The isomorphism problem for cominuscule Schubert varieties

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Schubert varieties:

Let G be a reductive linear algebraic group over $\mathbb{C}$.

- Let denote a Borel subgroup B.
- Let P denote a parabolic subgroup containing B.
- Let W denote the Weyl group of G.

The partial flag variety G/P has a Bruhat decomposition:

$$G/P = \bigsqcup_{w \in W/(W \cap P)} BwP/P$$

- Let $W^P \simeq W/(W \cap P)$ denote the set of minimal length reps.

For any $w \in W^P$, define the *Schubert variety*:

$$X_w^P := \overline{BwP}/P$$

Question: Given $X_w^P \subseteq G/P$ and $X_{w'}^{P'} \subseteq G'/P'$, when are these Schubert varieties algebraically isomorphic?

Combinatorial data:

If $X_w^P \cong X_{w'}^{P'}$, then:

- $\ell(w) = \ell(w')$ (lengths are equal)
- $[e, w] \cap W^P \cong [e, w'] \cap W^{P'}$ (Bruhat intervals are poset isomorphic)
- $H^*(X_w^P) \cong H^*(X_{w'}^{P'})$ (cohomologies are ring isomorphic)

In general, these properties are not sufficient to determine an algebraic isomorphism.

History of the Schubert isomorphism problem:

1. $G = \mathrm{GL}_n(\mathbb{C})$ and $P = B$: Partition Schubert varieties are determined by cohomology. (Develin-Martin-Reiner, 2007)

$$X_\lambda \cong X_\mu \;\Leftrightarrow\; H^*(X_\lambda) \simeq H^*(\mu)$$

2. $P = B$ and $P' = B'$: Schubert varieties are determined by "Cartan equivalence". (R-Slofstra, arXiv:2021)

$$X_w \cong X_{w'} \;\Leftrightarrow\; (w, A) \sim (w', A')$$

   where A, A' are the Cartan matricies of G, G'.

3. $G = \mathrm{GL}_n(\mathbb{C})$ and P maximal: Grassmaniann Schubert varieties are determined up to transpose of its partition. (Tarigradschi-Xu, arXiv:2022)

$$X_\lambda^P \cong X_\mu^P \;\Leftrightarrow\; \mu = \lambda \text{ or } \lambda^T$$

Cominuscule flag varieties:

Let $R \subseteq \mathfrak{h}^*$ denote the root system of G. The roots decompose:

$$R = R^+ \sqcup R^-.$$

Let $\Delta \subseteq R^+$ denote the set of simple roots. The Weyl group W is generated by the set

$$S := \{s_\alpha \mid \alpha \in \Delta\}.$$

For any subset $J \subseteq S$, define the parabolic subgroup

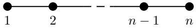
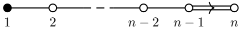
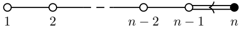
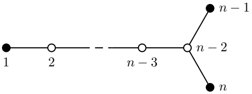
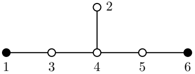
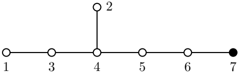
$$P_J := BW_JB.$$

Definition: We say the flag variety G/P_J is *cominuscule* if there exists $\gamma \in \Delta$ such that

1. $J = S \setminus \{\gamma\}$
2. γ appears with coefficient 1 in the longest root in R^+.

We call γ the cominuscule root of G/P_J. (set $P_\gamma := P_J$)

Classification of cominuscule flag varieties:

	Grassmannian: $A_n/P_k = \mathrm{Gr}(k, n+1)$
	Odd quadric: $B_n/P_1 = Q^{2n-1}$
	Lagrangian Gr: $C_n/P_n = \mathrm{LG}(n, 2n)$
	Even quadric: $D_n/P_1 = Q^{2n-2}$ Orthogonal Gr: $D_n/P_{n-1} \cong D_n/P_n = \mathrm{OG}(n, 2n)$
	Cayley plane: $E_6/P_1 \cong E_6/P_6$
	Freudenthal variety: E_7/P_7

Nice fact:
Cominuscule flag varieties share many common properties with Grassmannians.

Define a poset structure on R^+ by the relation:

$$\alpha \leq \beta \quad \Leftrightarrow \quad \beta - \alpha \text{ is a positive sum of simple roots.}$$

For any cominuscule flag variety X with cominuscule root $\gamma \in \Delta$, define the upper order ideal:

$$\mathcal{P}_X := \{\alpha \in R^+ \ \ |\alpha \geq \gamma\}$$

Theorem: Proctor (1984)

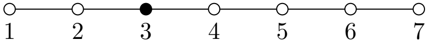
If $X = G/P$ is cominuscule, then

1. The poset $\mathcal{P}_X$ is a distributive lattice.
2. The set W^P is in bijection with lower order ideals of $\mathcal{P}_X$ under the map

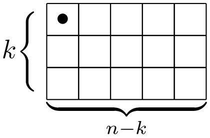
$$w \mapsto \lambda(w) := \{\alpha \in R^+ \mid w(\alpha) \in R^-\}.$$

The set $\lambda(w)$ is called the *inversion set* of w.

Grassmannian A_{n-1}/P_k:



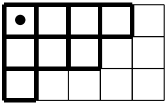
The poset $\mathcal{P}_{\mathrm{Gr}(k,n)}$ is isomorphic to



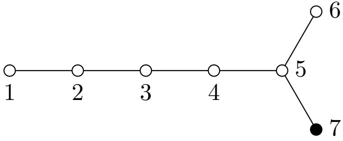
where $\square = \alpha_i + \cdots + \alpha_j$ if $\square$ is in the $(k-i+1)$-th row and $(k+j-1)$-th col. The partial order is given by

$$\boxed{\text{a}} \leq \boxed{\text{c}} \;\Leftrightarrow\; \boxed{\text{a}} \text{ is weakly northwest of } \boxed{\text{c}}.$$

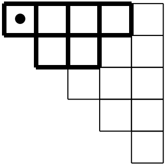
Lower order ideals (or equivalently, inversion sets) are given by Young diagrams of partitions:



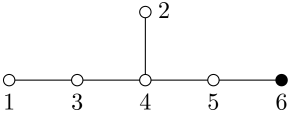
Orthogonal Grassmannian D_n/P_n:



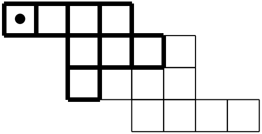
$\mathcal{P}_{\mathrm{OG}(n,2n)} =$



Caley plane E_6/P_6:



$\mathcal{P}_{E_6/P_6} =$



Grassmaniann like properties:

Proposition: Thomas-Yong (2009)

Let $X = G/P$ be cominuscule. Then

1. $w \leq u \in W^P \quad \Leftrightarrow \quad \lambda(w) \subseteq \lambda(u) \subseteq \mathcal{P}_X$
2. $\dim(X_w^P) = |\lambda(w)|$.

Notation: If $\mu = \lambda(w)$ is a lower order ideal in $\mathcal{P}_X$, the define

$$X_\mu = X_{\lambda(w)} := X_w^P$$

Tarigradschi-Xu (arXiv:2022)

For $X = \mathrm{Gr}(k, n)$, we have

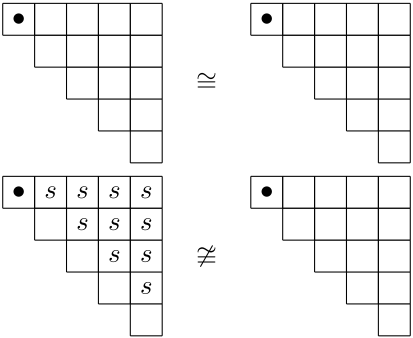
$$X_\lambda \cong X_\mu \quad \Leftrightarrow \quad \lambda = \mu \text{ or } \mu^T \quad \Leftrightarrow \quad \lambda \cong \mu \text{ as posets.}$$

Question: Does poset isomorphism work for cominuscule Schubert varieties?

Answer: If we are allowed to compare across types, then no:

$$\mathrm{LG}(n, 2n) \ncong \mathrm{OG}(n, 2n)$$

however



Fix: Label boxes by long/short roots.

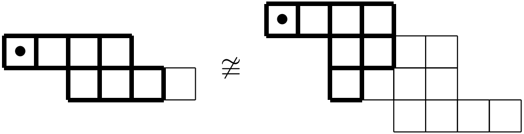
Updated definition: For any $w \in W^P$, let $\lambda = \lambda(w)$ be the poset of inversions labeled by long/short roots.

Theorem: R-Tarigradschi-Xu (arXiv:2023)

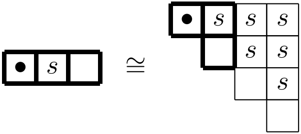
Let $X_\lambda \subset X$ and $X'_\mu \subseteq X'$ be cominuscule Schubert varieties. Then

$$X_\lambda \cong X'_\mu \quad \Leftrightarrow \quad \lambda \cong \mu \text{ as } \textbf{labeled} \text{ posets.}$$

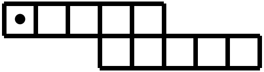
Example: $X_\lambda \subset D_5/P_1$ is not isomorphic to $X'_\mu \subset E_6/P_6$:



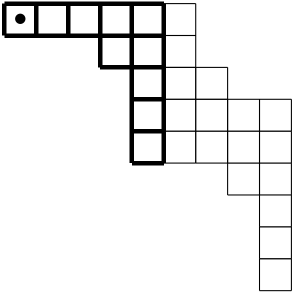
Example: $X_\lambda = B_2/P_1 = Q^3$ is isomorphic to $X'_\mu \subset C_n/P_n = \mathrm{LG}(n, 2n)$:



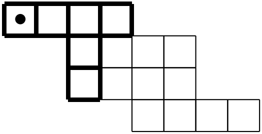
Example: $D_6/P_1 = Q^{10} \cong X'_\lambda \subseteq E_7/P_7$



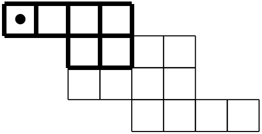
$\cong$



Example: $X_\lambda \not\cong X'_\mu$ in E_6/P_6



$\not\cong$



Proof of $X_\lambda \cong X'_\mu \;\Rightarrow\; \lambda \cong \mu$:

Suppose Y is isomorphic to a cominuscule Schubert variety.

1. Minimal extremal rays of the effective cone in the Chow group $A_*(Y)$ correspond to Schubert classes.
(Fulton–MacPherson–Sottile–Sturmfels 1995)

2. There is a unique effective generator in $\mathrm{Pic}(Y)$.

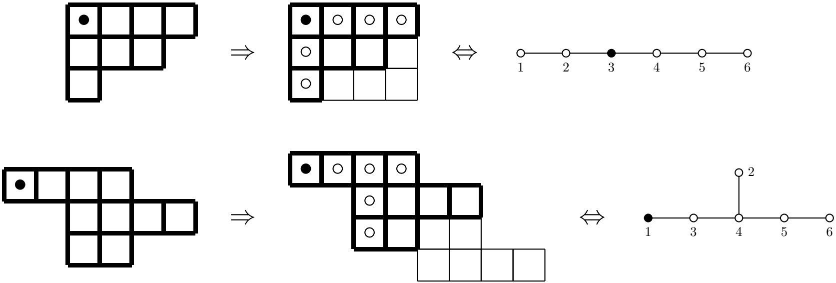
3. Apply a "blind" Chevalley formula using the intersection product to construct a labeled poset.

4. If $Y \cong X_\lambda \subseteq G/P$, this construction yields λ.

Proof of $\lambda \cong \mu \;\Rightarrow\; X_\lambda \cong X'_\mu$:

Suppose we have a labeled poset λ.

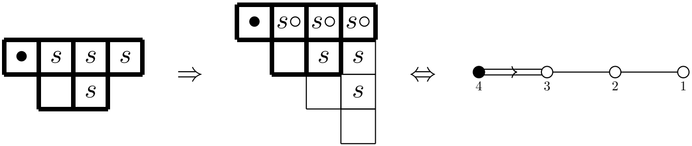
1. Construct a "minimal flag variety" corresponding to λ

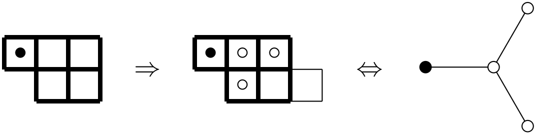


2. If $X_\lambda \subseteq G/P$, then "minimally" embed $X_\lambda \subseteq L/(L \cap P) \hookrightarrow G/P$.
3. If $\lambda \cong \mu$, then minimal flag varieties are isomorphic with

$$X_\lambda \hookrightarrow L/(L \cap P) \cong L'/(L' \cap P') \hookleftarrow X'_\mu.$$

This identifies X_λ with X'_μ.





Thank you!