ &= \int_{\mathbb{R}} \log|x-z| \, d\mu_{\mathbb{R}}(z) + \frac{1}{2} \int_{\mathbb{R}} \log\left(1 + \frac{y^2}{(x-z)^2}\right) \, d\mu_{\mathbb{R}}(z). \end{aligned}$$

One can check directly (via an analysis similar to that used in [9]) that the quantity $\int_{\mathbb{R}} \log|x-z| \, d\mu_{\mathbb{R}}(z) = 0$ when $x = \pm 1$. Combining this with the proof in [9, Theorem 1] that for all $x \neq \pm 1$, $U_\delta^{\mu_{\mathbb{R}}}(x) = {}^{7\zeta(3)}/_{2\pi^2}$, we see that in fact $U_\delta^{\mu_{\mathbb{R}}}(x) = {}^{7\zeta(3)}/_{2\pi^2}$ for all $x \in \mathbb{R}$, and so it follows that

$$\log^+|x| - \int_{\mathbb{R}} \log|x-z| \, d\mu_{\mathbb{R}}(z) = 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

Using $\log^+|x+iy| \geq \log^+|x|$ we obtain:

$$U_\delta^{\mu_{\mathbb{R}}}(x+iy) \geq \frac{7\zeta(3)}{2\pi^2} - \frac{1}{2} \int_{\mathbb{R}} \log\left(1 + \frac{y^2}{(x-z)^2}\right) \, d\mu_{\mathbb{R}}(z). \tag{13}$$

By (12) and Lemma 9 it follows immediately that

$$V_\delta(E) \geq U_\delta^{\mu_{\mathbb{R}}}(x+iy) \geq \frac{7\zeta(3)}{2\pi^2} - 0.95 \cdot \sqrt{y} \geq \frac{7\zeta(3)}{2\pi^2} - 0.95 \cdot \sqrt{\epsilon}.$$

This completes the proof of Theorem 7. □

3.2. **Non-archimedean results.** We will now prove analogous results for the p-adic setting. In particular, we assume a local field $L_p/\mathbb{Q}_p$ is fixed, for a finite rational prime p with notation as in §2.2 above.

As before, we let $O = O_{L_p}$ denote the ring of integers of L_p and we let π be a uniformizing element. We also let $q = p^f$ denote the order of the residue field $O/\pi O$ and let $\mathsf{P}^1(\mathbb{C}_p)$ denote the Berkovich projective line. Notice that $\mathbb{P}^1(L_p)$ is a compact subset of $\mathsf{P}^1(\mathbb{C}_p)$. We will in fact prove three approximation results, one for an ϵ-neighborhood of O in the sense of the retraction map of Favre and Rivera-Letelier [7, §4], another for a neighborhood of $O^\times$, and finally a result for a neighborhood of $\mathbb{P}^1(L_p)$ itself.

In the following, we extend the kernel $\delta(x, y)$ from the classical projective line over our local field $\mathbb{P}^1(L_p)$ to the Berkovich projective line $\mathsf{P}^1(\mathbb{C}_p)$ by viewing $-\log\delta(x, y)$ as the generalized Hsia kernel of [3, §4.4] with respect to the Gauss point $\zeta_{0,1}$ (the so-called *spherical kernel*). As we did above, we use the notation $\zeta_{x,r} \in \mathsf{A}^1(\mathbb{C}_p)$ for the type II or type III point of the Berkovich projective line which corresponds to the sup norm on the disc of radius r centered at $x \in \mathbb{C}_p$. Note that $\mathsf{P}^1(\mathbb{C}_p) = \mathsf{A}^1(\mathbb{C}_p) \cup \{\infty\}$.

Theorem 10. *Fix $n \in \mathbb{N}$ and let $0 < \epsilon < 1$ satisfy $|\pi^{n+1}| < \epsilon \le |\pi^n|$. Then the δ-Robin constant of the set*

$$F = \{\zeta \in \mathsf{P}^1(\mathbb{C}_p) : \zeta = \zeta_{x,\eta} \text{ for } x \in O,\ 0 \le \eta \le \epsilon\}.$$

satisfies

$$V_\delta(F) \ge V_\delta(O) + \frac{\log|\pi|}{q^{n-1}(q-1)} = -\left(1 - \frac{1}{q^{n-1}}\right)\frac{\log|\pi|}{q-1}. \tag{14}$$

Proof. We start by recalling that, as $\epsilon < 1$, the set F does not contain the canonical Gauss point $\zeta_{0,1}$ and so, it follows from [3, Cor. 6.19] that

$$V_\delta(F) \ge \inf_{\zeta\in F} U_\delta^\mu(\zeta)$$

for every Borel probability measure μ supported on F, where U_δ^μ denotes the δ-potential

$$U_\delta^\mu(\zeta) = \int -\log\delta(\zeta, y)\, d\mu(y).$$

The result will follow from finding a lower bound for U_δ^μ where μ is the equilibrium measure of O, namely, its normalized additive Haar measure. To compute a lower bound for the above potential, we note that for every $\zeta \in F$, we can choose an $x \in O$ such that $\zeta = \zeta_{x,\eta}$ for some $0 \le \eta \le \epsilon$. We can then compute, noting that $|x|, |y| \le 1$:

$$\delta(\zeta_{x,\eta}, y) = \begin{cases} \eta & \text{if } |x-y| \le \eta \\ |x-y| & \text{if } |x-y| \ge \eta \end{cases}.$$

It follows that

$$U_\delta^\mu(\zeta_{x,\eta}) = -\log\eta \cdot \mu(\{y \in O : |x-y| \le \eta\}) + \int_{\{y\in O:|x-y|>\eta\}} -\log|x-y|\, d\mu(y).$$

Using the fact that $\delta(x,y) = |x-y|$ for all $x \neq y \in O$, we see that for every $x \in O$,

$$\int_{\{y\in O:|x-y|>\eta\}} -\log|x-y|\,d\mu(y) + \int_{\{y\in O:|x-y|\leq\eta\}} -\log|x-y|\,d\mu(y) = \int_O -\log\delta(x,y)\,d\mu(y) = V_\delta(O).$$

It follows that we can rewrite the equation for $U^\mu_\delta(\zeta_{x,\eta})$ as:

$$U^\mu_\delta(\zeta_{x,\eta}) = V_\delta(O) - \mu(\{y\in O : |x-y|\leq\eta\})\log\eta + \int_{\{y\in O:|x-y|\leq\eta\}} \log|x-y|\,d\mu(y). \tag{15}$$

Since μ is translation invariant by O, we can assume $x=0$ above, and the quantity depends only on η. If we let $k\in\mathbb{N}$ such that

$$|\pi^k| \leq \eta < |\pi^{k-1}|$$

then

$$\begin{aligned}\int_{\{y\in O:|y|\leq\eta\}} \log|y|\,d\mu(y) &= \int_{\pi^k O} \log|y|\,d\mu(y)\\ &= \sum_{\ell=k}^{\infty}\int_{\pi^\ell O^\times}\log|y|\,d\mu(y)\\ &= \sum_{\ell=k}^{\infty}\log|\pi^\ell|\cdot\frac{1}{q^\ell}\frac{q-1}{q}\\ &= \frac{k\log|\pi|}{q^k} + \frac{\log|\pi|}{q^k(q-1)}.\end{aligned}$$

Using the fact that $-\log\eta \geq -\log|\pi^{k-1}|$ and

$$\mu(\{y\in O : |x-y|\leq\eta\}) = \mu(\{y\in O : |x-y|\leq|\pi^k|\}) = \frac{1}{q^k},$$

it follows that

$$-\mu(\{y\in O : |x-y|\leq\eta\})\log\eta \geq -(k-1)\log|\pi|\cdot\frac{1}{q^k},$$

so substituting both estimates into (15) now yields that

$$U^\mu_\delta(\zeta_{x,\eta}) \geq V_\delta(O) + \frac{\log|\pi|}{q^k} + \frac{\log|\pi|}{q^k(q-1)} = V_\delta(O) + \frac{\log|\pi|}{q^{k-1}(q-1)}$$

As $\log|\pi| < 0$, this expression is minimized by choosing k as small as possible. Since $\eta \leq \epsilon \leq |\pi^n|$ it follows that $k \geq n$, so we let $k = n$ to obtain a lower bound for the δ-potential on all of F. Now substituting the well-known result that $V_\delta(O) = {}^{-\log|\pi|}/_{(q-1)}$ (cf. [14, Example 4.1.24]) yields the desired result. □

Theorem 11. *Fix $n\in\mathbb{N}$ and let $0<\epsilon<1$ satisfy $|\pi^{n+1}| < \epsilon \leq |\pi^n|$. Then the δ-Robin constant of the set*

$$F' = \{\zeta\in\mathsf{P}^1(\mathbb{C}_p) : \zeta = \zeta_{x,\eta} \text{ for } x\in O^\times,\ 0\leq\eta\leq\epsilon\}.$$

satisfies

$$V_\delta(F') \geq V_\delta(O^\times) + \frac{q\log|\pi|}{q^{n-1}(q-1)^2} = -\left(1-\frac{1}{q^{n-1}}\right)\frac{q\log|\pi|}{(q-1)^2} \tag{16}$$

Proof. Our proof follows the same reasoning as in the previous theorem. We again note that as $\epsilon < 1$, the set F' does not contain the canonical Gauss point $\zeta_{0,1}$ and so it follows from [3, Cor. 6.19] that

$$V_\delta(F') \geq \inf_{\zeta\in F'} U_\delta^\mu(\zeta)$$

for every Borel probability measure μ supported on F', where U_δ^μ denotes the δ-potential

$$U_\delta^\mu(\zeta) = \int -\log \delta(\zeta, y)\, d\mu(y).$$

The result will follow from finding a lower bound for U_δ^μ where μ denotes the Haar measure of $O^\times$, which is both the usual logarithmic and δ-equilibrium measure of $O^\times$, as $-\log|x-y| = -\log\delta(x,y)$ for $x, y \in O^\times$ and the equilibrium measure must be invariant under multiplication by elements of $O^\times$ as it is unique. It also follows from this that the logarithmic potential is constant on all of $O^\times$, so we can compute that, letting ν denote the normalized mass 1 Haar measure of πO,

$$\begin{aligned}
V_\delta(O^\times) &= \int_{O^\times} -\log|1-x|\, d\mu(x) = \frac{1}{q-1}\int_{1+\pi O} -\log|1-x|\, d\mu(x) \\
&= -\frac{1}{q-1}\int_{\pi O} \log|x|\, d\nu(x) = -\frac{1}{q-1}\sum_{k=1}^{\infty} \log|\pi^k| \cdot \nu(\pi^k O^\times) \\
&= -\frac{1}{q-1}\sum_{k=1}^{\infty} \frac{q-1}{q}\cdot\frac{1}{q^{k-1}}\log|\pi^k| = -\frac{q\log|\pi|}{(q-1)^2}.
\end{aligned}$$

It follows exactly as in the previous theorem's proof that, for $x \in O^\times$ and $\zeta_{x,\eta} \in F'$, we can rewrite the equation for $U_\delta^\mu(\zeta_{x,\eta})$ as:

$$\begin{aligned}
U_\delta^\mu(\zeta_{x,\eta}) = V_\delta(O^\times) &- \mu(\{y \in O^\times : |x-y| \leq \eta\})\log\eta \\
&+ \int_{\{y\in O^\times:|x-y|\leq\eta\}} \log|x-y|\, d\mu(y).
\end{aligned} \tag{17}$$

Since μ is translation invariant under multiplication by $O^\times$, we can assume $x = 1$ above, and the quantity depends only on η. If we let $k \in \mathbb{N}$ such that

$$|\pi^k| \leq \eta < |\pi^{k-1}|$$

then, with ν the normalized mass 1 Haar measure of πO as before,

$$\begin{aligned}
\int_{\{y\in O^\times:|1-y|\leq\eta\}} \log|1-y|\, d\mu(y) &= \frac{1}{q-1}\int_{\pi^k O} \log|y|\, d\nu(y) \\
&= \frac{1}{q-1}\sum_{\ell=k}^{\infty}\int_{\pi^\ell O^\times} \log|y|\, d\nu(y) \\
&= \frac{1}{q-1}\sum_{\ell=k}^{\infty} \log|\pi^\ell| \cdot \frac{1}{q^{\ell-1}}\frac{q-1}{q} \\
&= \frac{q}{q-1}\left(\frac{k\log|\pi|}{q^k} + \frac{\log|\pi|}{q^k(q-1)}\right).
\end{aligned}$$

Using the fact that $-\log\eta \geq -\log|\pi^{k-1}|$ and

$$\mu(\{y \in O^\times : |x-y| \leq \eta\}) = \mu(\{y \in O^\times : |x-y| \leq |\pi^k|\}) = \frac{1}{q^{k-1}(q-1)},$$

it follows that

$$-\mu(\{y \in O : |x-y| \le \eta\}) \cdot \log \eta \ge -\frac{k-1}{q^{k-1}(q-1)} \log |\pi|,$$

so substituting both estimates into (17) now yields that

$$U_\delta^\mu(\zeta_{x,\eta}) \ge V_\delta(O^\times) + \frac{q \log |\pi|}{q^{k-1}(q-1)^2} = -\frac{q \log |\pi|}{(q-1)^2} + \frac{q \log |\pi|}{q^{k-1}(q-1)^2}.$$

As $\log |\pi| < 0$, this expression is minimized by choosing k as small as possible. Since $\eta \le \epsilon \le |\pi^n|$ it follows that $k \ge n$, so we let $k = n$ to obtain the desired lower bound for the δ-potential on all of F'. □

Theorem 12. *Fix $n \in \mathbb{N}$ and let $0 < \epsilon < 1$ satisfy $|\pi^{n+1}| < \epsilon \le |\pi^n|$. Then the δ-Robin constant of the set*

$$G = \{\zeta \in \mathsf{P}^1(\mathbb{C}_p) : \zeta = \zeta_{x,\eta} \text{ for } x \in L_p,\ 0 \le \eta \le \epsilon\} \cup \{\infty\}$$

satisfies

$$V_\delta(G) \ge V_\delta(\mathbb{P}^1(L_p)) + \frac{q \log |\pi|}{q^{n-1}(q^2-1)} = -\left(1 - \frac{1}{q^{n-1}}\right) \frac{q \log |\pi|}{q^2-1}.$$

Proof. Our proof follows the same reasoning as in the previous two theorems. We again note that as $\epsilon < 1$, the set G does not contain the canonical Gauss point $\zeta_{0,1}$ and so it follows from [3, Cor. 6.19] that

$$V_\delta(G) \ge \inf_{\zeta \in G} U_\delta^\mu(\zeta)$$

for every Borel probability measure μ supported on G, where U_δ^μ denotes the δ-potential

$$U_\delta^\mu(\zeta) = \int -\log \delta(\zeta, y)\, d\mu(y).$$

The result will follow from finding a lower bound for U_δ^μ where μ is the δ-equilibrium measure of $\mathbb{P}^1(L_p)$ determined in [9, Thm. 1(c)], namely, the unique $\mathrm{PGL}_2(O)$-invariant probability measure on $\mathbb{P}^1(L_p)$. To compute a lower bound for the above potential, we note that for every $\zeta \in G$, either $\zeta = \infty$, or we can choose an $x \in L_p$ such that $\zeta = \zeta_{x,\eta}$ for some $0 \le \eta \le \epsilon \le |\pi^n|$. As $\infty \in \mathbb{P}^1(L_p)$, it is easy to check classically (cf. the proof of Theorem 1(c) in [9]) that $U_\delta^\mu(\infty) = V_\delta(\mathbb{P}^1(L_p))$, and there is nothing to show in this case. We may therefore assume $\zeta = \zeta_{x,\eta}$ for some $x \in L_p$ and $0 \le \eta \le \epsilon$. We can then compute that:

$$\delta(\zeta_{x,\eta}, y) = \begin{cases} \dfrac{\eta}{\max\{|x|,1\} \cdot \max\{|y|,1\}} & \text{if } |x-y| \le \eta \\ \dfrac{|x-y|}{\max\{|x|,1\} \cdot \max\{|y|,1\}} & \text{if } |x-y| \ge \eta. \end{cases}$$

It follows that

$$U_\delta^\mu(\zeta_{x,\eta}) = V_\delta(\mathbb{P}^1(L_p)) - \log \eta \cdot \mu(\{y \in \mathbb{P}^1(L_p) : |x-y| \le \eta\}) + \int_{\{y \in \mathbb{P}^1(L_p) : |x-y| \le \eta\}} \log |x-y|\, d\mu(y).$$

Let $k \in \mathbb{N}$ be such that $|\pi^k| \leq \eta < |\pi^{k-1}|$ and let ν denote the unit Haar measure of πO. We can compute the integral as:

$$\begin{aligned}\int_{\{y\in\mathbb{P}^1(L_p):|x-y|\leq\eta\}} \log|x-y|\,d\mu(y) &= \mu(x+\pi O)\cdot\int_{\{y\in\pi O:|y|\leq\eta\}}\log|y|\,d\nu(y)\\ &= \mu(x+\pi O)\cdot\sum_{\ell=k}^{\infty}\int_{\pi^\ell O^\times}\log|y|\,d\nu(y)\\ &= \mu(x+\pi O)\cdot\sum_{\ell=k}^{\infty}\log|\pi^\ell|\cdot\frac{1}{q^{\ell-1}}\frac{q-1}{q}\\ &= \mu(x+\pi O)\cdot\frac{\log|\pi|(k(q-1)+1)}{q^{k-1}(q-1)}.\end{aligned}$$

If $|x| \leq 1$, then $\mu(x+\pi O) = 1/q+1$ and $-\log\eta \geq -\log|\pi^{k-1}|$. If $|x| > 1$, then $x + \pi O$ is a proper subset of the set $\{z \in \mathbb{P}^1(L_p) : |z| > 1\}$, so

$$\mu(x+\pi O) \leq \mu(\{z\in\mathbb{P}^1(L_p) : |z|>1\}) = 1/q+1 \quad \text{when} \quad |x|>1.$$

the same arguments yield

$$-\log\eta\cdot\mu(\{y\in\mathbb{P}^1(L_p) : |x-y|\leq\eta\}) \geq -(k-1)\log|\pi|\cdot\frac{1}{q^{k-1}(q+1)}.$$

With these equations we achieve the lower bound

$$\begin{aligned}U_\delta^\mu(\zeta_{x,\eta}) &\geq V_\delta(\mathbb{P}^1(L_p)) - \frac{\log|\pi|(k-1)}{q^{k-1}(q+1)} + \frac{\log|\pi|(k(q-1)+1)}{q^{k-1}(q^2-1)}\\ &= V_\delta(\mathbb{P}^1(L_p)) + \frac{q\log|\pi|}{q^{k-1}(q^2-1)}.\end{aligned}$$

Since $\eta \leq \epsilon \leq |\pi^n|$ it follows that $k \geq n$, so we let $k = n$ to obtain a lower bound. Combined with the result from [9, Theorem 1(c)] that $V_\delta(\mathbb{P}^1(L_p)) = -q\log|\pi|/q^2-1$ we obtain the desired result. □

4. Proof of Theorem 1 and generalization

For simplicity of notation we will prove first Theorem 1 as stated, which takes as the base field $\mathbb{Q}$. We will then give the statement of the generalization of this result over a base field K in Theorem 13 below, and describe what routine changes need to be made for the proof to work over an arbitrary base field.

Proof of Theorem 1. To ease notation, we will only prove (2) and (3). To prove (4) one only has to replace Theorem 10 by Theorem 11 in the following argumentation. Recall that

$$h(\alpha) = \frac{1}{2}\sum_{p\in M_\mathbb{Q}}(\lambda_p - [\alpha], \lambda_p - [\alpha])_p.$$

For all finite p, by [7, Lemma 5.4] we have

$$(\lambda_p - [\alpha], \lambda_p - [\alpha])_p \geq 0,$$

therefore, we can say

$$h(\alpha) \geq \frac{1}{2}\sum_{\substack{p\in S\\ p\nmid\infty,\ p^{1/e}<d}}(\lambda_p - [\alpha], \lambda_p - [\alpha])_p + \frac{1}{2}(\lambda_\infty - [\alpha], \lambda_\infty - [\alpha])_\infty, \tag{18}$$

where $d = |G_{\mathbb{Q}}\alpha|$ denotes the degree of α over $\mathbb{Q}$. Let $\epsilon = 1/d$. For each finite place $p \in S$ with $p^{1/e} < d$, we let

$$n_p = \left\lfloor \frac{\log d}{\log p^{1/e}} \right\rfloor = \left\lfloor \frac{e \log d}{\log p} \right\rfloor \in \mathbb{N}.$$

Then, for π a uniformizing parameter of L_p, the constant ϵ satisfies

$$|\pi^{n_p+1}| < \epsilon \leq |\pi^{n_p}| < 1.$$

The condition that $p^{1/e} < d$ ensures that $n_p \geq 1$, which is necessary to apply Theorems 10 and 12.

Let $[\alpha]_\epsilon$ be the regularized measure on $\mathsf{P}^1(\mathbb{C}_p)$ as defined in Section 2.2 above. Suppose that the conjugates of α all lie in O_{L_p} (respectively, L_p). Then the measure $[\alpha]_\epsilon$ is supported on the set F (resp. G) defined in Theorem 10 (resp. Theorem 12) above, and further,

$$(\lambda_p - [\alpha]_\epsilon, \lambda_p - [\alpha]_\epsilon)_p = I([\alpha]_\epsilon) \geq \begin{cases} V_\delta(F) \geq (1 - \dfrac{1}{q^{n_p-1}})\dfrac{\log p}{e(q-1)} \\ V_\delta(G) \geq (1 - \dfrac{1}{q^{n_p-1}})\dfrac{q \log p}{e(q^2-1)}, \quad \text{resp.,} \end{cases}$$

where we have used the fact that $\log|\pi|_p = -(\log p)/e$. Applying Proposition 6 to the measures $[\alpha]$ and $[\alpha]_\epsilon$ we obtain:

$$\sum_{\substack{p\in S \\ p\nmid\infty,\ p^{1/e}<d}} (\lambda_p - [\alpha], \lambda_p - [\alpha])_p \geq \sum_{\substack{p\in S \\ p\nmid\infty,\ p^{1/e}<d}} \left((1 - \frac{1}{q^{n_p-1}})\frac{\log p}{e(q-1)} - \frac{\log d}{d} \right) \tag{19}$$

if all conjugates of α lie in O_{L_p}, and

$$\sum_{\substack{p\in S \\ p\nmid\infty,\ p^{1/e}<d}} (\lambda_p - [\alpha], \lambda_p - [\alpha])_p \geq \sum_{\substack{p\in S \\ p\nmid\infty,\ p^{1/e}<d}} \left((1 - \frac{1}{q^{n_p-1}})\frac{q\log p}{e(q^2-1)} - \frac{\log d}{d} \right) \tag{20}$$

else.

The remainder of the proof now has two cases, depending on whether $\infty \in S$ (in which case our number is totally real, as all conjugates lie in $L_\infty = \mathbb{R}$) or not. First, suppose $\infty \notin S$. We use M. Baker's reformulation [1] of Mahler's inequality [11] to bound the archimedean term of the height:

$$(\lambda_\infty - [\alpha], \lambda_\infty - [\alpha])_\infty \geq -\frac{\log d}{d-1}, \tag{21}$$

and at the places of S, we use our local bounds at each p. If on the other hand we have $\infty \in S$, then we apply Proposition 5 to say that

$$(\lambda_\infty - [\alpha], \lambda_\infty - [\alpha])_\infty \geq (\lambda_\infty - [\alpha]_\epsilon, \lambda_\infty - [\alpha]_\epsilon)_\infty - 2\epsilon + \frac{\log \epsilon}{d}.$$

Now, $[\alpha]_\epsilon$ has support in the strip $E_\epsilon \subset \mathbb{P}^1(\mathbb{C})$ with imaginary part bounded in absolute value by ϵ, so applying Theorem 7 to $[\alpha]_\epsilon$, we see that:

$$(\lambda_\infty - [\alpha]_\epsilon, \lambda_\infty - [\alpha]_\epsilon)_\infty = I([\alpha]_\epsilon) \geq V_\delta(E_\epsilon) \geq \frac{7\zeta(3)}{2\pi^2} - 0.95\sqrt{\epsilon}.$$

Take $\epsilon = \frac{1}{d^2}$ and set

$$V_\infty = \frac{7\zeta(3)}{4\pi^2} - \frac{0.95d+2}{2d^2} - \frac{(d-2)\log d}{2d(d-1)}.$$

Then combining these two results, we see that

$$(\lambda_\infty - [\alpha], \lambda_\infty - [\alpha])_\infty \geq \frac{7\zeta(3)}{2\pi^2} - \frac{0.95d+2}{d^2} - \frac{2\log d}{d} = -\frac{\log d}{d-1} + 2V_\infty. \tag{22}$$

Of course we can ignore the fact that $\infty \in S$, when we do not benefit from this contribution to the height. Hence, we can as good replace $2V_\infty$ in (22) by $\max\{2V_\infty, 0\}$. Combining the above inequalities (19), respectively (20), and (21) or (22) with equation (18) gives the desired result. □

We will now state the generalization of Theorem 1 to arbitrary base number field K.

Theorem 13. *Fix a number field K and M_K denote the set of places of K. Let $S \subset M_K$ and for each $v \in S$ suppose we are given a local finite normal extension L_v/K_v, taking $L_v = \mathbb{R}$ if $v \mid \infty$. For $v \nmid \infty$ we let*

q_v	*the order of the residue field of L_v,*
p_v	*the rational prime over which v lies, and*
e_v	*the local ramification index of $L_v/\mathbb{Q}_v$.*

Let L_S denote the field of all $\alpha \in \overline{K}$ for which the minimal polynomial of α splits over L_v for every $v \in S$ and let $N_v = {}^{[K_v:\mathbb{Q}_v]}/{}_{[K:\mathbb{Q}]}$ for each $v \in S$. Suppose that $\alpha \in L_S$ with $d = [K(\alpha) : K] > 1$, and let V_∞ be the constant, depending on d, from Theorem 1. Then

$$h(\alpha) \geq -\sum_{\substack{v \notin S \\ v \mid \infty}} \frac{N_v \log d}{2(d-1)} + \sum_{\substack{v \in S \\ v \mid \infty}} N_v V_\infty + \frac{1}{2} \sum_{\substack{v \in S \\ v \nmid \infty,\ p_v^{1/e_v} < d}} N_v \cdot \left((1 - \frac{1}{q_v^{n_v - 1}}) \frac{q_v \log p}{e_v(q_v^2 - 1)} - \frac{\log d}{d} \right)$$

where

$$n_v = \left\lfloor \frac{e_v \log d}{\log p_v} \right\rfloor.$$

If in addition α is an algebraic integer, then we have

$$h(\alpha) \geq -\sum_{\substack{v \notin S \\ v \mid \infty}} \frac{N_v \log d}{2(d-1)} + \sum_{\substack{v \in S \\ v \mid \infty}} N_v V_\infty + \frac{1}{2} \sum_{\substack{v \in S \\ v \nmid \infty,\ p_v^{1/e_v} < d}} N_v \left((1 - \frac{1}{q_v^{n_v - 1}}) \frac{\log p_v}{e_v(q_v - 1)} - \frac{\log d}{d} \right).$$

Sketch of proof. The proof is essentially the same as that for Theorem 1 above with the usual modifications: we start with the standard normalized expression for the Weil height written over K:

$$h(\alpha) = \frac{1}{2} \sum_{v \in M_K} N_v \cdot ([\alpha] - \lambda_v, [\alpha] - \lambda_v)_v$$

where above the v-adic absolute value in the energy pairing is chosen to extend the usual absolute value on $\mathbb{Q}$ over which it lies, and apply the same bounds locally as above, but when applying Theorems 10 and 12 for finite places in computing the δ-Robin constants we replace the former residue field order q of L_p with the order q_v of the residue field L_v and we observe that $\log |\pi|_v = {}^{-\log p_v}/{}_{e_v}$ where π is the uniformizing parameter of L_v. The remainder of the proof is unchanged. □

5. Proof of Theorem 2

Now we will use Theorem 1 to calculate absolute lower bounds for non roots of unity in $L_S^\times$. We can use Schinzel's lower bound mentioned in the introduction, whenever $\infty \in S$. Hence, for simplicity we assume from now on that $\infty \notin S$. For totally p-adic algebraic units, Petsche [12] gave an easy argument to verify an effective lower height bound. We will summarize and extend his idea in our setting.

Proposition 14 (Petsche)**.** *Let S be a finite set of primes, and $\alpha \in L_S^\times$ be an algebraic unit which is not a root of unity. Denote by l the least common multiple of 2 and the elements in $\{p^{f_p} - 1 | p \in S\}$. Then we have*

$$h(\alpha) \geq \frac{1}{l}\left(\sum_{p\in S} \frac{v_p(l)+1}{e_p}\log p - \log 2\right).$$

Proof. Let K be a normal extension of $\mathbb{Q}$, with $\alpha \in K$ and $e_p = e(K,p)$, $f_p = f(K,p)$ for all $p \in S$. Fix a prime $p \in S$ and let $\mathfrak{p}$ be any prime in K lying above p. By $N_{K/\mathbb{Q}}(.)$ we denote the norm on the field extension $K/\mathbb{Q}$ and set $q = p^{f_p} = N_{K/\mathbb{Q}}(\mathfrak{p})$. Then, since no power of α is in $\mathfrak{p}$ and α is not a root of unity, we have $\alpha^{q-1} - 1 \in \mathfrak{p} \setminus \{0\}$. Therefore, for any $n \in \mathbb{N}$ we have

$$\alpha^{(q-1)p^n} - 1 = ((\alpha^{q-1}-1)+1)^{p^n} - 1 = \sum_{i=1}^{p^n} \binom{p^n}{i}(\alpha^{q-1}-1)^i \equiv 0 \mod \mathfrak{p}^{n+1}. \tag{23}$$

Here we have used the well known formula $v_p(\binom{p^n}{i}) = n - v_p(i) \geq n - i + 1$.

Let $l \in \mathbb{N}$ be as in the assumption. Since (23) is true for all pairs $\mathfrak{p}|p$, with $p \in S$, and $K/\mathbb{Q}$ is Galois we get

$$|N_{K/\mathbb{Q}}(\alpha^l - 1)| \geq \prod_{p\in S}\prod_{\mathfrak{p}|p} N_{K/\mathbb{Q}}(\mathfrak{p})^{v_p(l)+1} = \left(\prod_{p\in S} p^{v_p(l)+1/e_p}\right)^{[K:\mathbb{Q}]}.$$

By basic height estimates it follows

$$lh(\alpha) + \log 2 = h(\alpha^l) + \log 2 \geq h(\alpha^l - 1) \geq \sum_{p\in S}\frac{v_p(l)+1}{e_p}\log p.$$

The statement of the proposition follows immediately. □

Dubickas and Mossinghoff [6] have slightly strengthened Petsche's result. However, both estimates are only positive for small ramification degrees, and in the special case of an algebraic unit. In the remainder of this paper we calculate lower height bounds for all elements in $L_S^\times$ which are not roots of unity.

Therefore we define real valued functions on the interval $(1,\infty)$ by

$$f_S(x) = -\frac{\log x}{2(x-1)} + \frac{1}{2}\sum_{\substack{p\in S\\ p^{1/e}<x}}\left((1-\frac{1}{q^{n_p-1}})\frac{q\log p}{e(q^2-1)} - \frac{\log x}{x}\right)$$

and

$$g_S(x) = -\frac{\log x}{2(x-1)} + \frac{1}{2}\sum_{\substack{p\in S\\ p^{1/e}<x}}\left((1-\frac{1}{q^{n_p-1}})\frac{q\log p}{e(q-1)^2} - \frac{\log x}{x}\right),$$

where $n_p = \lfloor {}^{e\log x}/_{\log p} \rfloor$.

Lemma 15. *Let $\alpha \in L_S^\times$ be not an algebraic unit and let $y \in (1,\infty)$ be such that $f_S(y) \geq {}^{\log(2)}/_{y}$. Then we have*

$$h(\alpha) \geq \min\left\{f_{S'}(\lceil y \rceil), \frac{\log 2}{\lfloor y \rfloor}\right\} \geq \frac{\log 2}{y}, \tag{24}$$

where $S' = \{p \in S | p^{1/e} \leq y\}$.

Proof. The function $f_S(x)$ tends to a positive value as x tends to infinity and is negative in the interval $(1,\epsilon)$ for $\epsilon = \min_{p\in S}\{p^{1/e}\}$. Since ${}^{\log 2}/_{x}$ is a positive monotonically decreasing function approaching zero, a y as in the assumption exists. Set $S' = \{p \in S | p^{1/e} \leq y\}$, then S' is not empty and we have $\alpha \in L_S \subseteq L_{S'}$. Moreover, $f_{S'}(x)$ is monotonically increasing in the interval $[y,\infty)$. In particular, we have $f_{S'}(x) \geq {}^{\log 2}/_{x}$ for all $x \geq y$.

Denote the degree of α by d. It is well known that the height of α is bounded from below by ${}^{\log 2}/_{d}$ and equation (2) tells us $h(\alpha) \geq f_{S'}(d)$. Therefore we have

$$h(\alpha) \geq \min_{n\in\mathbb{N}}\left\{\max\left\{f_{S'}(n), \frac{\log 2}{n}\right\}\right\}.$$

For $n \geq y$ we have $f_{S'}(n) \geq f_{S'}(\lceil y \rceil)$ and for $n \leq y$ we have ${}^{\log 2}/_{n} \geq {}^{\log 2}/_{\lfloor y \rfloor}$. This proves the lemma, as the last inequality in (24) is trivial. □

Let $\theta = 1.324...$ denote the smallest Pisot number; i.e. the real root of $x^3 - x - 1$. We define the function $\mathrm{Do}(x) : (1,\infty) \to \mathbb{R}$ as

$$\mathrm{Do}(x) = \begin{cases} \dfrac{\log\theta}{x} & \text{for } x \leq 7 \\ \dfrac{1}{4x}\left(\dfrac{\log\log x}{\log x}\right)^3 & \text{for } x > 7 \end{cases}.$$

Lemma 16. *Let $\alpha \in L_S^\times$ be an algebraic unit which is not a root of unity and let $z \in (1,\infty)$ be such that $g_S(z) \geq \mathrm{Do}(z)$. Then we have*

$$h(\alpha) \geq \min\{g_{S'}(\lceil z \rceil), \mathrm{Do}(\lfloor z \rfloor)\} \geq \mathrm{Do}(z), \tag{25}$$

where $S' = \{p \in S | p^{1/e} \leq z\}$.

Proof. We denote the degree of α by d. The best known general lower bound for the height of α is due to Voutier [16], who improved the constant of a previous lower bound due to Dobrowolski [5]. This bound is

$$h(\alpha) \geq \frac{1}{4d}\left(\frac{\log\log d}{\log d}\right)^3. \tag{26}$$

Using a complete list of algebraic units of degree ≤ 37 of small Mahler measure due to Flammang, Rhin, and Sac-Épée [10], we find $h(\alpha) \geq {}^{\log\theta}/_{d}$, whenever $d \leq 7$. This leads to the estimate $h(\alpha) \geq \mathrm{Do}(d)$. By (4) we also know $h(\alpha) \geq g_{S'}(d)$. Since $\mathrm{Do}(x)$ is monotonically decreasing, the Lemma follows with the same argument as in Lemma 15. □

In order to prove Theorem 2 it remains to find elements y and z satisfying the assumptions of Lemmas 15 and 16.

Example 17. Let $S = \{2, 3\}$ and set $L_p = \mathbb{Q}_p$ for $p \in S$. Then L_S is the subfield of $\overline{\mathbb{Q}}$ consisting of all algebraic numbers which are totally 2-adic and totally 3-adic. Note in the following, that by classical algebraic number theory there are no non-trivial totally 2-adic roots of unity. With Proposition 14 we find that the height of an algebraic unit in $L_S \setminus \{\pm 1\}$ is bounded from below by $(\log 2+\log 3)/2 = 0.89587...$. We apply Lemma 15 with $y = 15.9$ to deduce

$$h(\alpha) \geq \frac{\log 2}{15} = 0.04620\ldots$$

for every $\alpha \in L_S^\times \setminus \{\pm 1\}$.

Lemma 18. *Let a and b be real numbers with $a > 0$ and $b \geq 1 + \log a$. Then $ax - b - \log x$ is positive for all real $x \geq (8/5)a^{-1}(\log(a^{-1}) + b)$.*

Proof. This is the second inequality of [13, Lemma 3.3]. Note, that in the proof of this part of the lemma it is only required that $-\exp(-1) \leq -a\exp(-b)$ which is equivalent to our assumption $b \geq 1 + \log a$. □

Proposition 19. *Let p be a rational prime number and $L_p/\mathbb{Q}_p$ and $L_{\{p\}}/\mathbb{Q}$ as usual in this paper. For every $\alpha \in L_{\{p\}}^\times$, which is not an algebraic unit, one has*

$$h(\alpha) \geq \frac{\log(2)\log(p)}{5e'(q+1)\log\left(\frac{5e'(q+1)}{\log(p)}\right)}. \tag{27}$$

Here we set $e' = \max\{2, e\}$.

Proof. After a possible extension of L_p by a totally ramified extension of degree two, we can assume $e \geq 2$. Hence we can work with e replaced by $e' = \max\{2, e\}$. In order to apply Lemma 15 it is sufficient to find a real $y > p \geq p^{1/e}$ with

$$-\frac{\log y}{2(y-1)} + \frac{1}{2}\left((1 - q^{1-e'})\frac{q\log p}{e'(q^2-1)} - \frac{\log y}{y}\right) - \frac{\log 2}{y} \geq 0. \tag{28}$$

This is equivalent to

$$\frac{q - q^{2-e'}}{q^2 - 1} \cdot \frac{\log p}{e'} y - 2\log 2 - \frac{2y-1}{y-1}\log y \geq 0.$$

Since $2y-1/y-1 \leq 3$ for $y \geq p$, and $q-q^{2-e'}/q^2-1 \geq 1/q+1$ this equation holds true if

$$\frac{\log p}{3e'(q+1)} y - \frac{2\log 2}{3} - \log y \geq 0.$$

The term $1+\log(\log p/3e'(q+1))$ is always negative. In particular we can apply Lemma 18 to deduce that this last inequality, and hence (28), is satisfied for

$$y = \frac{24e'(q+1)}{5\log p}\left(\log\left(\frac{3e'(q+1)}{\log p}\right) + \frac{2\log 2}{3}\right),$$

which is indeed greater than p. The expression

$$\frac{e'(q+1)}{5\log p}\left(24\log\left(\frac{3e'(q+1)}{\log p}\right) - 25\log\left(\frac{5e'(q+1)}{\log p}\right) + 16\log 2\right)$$

is always negative (it is maximized for $p = 3, e = 2, f = 1$ which yields $-6.9386\ldots$). Hence, we have

$$\begin{aligned} y &= \frac{24e'(q+1)}{5\log p}\left(\log\left(\frac{3e'(q+1)}{\log p}\right)+\frac{2\log 2}{3}\right)\\ &= \frac{5e'(q+1)}{\log p}\log\left(\frac{5e'(q+1)}{\log p}\right)\\ &+\frac{e'(q+1)}{5\log p}\left(24\log\left(\frac{3e'(q+1)}{\log p}\right)-25\log\left(\frac{5e'(q+1)}{\log p}\right)+16\log 2\right)\\ &< \frac{5e'(q+1)}{\log p}\log\left(\frac{5e'(q+1)}{\log p}\right). \end{aligned}$$

Therefore, by Lemma 15 we can conclude that every $\alpha \in L^{\times}_{\{p\}}$ which is not an algebraic unit satisfies

$$h(\alpha) \geq \frac{\log 2}{y} \geq \frac{\log(2)\log(p)}{5e'(q+1)\log\left(\frac{5e'(q+1)}{\log p}\right)},$$

proving the proposition. □

Proposition 20. *Let p be a rational prime number and $L_p/\mathbb{Q}_p$ and $L_{\{p\}}/\mathbb{Q}$ as above. For every algebraic unit $\alpha \in L^{\times}_{\{p\}}$, which is not a root of unity, one has*

$$h(\alpha) \geq \frac{\log p}{15e'(q-1)\log\left(\frac{5e'(q-1)}{\log p}\right)^4}. \tag{29}$$

Here we set $e' = \max\{2, e\}$.

Proof. The proof is almost the same as the proof of Proposition 19. Again we may, after a possible extension of L_p by a totally ramified extension of degree two, assume $e \geq 2$. Hence we can work with e replaced by $e' = \max\{2, e\}$. We want to apply Lemma 16. To do so it is sufficient to find a real $z > p \geq p^{2/e'}$ with

$$-\frac{\log z}{2(z-1)}+\frac{1}{2}\left((1-q^{1-e'})\frac{q\log p}{e'(q-1)^2}-\frac{\log z}{z}\right)-\frac{1}{4z}\left(\frac{\log\log z}{\log z}\right)^3 \geq 0. \tag{30}$$

In case $z > 7$ we have ${}^{2z-1}/_{z-1} \leq {}^{13}/_{6}$. Using this, $({}^{\log\log z}/_{\log z})^3 \leq {}^{1}/_{20}$, and $(1-q^{1-e'})^{q}/_{(q-1)^2} \geq {}^{1}/_{q-1}$ we see that (30) follows if

$$-\log z + \frac{6\log p}{13e'(q-1)}z - \frac{3}{260} > 0.$$

Hence, by Lemma 18, this is satisfied for any $z > \max\{p, 7\}$ with

$$z \geq \frac{52}{15}\frac{e'(q-1)}{\log p}\left(\log\left(\frac{13}{6}\frac{e'(q-1)}{\log p}\right)+\frac{3}{260}\right). \tag{31}$$

We claim that $z = {}^{7e'(q-1)}/_{2\log p} \cdot \log({}^{5e'(q-1)}/_{\log p})$ is a valid choice for such a constant. One easily checks that inequality (31) is satisfied by this z. Moreover, one can calculate directly that $z > 7$ for all $p \leq 7$. Therefore, it remains to prove that we have $z > p$ for all $p > 7$. It is clear that z is minimized for $e' = 2$ and $f = 1$.

For any prime $p > 7$ it is $^{59}/_{10} \log(10(p-1)) - 7 \log\log p > 0$ and $\log(^{p}/_{10(p-1)}) < 0$. Hence we have for $p > 7$

$$\begin{aligned}
&\frac{59}{10}\log(10(p-1)) - 7\log\log p > \frac{11}{10}\log\left(\frac{p}{10(p-1)}\right)\\
\Rightarrow &7\log(10(p-1)) - 7\log\log p > \frac{11}{10}\log p\\
\Rightarrow &7\log\left(\frac{10(p-1)}{\log p}\right) > \frac{p}{p-1}\log p\\
\Rightarrow &z = \frac{7}{2}\frac{e'(q-1)}{\log p}\log\left(\frac{5e'(q-1)}{\log p}\right) > p
\end{aligned}$$

proving the claim. Since z satisfies (30), Lemma 16 tells us that for any algebraic unit $\alpha \in L_{\{p\}}$ which is not a root of unity we have

$$h(\alpha) \geq \frac{\log p}{14e'(q-1)\log\left(\frac{5e'(q-1)}{\log p}\right)}\left(\frac{\log\log\left(\frac{7}{2}\frac{e'(q-1)}{\log p}\log\left(\frac{5e'(q-1)}{\log p}\right)\right)}{\log\left(\frac{7}{2}\frac{e'(q-1)}{\log p}\log\left(\frac{5e'(q-1)}{\log p}\right)\right)}\right)^3.$$

The function

$$x \mapsto \frac{\log\log\left(\frac{7}{2}x\log(5x)\right)\log(5x)}{\log\left(\frac{7}{2}x\log(5x)\right)}$$

is monotonically increasing in the interval $(1,\infty)$. As the minimum of $^{e'(q-1)}/_{\log p}$ is attained for $e' = 2$, $f = 1$ and $p = 2$, it follows

$$\begin{aligned}
h(\alpha) &\geq \frac{\log(p)}{14e'(q-1)\log\left(\frac{5e'(q-1)}{\log p}\right)^4}\left(\frac{\log\log\left(\frac{7}{\log 2}\log\left(\frac{10}{\log 2}\right)\right)\log\left(\frac{10}{\log 2}\right)}{\log\left(\frac{7}{\log 2}\log\left(\frac{10}{\log 2}\right)\right)}\right)^3\\
&\geq \frac{\log p}{15e'(q-1)\log\left(\frac{5e'(q-1)}{\log p}\right)^4}
\end{aligned}$$

for any algebraic unit $\alpha \in L_{\{p\}}$ which is not a root of unity. □

Proof of Theorem 2. In order to prove the first part of the theorem, we have to show that the lower bound (27) is always greater than the bound (29). This is equivalent to the statement

$$\frac{\log\left(\frac{5e'(q+1)}{\log p}\right)}{\log(2)\log\left(\frac{5e'(q-1)}{\log p}\right)^4} \leq \frac{3(q-1)}{q+1}.$$

Since the right hand side is obviously greater or equal to 1, we are left to prove that $\log(^{5e'(q+1)}/_{\log p}) \leq \log(2)\log(^{5e'(q-1)}/_{\log p})^4$.

The polynomial $\log(2)(x - \log(3))^4 - x$ takes positive values for all $x \geq 3$. Moreover, we have $\log(^{5e'(q+1)}/_{\log p}) \geq 3$ for all choices of $e', q \geq 2$. It follows

$$\log\left(\frac{5e'(q+1)}{\log p}\right) \leq \log(2)\left(\log\left(\frac{5e'(q+1)}{3\log p}\right)\right)^4 \leq \log(2)\left(\log\left(\frac{5e'(q-1)}{\log p}\right)\right)^4$$

proving the claim.

In case $e = 1$ we can apply Proposition 14 to bound the height of algebraic units by ${}^{\log(p/2)}/_{q-1}$ for an odd prime p and ${}^{\log 2}/_{2(2^f-1)}$ for $p = 2$. As these estimates are always greater than (27), the theorem follows.

□

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