A generating function for left keys and its associated representation theory

"Ti esrever dna ti pilf, nwod gniht ym tup"

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AMS Fall Southeastern Sectional Meeting
Gainesville, FL
November, 2019

Objects of study

$$\mathrm{Sym} \subset \mathrm{QSym} \subset \mathbb{Q}[x_1, x_2, \ldots x_n]$$

- Sym: polynomials s.t. the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ equals the coefficient of $x_{\sigma(1)}^{\alpha_1} x_{\sigma(2)}^{\alpha_2} \cdots x_{\sigma(k)}^{\alpha_k}$ for $\sigma \in S_n$

$$f_1(x_1, x_2, \ldots, x_n) = x_1^3 x_2^5 + x_1^5 x_2^3 + x_1^3 x_3^5 + x_1^5 x_3^3 + \ldots$$

- QSym: polynomials s.t. the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ equals the coefficient of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ for $i_1 < i_2 < \cdots < i_k$.

$$f_2(x_1, x_2, \ldots, x_n) = x_1^3 x_2^5 x_3^0 + x_1^3 x_2^0 x_3^5 + \ldots + x_1^0 x_2^3 x_3^5 + \ldots$$

- $\mathbb{Q}[x_1, x_2, \ldots]$: polynomials in n variables

$$f_3(x_1, x_2, \ldots, x_n) = x_1 x_4^2 + 2 x_3^7 x_8^2 + \ldots$$

Bases

- **Bases for Sym:** (indexed by partitions)

  monomial, power sum, elementary, homogeneous, Schur, ...
- **Bases for QSym:** (indexed by compositions)

  monomial, fundamental (Gessel), quasisymmetric Schur (HLMvW), dual immaculate (BBSSZ), ...
- **Bases for $\mathbb{Q}[x_1, x_2, \ldots]$:** (indexed by weak compositions)

  monomials, key polynomials (Lascoux-Schützenberger), Demazure atoms (M), slide polynomials (Assaf-Searles), ...

⋆ Many of these have combinatorial descriptions in terms of tableaux-like diagrams.

5	6	
1	1	3

,

5	6	
3		
1	2	4

,

6	4	3
3		
2	1	1

,

4	3	
5	4	2

, ...

What happens when you flip the variables?

Symmetric functions (such as Schur functions) stay the same.

$$s_\lambda(x_1, x_2, \ldots, x_n) = s_\lambda(x_n, x_{n-1}, \ldots, x_2, x_1)$$

Decreasing

4			
7	5	5	
8	7	6	3

Increasing

5			
2	4	4	
1	2	3	6

"Plays" well together		**"Plays" well together**
Nonsymmetric Macdonald		
Quasisymmetric Schur	$\leftrightarrow$	Young quasisymmetric Schur
Demazure atoms		
		dual immaculate
key polynomials		

Macdonald polynomials $P_\lambda(X; q, t)$ (I.G. Macdonald, 1988)

1. (Triangular). $P_\lambda = m_\lambda +$ lower terms in dominance order.
2. (Orthonormal). $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ if $\lambda \neq \mu$, where

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}.$$

- Schur positivity conjecture
- Connection to symmetric function bases (specializations)
- All Lie types - alcove walks (Ram-Yip)
- Geometry of Hilbert Schemes (Mark Haiman)

Theorem (Haglund, Haiman, Loehr (2008))

$$\tilde{H}_\mu(X; q, t) = \sum_{\sigma:\ \mu \to \mathbb{Z}^+} x^\sigma q^{\mathrm{inv}(\sigma,\mu)} t^{\mathrm{maj}(\sigma,\mu)}.$$

5	8		
2	6	1	
3	3	8	1

↻

$$weight = x_1^2 x_2 x_3^2 x_5 x_6 x_8^2 q^5 t^4$$

Nonsymmetric Macdonald polynomials (Cherednik, Macdonald, Opdam-Heckman)

- triangularity and orthogonality
- Eigenfunctions
- Symmetrize to Macdonald polynomials

Specializations of nonsymmetric Macdonald polynomials

$$P_\lambda(X;q,t) = \prod_u (1 - q^{l(u)+1} t^{a(u)}) \sum_{inc(a)=\lambda} \frac{E_\mu(X;q^{-1},t^{-1})}{\prod_u (1 - q^{l(u)+1} t^{a(u)})}$$

specializes to:

$$s_\lambda(X) = \sum_{inc(a)=\lambda} E_a(X;\infty,\infty),$$

where the $E_a(X;\infty,\infty) = \mathcal{A}_a(X)$ are the "Demazure atoms".

Definition

The quasisymmetric Schur function QS_α is the sum of Demazure atoms over all weak compositions collapsing to α:

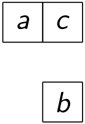
$$QS_\alpha = \sum_{a^+=\alpha} \mathcal{A}_a.$$

$$QS_{12} = \mathcal{A}_{120} + \mathcal{A}_{102} + \mathcal{A}_{012}$$

Combinatorial description

A semi-standard reverse composition tableau (SSRCT) of shape α is a filling of α (French notation) such that:

1. Row entries weakly decrease from left to right.
2. Entries in the leftmost column strictly increase bottom to top.
3. (Triple Rule) For a triple of entries as shown below, $a \leq b \Rightarrow c < b$:



Definition

The quasisymmetric Schur function QS_α is defined by

$$QS_\alpha(X) = \sum_{T \in SSRCT(\alpha)} X^T$$

Quasisymmetric Schur functions refine Schur functions

$$s_\lambda = \sum_{inc(\alpha)=\lambda} QS_\alpha$$

$$s_{322} = QS_{322} + QS_{232} + QS_{232}$$

1	1	
4	2	
5	4	3

$RSSYT(3,2,2)$

1	1	
4	4	3
5	2	

$SSRCT(2,3,2)$

($\star$) If a symmetric function is quasisymmetric Schur-positive, then it is Schur positive!

Young quasisymmetric Schur functions (flip and reverse)

A semi-standard composition tableau (SSCT) of shape α is a filling of α (French notation) such that:

1. Row entries weakly increase from left to right.
2. Entries in the leftmost column strictly increase bottom to top.
3. (Triple Rule)

Definition (Luoto, Mykytiuk, van Willigenburg)

The Young quasisymmetric Schur function $\mathcal{YS}_\alpha$ is defined by

$$\mathcal{YS}_\alpha(X) = \sum_{T \in SSCT(\alpha)} X^T.$$

$$\mathcal{YS}_\alpha(x_1, x_2, \ldots, x_n) = QS_{rev(\alpha)}(x_n, x_{n-1}, \ldots, x_2, x_1)$$

Creation operators $\leftrightarrow$ NSym $\leftrightarrow$ dual immaculates

A semi-standard immaculate tableau (SSIT) of shape α is a filling of α (French notation) such that:

1. Row entries weakly increase from left to right.
2. Entries in the leftmost column strictly increase bottom to top.

Definition (Berg, Bergeron, Saliola, Serrano, Zabrocki)

The dual immaculate quasisymmetric function is defined by:

$$\mathcal{D}_\alpha = \sum_{F \in SSIT(\alpha)} x^F.$$

3	3
2	
1	1

3	3
2	
1	2

3	3
2	
1	3

$$\mathcal{D}_{212}(x_1, x_2, x_3) = x_1^2 x_2 x_3^2 + x_1 x_2^2 x_3^2 + x_1 x_2 x_3^3$$

Theorem (Allen-Hallam-M)

The dual immaculate quasisymmetric functions expand positively into the Young quasisymmetric Schur basis.

A polynomial analogue of dual immaculates...

- Looking for a polynomial analogue of the dual immaculate quasisymmetric functions
- Nothing we tried "played nicely" with the other polynomial bases
- Had to look to the flipped and reversed picture! (Quasisymmetric Schurs play well with key polynomials, Demazure atoms, etc...Young quasisymmetric Schurs do not.)

Definition (M-Searles)

The reverse dual immaculate quasisymmetric function is defined by:

$$\hat{\mathcal{D}}_\alpha = \sum_{F \in RSSIT(\alpha)} x^F,$$

where $RSSIT(\alpha)$ is the fillings of α whose entries:

1. weakly decrease from left to right
2. leftmost column entries strictly increase from bottom to top.

$$\hat{\mathcal{D}}_\alpha(x_1, x_2 \ldots, x_n) = \mathcal{D}_{rev(\alpha)}(x_n, \ldots, x_2, x_1)$$

Proposition (M-Searles)

The reverse dual immaculate quasisymmetric functions expand positively into the quasisymmetric Schur basis.

Key polynomials (Demazure characters)

Let a be a weak (allowing zeros) composition. Then the key polynomial κ_a is given by:

$$\kappa_a = \sum_{b \leq a} \mathcal{A}_b,$$

under the Bruhat order.

Definition

A key is a semi-standard Young tableau whose entries in the $(j+1)^{th}$ column are a subset of the entries in the j^{th} column.

$key(1,0,3,2) =$

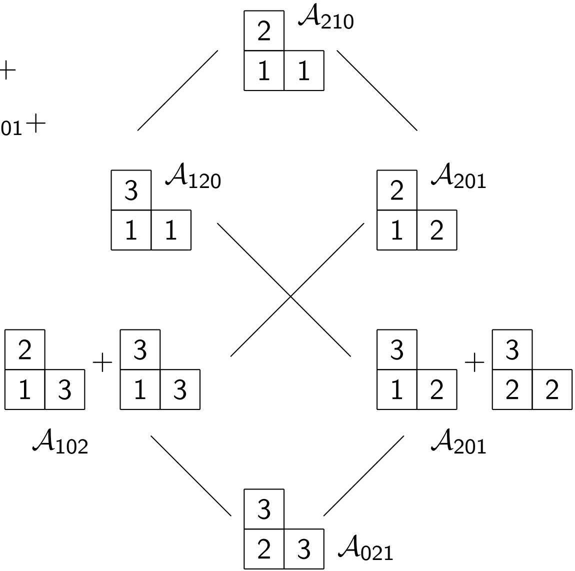
4		
3	4	
1	3	3

$$K_+\left(\begin{array}{|c|c|c|c|} \hline 5 & 7 & & \\ \hline 3 & 6 & 8 & \\ \hline 2 & 4 & 5 & \\ \hline 1 & 1 & 3 & 4 \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|} \hline 8 & 8 & & \\ \hline 6 & 6 & 8 & \\ \hline 5 & 5 & 5 & \\ \hline 4 & 4 & 4 & 4 \\ \hline \end{array} \qquad K_-\left(\begin{array}{|c|c|c|c|} \hline 5 & 7 & & \\ \hline 3 & 6 & 8 & \\ \hline 2 & 4 & 5 & \\ \hline 1 & 1 & 3 & 4 \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|} \hline 5 & 5 & & \\ \hline 3 & 3 & 5 & \\ \hline 2 & 2 & 3 & \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array}$$

Theorem (Lascoux-Schützenberger)

$$\kappa_a = \sum_{\substack{T \in SSYT(a) \\ K_+(T) \le key(a)}} x^T$$

$$\kappa_{102} = \mathcal{A}_{210} + \mathcal{A}_{120} + \mathcal{A}_{201} + \mathcal{A}_{102}$$



Young key polynomials

Let a be a weak (allowing zeros) composition. Then the Young key polynomial $\hat{\kappa}_a$ is given by:

$$\hat{\kappa}_a(x_1, x_2, \ldots, x_n) = \kappa_{rev(a)}(x_1, x_2, \ldots, x_n).$$

Theorem (M-Searles)

$$\hat{\kappa}_a = \sum_{\substack{T \in SSYT(a) \\ K_-(T) \geq key(a)}} x^T$$

⋆ The roles of left $K_-(T)$ and right $K_+(T)$ keys are switched when the variables are reversed.

Young key module

- $\hat{B} = \{\text{lower triangular matrices}\}$
- $x = \begin{bmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & 0 & x_{n-1} & 0 \\ 0 & \dots & 0 & 0 & x_n \end{bmatrix}$
- x acts on $\hat{B}$ by left multiplication
- Young key module $\hat{\mathcal{K}}_a$: $\hat{\kappa}_a =$ trace of the action of x on $\hat{\mathcal{K}}_a$.
- Parallels the generalized flagged Schur module and key module construction in "Key polynomials and a flagged Littlewood-Richardson Rule" by Reiner and Shimozono.

Young key module

- <u>diagram D</u> - finite subset of $\mathbb{P} \times \mathbb{P}$
- <u>row group $R(D)$</u> - permutations fixing row entries
- <u>column group $C(D)$</u> - permutations fixing column entries

$$e_T = \sum_{\substack{\alpha \in R(D) \\ \beta \in C(D)}} sgn(\beta)\, T_{\alpha\beta},$$

where $T_{\alpha\beta}$ means apply α to T and then apply β to the result.

$$T = \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{2}{c}{} \\ \hline 4 & \multicolumn{1}{c|}{2} \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \quad e_T = \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{2}{c}{} \\ \hline 4 & \multicolumn{1}{c|}{2} \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} + \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{4} \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} - \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{2}{c}{} \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} + \ldots$$

The Young key module $\hat{\mathcal{K}}_a$ is the $\hat{B}$-module with basis

$$\{e_{T(u)}\}_{u \in \hat{\mathcal{W}}(a)},$$

where $\hat{\mathcal{W}}(a)$ is the set of all words $u = \cdots u^{(2)} u^{(1)}$ such that:

- $|u^{(i)}| = a_i$
- $u \leftrightarrow (P, std(key(a)))$ under RSK
- Each entry in $u^{(i)}$ is greater than or equal to i

Theorem (M-Searles)

The Young key polynomial $\hat{\kappa}_a$ is the trace of x acting on the Young key module $\hat{\mathcal{K}}_a$.

Further directions

- Fill out the picture of flipping and reversing for other families of polynomials
- Connection to nonsymmetric Macdonald polynomials, Schubert polynomials
- Understand the creation operator picture in the dual of the reverse dual immaculates

"Is it worth it? Let me work it.
Put my thing down, flip it and reverse it.
Ti esrever dna ti pilf, nwod gniht ym tup."

- Missy Elliot