Enumerating permutations that avoid split patterns $23|1$ and $3|12$

Travis Grigbsy and Edward Richmond

Oklahoma State University

April 20, 2024

Let $r < n \in \mathbb{Z}_+$ and $\mathbb{C}^n = \mathrm{Span}_{\mathbb{C}}\{e_1, \ldots, e_n\}$.

Complete flag variety:

$$\mathrm{Fl}(n) := \{V_\bullet = (V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n) \mid \dim(V_i) = i\}$$

Grassmannian:

$$\mathrm{Gr}(r, n) := \{V \subset \mathbb{C}^n) \mid \dim(V) = r\}$$

Consider the projection $\pi_r : \mathrm{Fl}(n) \twoheadrightarrow \mathrm{Gr}(r, n)$ given by

$$\pi_r(V_\bullet) = V_r.$$

The map π_r is a fiber bundle map on $\mathrm{Fl}(n)$ with fibers

$$\begin{aligned}\pi_r^{-1}(V) &\simeq (V_1 \subset \cdots \subset V_{r-1} \subset V) \times (V_{r+1}/V \subset \cdots \subset V_{n-1}/V \subset \mathbb{C}^n/V) \\ &\simeq \mathrm{Fl}(r) \times \mathrm{Fl}(n-r).\end{aligned}$$

Question: When is π_r restricted to a Schubert variety of $\mathrm{Fl}(n)$ a fiber bundle?

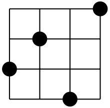
For any $n \times n$ permutation matrix w, define the **Schubert variety:**

$$X(w) := \{V_\bullet \mid \dim(E_i \cap V_j) \geq \mathrm{rk}(w[i,j])\}$$

where $E_i := \mathrm{Span}\{e_1, \ldots, e_i\}$ and $w[i,j]$ is the $(i \times j)$ NW-submatrix of w.

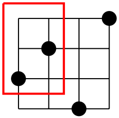
Conventions:

- The matrix entries of w mark the points $(i, w(i))$ (rows $\rightarrow$, cols $\downarrow$).
- $(1,1)$ represents the NW corner of the matrix.

Example: $n = 4$ and $w = 3241 =$ 

Example: Consider the Schubert variety

$$X(3241) = \{V_\bullet \mid \dim(E_3 \cap V_2) \geq 2\} = \{V_\bullet \mid V_2 \subset E_3\}$$



and the projection $\pi_3 : (V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4) \mapsto (\cancel{V_1} \subset \cancel{V_2} \subset V_3 \subset \mathbb{C}^4)$.

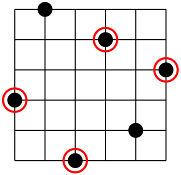
Restricting to $X(3241)$, we get $\pi_3 : X(3241) \twoheadrightarrow \mathrm{Gr}(3,4)$.

The fiber over V is

$$\pi_3^{-1}(V) = \{(V_1 \subset V_2 \subset V \subset \mathbb{C}^4) \mid V_1 \subset V_2 \subseteq E_3 \cap V\}$$
$$\cong \begin{cases} \mathrm{Fl}(2) & \text{if } \dim(E_3 \cap V) = 2 \\ \mathrm{Fl}(3) & \text{if } E_3 = V. \end{cases}$$

So π_3 is not a fiber bundle on $X(3241)$. (But π_1 and π_2 are fiber bundles!)

Pattern avoidance: Let $m \leq n$. We say a permutation $w = w(1) \cdots w(n)$ *contains* the pattern $u = u(1) \cdots u(m)$ if there is a subsequence of w with the same relative order as u. Otherwise, w *avoids* the pattern u.

Example: $w = 416253 =$ 

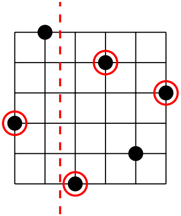
contains the pattern 3412, but avoids the pattern 1234.

Remark: Pattern avoidance has been a useful tool to describe many geometric properties of Schubert varieties. (Survey article by Abe-Billey (2016))

Split pattern avoidance:
We say a permutation w *contains the split pattern* $u = u_1|u_2$ with respect to position r if

- there is a subsequence of w with the same relative order as u such that
- $w(1)\cdots w(r)$ contains u_1 and $w(r+1)\cdots w(n)$ contains u_2.

Otherwise, w *avoids the split pattern* $u = u_1|u_2$ with respect to position r.

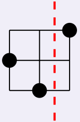
Example: $w = 416253 =$ 

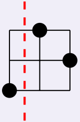
contains the split pattern $3|412$ with respect to positions $r = 1, 2$ but avoids $3|412$ with respect to $r = 3, 4, 5$.

Theorem 1: Alland-R (2018)

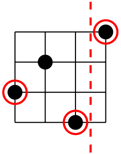
The following are equivalent:

- The projection π_r is a fiber bundle on $X(w)$.
- w avoids the split patterns $23|1$ and $3|12$ with respect to position r.





Example: Let $w = 3241$ and consider $X(w) = \{V_\bullet \mid V_2 \subset E_3\}$.

We have $w =$  containing $23|1$ with respect to position $r = 3$.

Hence π_3 is not a fiber bundle on $X(w)$.

Counting question: How many $n \times n$ permutations avoid the split patterns $23|1$ and $3|12$ with respect to position r?

Let $K(r,n)$ denote this set of permutations and let $k(r,n) := |K(r,n)|$.

Facts:

- $k(0,n) = k(n,n) = n!$
- $k(r,n) = k(n-r,n)$
- $k(1,n) = \sum_{j=1}^{n} \frac{(n-1)!}{(j-1)!}$

For $m \geq j \geq 1$, let $(m)_j := m(m-1)\cdots(m-j+1)$ denote falling factorial.

Theorem: Grigsby-R (arXiv:2024)

For any, $0 \leq r \leq n$, we have

$$k(r,n) = r!(n-r)! + \sum_{i=1}^{r}\sum_{j=1}^{n-r} \binom{n-i-j}{r-i} \cdot (r)_{i-1} \cdot (n-r)_{j-1}.$$

Table of values of $k(r,n)$:

$r\backslash n$	1	2	3	4	5	6	7	8	9
0	1	2	6	24	120	720	5040	40320	362880
1	1	2	5	16	65	326	1957	13700	109601
2		2	5	14	47	194	977	5870	41099
3			6	16	47	162	676	3416	20541
4				24	65	194	676	2836	14359

Proof of Theorem:

- Decompose $K(r,n) = K_L(r,n) \sqcup K_R(r,n)$ where

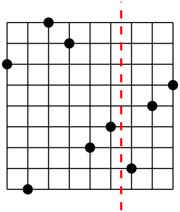
$$K_L(r,n) = \{w \in K(r,n) \mid w^{-1}(n) \leq r\}$$

$$K_R(r,n) = \{w \in K(r,n) \mid w^{-1}(n) > r\}.$$

- Show that $|K_R(r,n)| = (n-r) \cdot k(r,n-1)$.

Proof of Theorem:

- Let $S(i) := \{w \in K_L(r,n) \mid w(n) = i\}$, so $K_L(r,n) = \bigsqcup_i S(i)$.

Ex: $w = 391276|854 =$  $\in S(4) \subseteq K_L(6,9)$.

- Show that $|S(i)| = \binom{n-i-1}{r-i} \cdot (r)_{i-1}$.

$n=7, r$	0	1	2	3	4	5	6	7
$\lvert K_L(r,7)\rvert$	0	1	7	28	94	325	1237	5040
$\lvert K_R(r,7)\rvert$	5040	1956	970	648	582	652	720	0
$k(r,7)$	5040	1957	977	676	676	977	1957	5040

Observation: $r!(n-r)! \leq k(r,n) \leq n!$.

Define the generating function:

$$\mathcal{K}(x,y) := \sum_{n=0}^{\infty} \sum_{r=0}^{n} k(r,n) \frac{x^r y^{n-r}}{r!(n-r)!}.$$

Goal: Find a formula for $\mathcal{K}(x,y)$.

Proposition: Grigsby-R (arXiv:2024)

Let $a(r,s) := \dfrac{k(r,r+s)}{r! \cdot s!} - 1$. Then

$$a(r,s) = a(r,s-1) + a(r-1,s) - a(r-1,s-1) + \binom{r+s-2}{r-1} \frac{1}{r!\,s!}$$

and

$$\mathcal{K}(x,y) = \frac{\mathcal{L}(x,y)+1}{1-x-y+xy} \quad \text{where} \quad \mathcal{L}(x,y) := \sum_{r,s=0}^{\infty} \binom{r+s-2}{r-1} \frac{x^r y^s}{r!\,s!}.$$

New goal: Find a formula for $\mathcal{L}(x,y) := \displaystyle\sum_{r,s=0}^{\infty} \binom{r+s-2}{r-1} \frac{x^r y^s}{r!\, s!}$.

Observation: $\partial_{xy}(\mathcal{L}(x,y)) = \displaystyle\sum_{r,s=0}^{\infty} \binom{r+s}{r} \frac{x^r y^s}{r!\, s!}$.

Common bivariate generating functions for binomial coefficients:

$$\sum_{r,s=0}^{\infty} \binom{r+s}{r} x^r y^s = \frac{1}{1-x-y} \quad \text{and} \quad \sum_{r,s=0}^{\infty} \binom{r+s}{r} \frac{x^r y^s}{(r+s)!} = e^{x+y}.$$

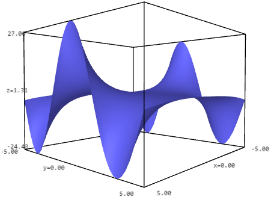
Question: What is the “middle” bivariate generating function for binomials??

Consider the complex valued function

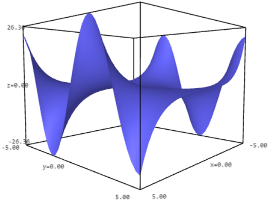
$$I_0(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta} d\theta.$$

The function $I_0(z)$ is called the **modified Bessel function of first kind** (at $\alpha = 0$) and is a solution to the modified Bessel equation

$$z^2 u_{zz} + z u_z - z^2 u = 0.$$



$\mathrm{Re}(I_0(x+iy))$



$\mathrm{Im}(I_0(x+iy))$

Bessel functions have applications to modeling heat conduction, membrane vibration, hydrodynamics and electrostatics.

Key property: The modified Bessel function has power series expansion

$$I_0(z) = \sum_{m=0}^{\infty} \frac{1}{(m!)^2} \left(\frac{z^2}{4}\right)^m.$$

Proposition: Grigsby-R (arXiv:2024)

The binomial generating function

$$\sum_{r,s=0}^{\infty} \binom{r+s}{r} \frac{x^r y^s}{r!\, s!} = e^{x+y} I_0(2\sqrt{xy}).$$

Theorem: Grigsby-R (arXiv:2024)

The generating function

$$\mathcal{K}(x,y) = \frac{\mathcal{L}(x,y)+1}{1-x-y+xy} \quad \text{where} \quad \mathcal{L}(x,y) = \iint e^{x+y} I_0(2\sqrt{xy})\, dx dy.$$

Proposition: Grigsby-R (arXiv:2024)

The binomial generating function

$$\sum_{r,s=0}^{\infty} \binom{r+s}{r} \frac{x^r y^s}{r!\, s!} = e^{x+y} I_0(2\sqrt{xy}).$$

Observation: Setting $x = y$ and applying Vandermonde's identity gives

$$e^{2x} I_0(2x) = \sum_{r,s=0}^{\infty} \binom{r+s}{r} \frac{x^{r+s}}{r!\, s!} = \sum_{r,s=0}^{\infty} \binom{r+s}{r}^2 \frac{x^{r+s}}{(r+s)!} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n!}.$$

Remark: This formula for the g.f. for central binomial coefficients is classically known.

Work in progress on split patterns:

1. Split pattern avoidance can be used to detect when the projection π_r restricted to $X(w)$ has at most two fiber types.
   Conjecture: (with Suho Oh) A permutation needs to avoid 20 split patterns:

$$4|123, 34|12, 234|1$$

$$5|3412, 5|3142, 4253|1, 4523|1$$

$$25|314, 25|341, 53|142, 54|213, 425|31, 354|21, 523|14, 253|14$$

$$263|451, 623|451, 263|415, 624|351, 623|415$$

2. Consider the projection of $X(w)$ onto a partial flag variety. Multi-bar split pattern avoidance can be used to characterized fiber bundle structures (with Reuven Hodges).

   The 2-step case may characterize fiber bundle structures on spherical Schubert varieties.

Thank you!