Divided symmetrization, quasisymmetric functions and Schubert polynomials

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Postnikov's divided symmetrization

Given $f \in \mathbb{C}[x_1, \ldots, x_n]$, the divided symmetrization of f, denoted by $\langle f \rangle_n$, is:

$$\langle f \rangle_n = \sum_{\sigma \in S_n} \sigma \cdot \left(\frac{f}{\prod_{1 \leq i \leq n-1} (x_i - x_{i+1})} \right).$$

Example

$$\langle 1 \rangle_2 = \frac{1}{x_1 - x_2} + \frac{1}{x_2 - x_1} = 0.$$

$$\langle x_1 \rangle_2 = \frac{x_1}{x_1 - x_2} + \frac{x_2}{x_2 - x_1} = 1.$$

$$\langle x_1^2 x_2 \rangle_3 = \sum_{\sigma \in S_3} \sigma \cdot \left(\frac{x_1^2 x_2}{(x_1 - x_2)(x_2 - x_3)} \right) = x_1 + x_2 + x_3.$$

Some basic observations

$$\langle f \rangle_n = \sum_{\sigma \in S_n} \sigma \cdot \left(\frac{f}{\prod_{1 \le i \le n-1} (x_i - x_{i+1})} \right).$$

1. $\langle f \rangle_n$ is a symmetric polynomial in $x_1, \ldots, x_n$.
2. If $f = gh$ where g is symmetric, then $\langle f \rangle_n = g \langle h \rangle_n$.
3. $\deg f < n - 1 \implies \langle f \rangle_n = 0$.
4. $\deg f = n - 1$ implies $\langle f \rangle_n$ is a scalar!

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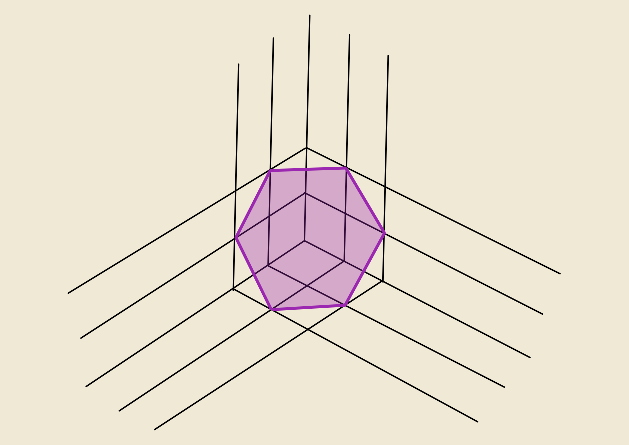
If $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and $\deg f = n - 1$, does $\langle f \rangle_n$ have a deeper meaning?

Usual permutahedra

Definition (Usual permutahedra)

For $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{R}^n$, the permutahedron $\mathcal{P}_\lambda$ is the convex hull of the S_n-orbit of λ.



Volumes of permutahedra

$\mathcal{P}_\lambda$ lies on the hyperplane defined by the sum of the λ_i.

Thus, the dimension of $\mathcal{P}_\lambda$ is at most $n-1$.

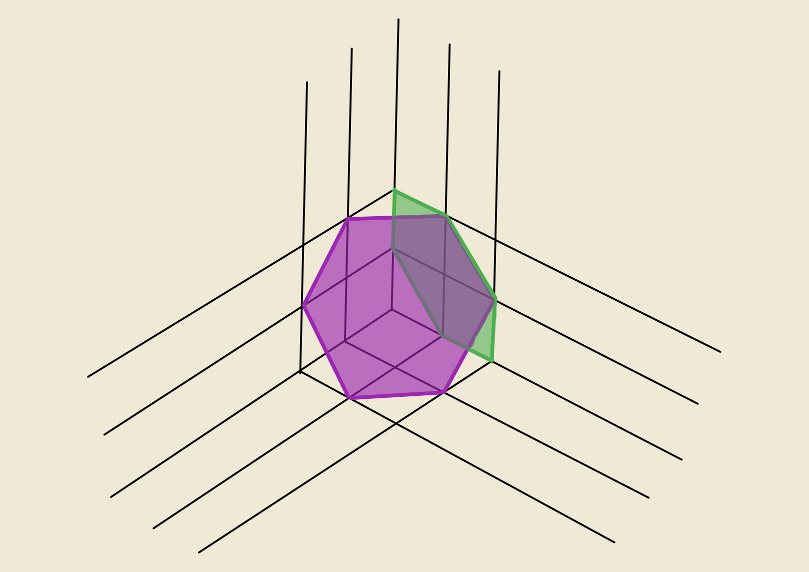
Definition

For a polytope P lying in a hyperplane in $\mathbb{R}^n$, define its volume $\mathrm{vol}(P)$ as the usual $(n-1)$-dimensional volume of the projection of P onto $x_n = 0$.

Given $\lambda = (\lambda_1, \dots, \lambda_n)$, set

$$V(\lambda) := \mathrm{vol}(P_\lambda).$$

Permutahedron $\mathcal{P}_{210}$



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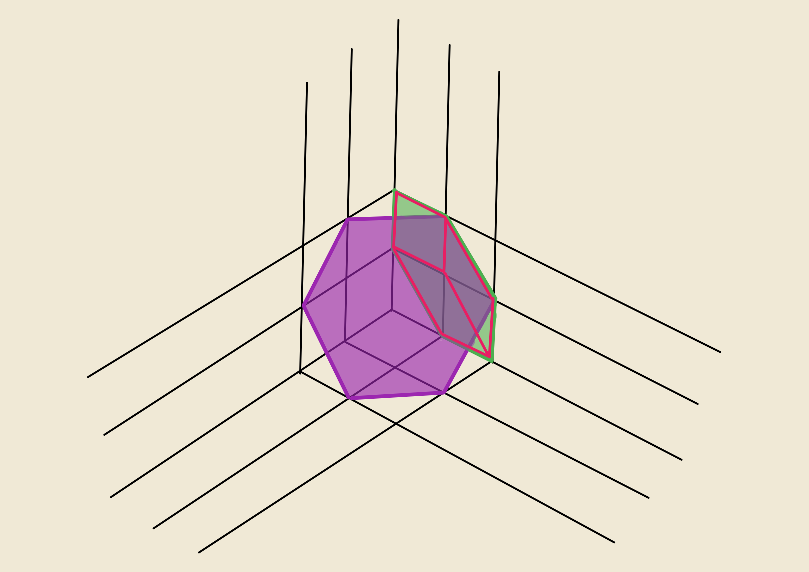

Divided symmetrization and volumes

Theorem (Postnikov'05)

$$(n-1)!V(\lambda) = \langle (\sum_{i=1}^{n} \lambda_i x_i)^{n-1} \rangle_n$$

Example

In the case $n = 3$ and $\lambda_3 = 0$, we get

$$\begin{aligned} 2V(\lambda_1, \lambda_2, 0) &= \langle \lambda_1^2 x_1^2 + 2\lambda_1\lambda_2 x_1 x_2 + \lambda_2^2 x_2^2 \rangle_3 \\ &= \lambda_1^2 \langle x_1^2 \rangle_3 + 2\lambda_1\lambda_2 \langle x_1 x_2 \rangle_3 + \lambda_2^2 \langle x_2^2 \rangle_3 \\ &= \lambda_1^2 + 2\lambda_1\lambda_2 - 2\lambda_2^2. \end{aligned}$$

The class of the Peterson variety

Alex Woo: Compute the class of the Peterson variety in terms of Schubert classes.

We translate this question into an equivalent form that involves studying polynomials modulo a specific ideal.

The coinvariant algebra

Let $e_k(x_1, \ldots, x_n)$ denote the kth elementary symmetric polynomial.

$$e_k(x_1, \ldots, x_n) := \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}$$

$$I_n = \text{ideal in } \mathbb{Z}[x_1, \ldots, x_n] \text{ generated by the } e_k.$$

The (type A) coinvariant algebra is the quotient $\mathbb{Z}[x_1, \ldots, x_n]/I_n$.

By work of Borel, this quotient is isomorphic to the integer cohomology ring of the complete flag variety.

The class of the Peterson variety

Theorem (Anderson-Tymoczko'07)

The class of the Peterson variety is represented by

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Reduce the product above mod I_n and expand in terms of representatives of Schubert classes aka Schubert polynomials.

(Reduced) Pipe dreams aka rc-graphs

To build a pipe dream for $w \in S_n$, draw the staircase $(n, \dots, 1)$, enumerate its rows from 1 through n and columns from $w(1)$ through $w(n)$. Fill in the internal squares with crossing tiles  or elbow tiles  so that

- i connects to $w(i)$,
- two strands intersect at most once.

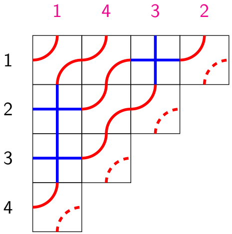


Figure: A reduced pipe dream for $w = 1432$.

Reduced pipe dreams and associated monomials

Given a reduced pipe dream D, set

$$\mathbf{x}_D := \prod_{\text{crossings } c \in D} x_{\mathrm{row}(c)}$$

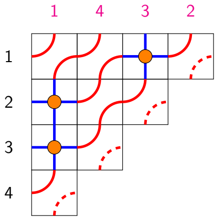


Figure: A pipe dream D with $\mathbf{x}_D = x_1 x_2 x_3$.

Bottom pipe dreams and codes

The code of $w \in S_n$ is the weak composition $(c_1, \ldots, c_n)$ where

$$c_i = \{j > i | w_i > w_j\}.$$

For instance, if $w = 1432$, then $\mathrm{code}(w) = (0, 2, 1, 0)$.

The bottom pipe dream for w is attached naturally to $\mathrm{code}(w)$.

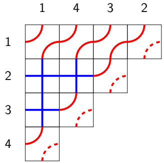


Figure: The bottom pipe dream for $w = 1432$.

Schubert polynomial (BJS or BB definition)

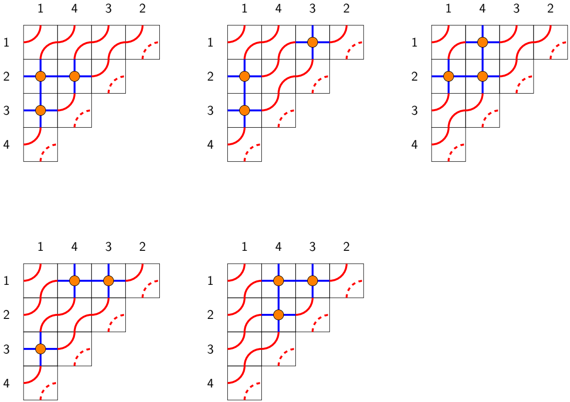
$$\mathrm{PD}(w) := \{\text{reduced pipe dreams for } w\}.$$

Definition

For $w \in S_n$, the Schubert polynomial $\mathfrak{S}_w(x_1, \ldots, x_n)$ is defined as

$$\mathfrak{S}_w(x_1, \ldots, x_n) := \sum_{D \in \mathrm{PD}(w)} \mathbf{x}_D.$$

Reduced pipe dreams for $w = 1432$



$$\mathfrak{S}_{1432} = x_2^2 x_3 + x_1 x_2 x_3 + x_1 x_2^2 + x_1^2 x_3 + x_1^2 x_2$$

Some empirical observations

Henceforth, for $w \in S_n$ with $\ell(w) = n - 1$, set

$$a_w := \langle \mathfrak{S}_w \rangle_n.$$

Theorem (Nadeau-T.'19)

The a_w give coefficients for Schuberts in the class of the Peterson.

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The a_w give coefficients for Schuberts in the class of the Peterson.

Geometry says $a_w \geq 0$. In fact

- $a_w > 0$ conjecturally. Also conjectured by Harada et al.
- $a_w = a_{w^{-1}}$ conjecturally.
- $a_w = a_{w_0 w w_0}$. Straightforward to establish.

An example: $n = 3$

We need to reduce

$$\prod_{j-i>1} (x_i - x_j) = x_1 - x_3$$

modulo ideal generated by $x_1 + x_2 + x_3$, $x_1x_2 + x_1x_3 + x_2x_3$, and $x_1x_2x_3$.

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$$x_1 - x_3 \equiv x_1 + (x_1 + x_2) = 1\mathfrak{S}_{213} + 1\mathfrak{S}_{132}.$$

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$$x_1 - x_3 \equiv x_1 + (x_1 + x_2) = 1\mathfrak{S}_{213} + 1\mathfrak{S}_{132}.$$

$$\langle \mathfrak{S}_{231} \rangle_3 = \langle x_1x_2 \rangle_3 = 1$$
$$\langle \mathfrak{S}_{312} \rangle_3 = \langle x_1^2 \rangle_3 = 1.$$

The real question

Find a manifestly positive combinatorial rule for a_w. Bonus points if it reflects invariance under inverses and/or conjugation by w_0.

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Find a manifestly positive combinatorial rule for a_w. Bonus points if it reflects invariance under inverses and/or conjugation by w_0.

Naive idea: Use divided symmetrization of monomials and BJS expansion of Schuberts.

Results in a signed formula, and yet instructive.

Catalan permutations

Call $w \in S_n$ with $\ell(w) = n - 1$ a Catalan permutation if the bottom pipe dream of w has at least i crosses in the first i diagonals for $1 \leq i \leq n - 1$.

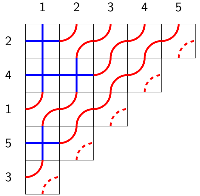

Figure: The bottom pipe dream for $w = 24153$.

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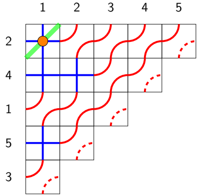

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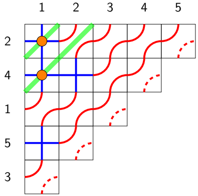

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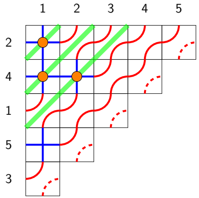

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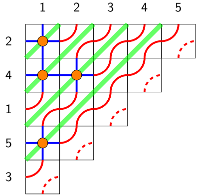

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Theorem (Nadeau-T.'19)

If $w \in S_n$ is Catalan, then

$$\langle \mathfrak{S}_w \rangle_n = |\mathrm{PD}(w)| = \mathfrak{S}_w(1^n).$$

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If w is Catalan then so is w^{-1}, and so $\langle \mathfrak{S}_w \rangle_n = \langle \mathfrak{S}_{w^{-1}} \rangle_n$ says that $\mathrm{PD}(w) = \mathrm{PD}(w^{-1})$.

Schuberts of grassmannian permutations (Schurs)

Given partition $\lambda = (\lambda_1 \leq \cdots \leq \lambda_k)$, a semistandard Young tableau T of shape λ is a filling of the Young diagram of λ so that rows increase weakly and columns increase strictly.

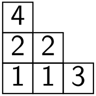

Figure: A semistandard Young tableau of shape $(1, 2, 3)$.

$$s_\lambda(x_1, \ldots, x_m) := \sum_{T \in \mathrm{SSYT}_{\leq m}(\lambda)} \mathbf{x}^{\mathrm{cont}(T)}.$$

DS of Schur polynomials

A standard Young tableau of shape λ is one where all numbers from 1 through $|\lambda|$ are used precisely once.

A descent in a standard Young tableau is an entry i such that $i+1$ occupies a row strictly above.

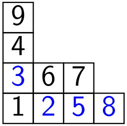

Figure: An SYT with descent set $\{2, 3, 5, 8\}$.

DS of Schur polynomials

Theorem (Nadeau-T.'19)

$$\langle s_\lambda(x_1, \ldots, x_k) \rangle_n = \#\{\text{SYTs } \textit{of shape } \lambda \textit{ with } k-1 \textit{ descents}\}$$

Example

3	
1	2

2	
1	3

$$s_{12}(x_1, x_2) = 2,$$
$$s_{12}(x_1, x_2, x_3) = 0.$$

Descents showing up is a hint that quasisymmetric functions are in the background.

$$\mathbf{x} := \{x_1, x_2, \ldots\}.$$

The ring of quasisymmetric functions QSym is the $\mathbb{Z}$-linear span of M_α defined as

$$M_{(\alpha_1,\ldots,\alpha_k)} := \sum_{i_1<\cdots<i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}.$$

The fundamental quasisymmetric function F_α is defined as

$$F_\alpha := \sum_{\beta \preccurlyeq \alpha} M_\beta.$$

$$M_{312} = x_1^3 x_2^1 x_3^2 + x_1^3 x_3^1 x_4^2 + x_2^3 x_3^1 x_4^2 + \cdots,$$
$$F_{311} = M_{311} + M_{1211} + M_{2111} + M_{11111}.$$

Quasisymmetric functions and their truncations

If we set $x_i = 0$ for all $i > m$ in a quasisymmetric function, we obtain a quasisymmetric polynomial in $\mathbf{x}_m = \{x_1, \ldots, x_m\}$.

Theorem (Nadeau-T.'19)

Given $\alpha \vDash n-1$ and $m \leq n$, we have

$$\langle F_\alpha(\mathbf{x}_m) \rangle_n = \delta_{m,\ell(\alpha)}.$$

DS of Schur polynomials

The divided symmetrization for Schur polynomials follows from the well-known expansion of Schur functions into fundamental quasisymmetrics.

$$s_\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} F_{\mathrm{comp}(\mathrm{des}(T))}$$

More quasisymmetric functions

$$J_n = \langle F_\alpha(x_1, \ldots, x_n) \text{ where } |\alpha| \geq 1 \rangle.$$

Note that the coinvariant ideal $I_n \subset J_n$.

Theorem (Nadeau-T.'19)

For $f \in J_n$ homogeneous of degree $n-1$,

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For $f \in J_n$ homogeneous of degree $n-1$,

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Upshot: Computing $\langle f \rangle_n$ could be facilitated by expanding f mod J_n in a DS-friendly basis for the quotient $\mathbb{Q}[x_1, \ldots, x_n]/J_n$.

The Aval-Bergeron-Bergeron basis

Theorem (Aval-Bergeron-Bergeron'04)

We have the following monomial basis $\mathcal{B}_n$ *for* $\mathbb{Q}[x_1, \ldots, x_n]/J_n$*:*

$$\mathcal{B}_n = \{\mathbf{x}_P \text{ where } P \text{ is } (n-1,k)\text{-subdiagonal for some } k\}.$$

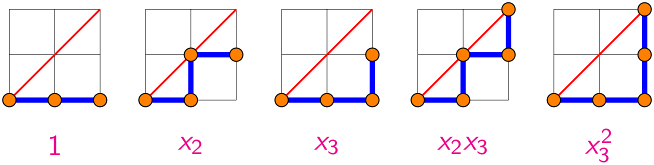

Figure: The ABB monomial basis for $\mathbb{Q}[x_1, x_2, x_3]/J_3$.

$$|\mathcal{B}_n| = \mathrm{Cat}_n.$$

General theorem

Call degree $n-1$ monomials in $\mathcal{B}_n$ as anti-Catalan monomials.
Their DS is $(-1)^{n-1}$.

Theorem (Nadeau-T.'19)

Given homogeneous f of degree $n-1$, express it as $g+h$ where $h \in J_n$ and g is a linear combination of anti-Catalan monomials. Then

$$\langle f \rangle_n = (-1)^{n-1} g(1^n).$$

A 'strange' relation

Theorem (Nadeau-T.'19)

$$\sum_{\substack{w\in S_n\\ \ell(w)=n-1}} \mathfrak{S}_{ww_0}(1^n)\langle\mathfrak{S}_w\rangle_n = n^{n-2}.$$

Proof involves:

- Cauchy identity of double Schuberts;
- LHS is the constant term in $\langle\prod_{1\le i\le n}(1+x_i)^{n-i}\rangle_n$;
- Eventually need to count lattice points in the permutahedron $P_{(n-2,\dots,1,0,0)}$, which equals the volume of the standard permutahedron $P_{(n-1,\dots,1,0)}$.

Thank you for listening!

Subdiagonal paths

Given nonnegative integers $k \leq n$, a lattice path from $(0, 0)$ to (n, k) is called (n, k)-subdiagonal if it stays below the line $y = x$.

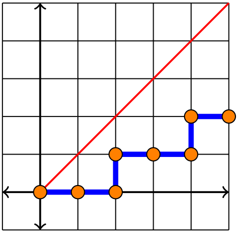

Figure: A $(5, 2)$-subdiagonal path P.

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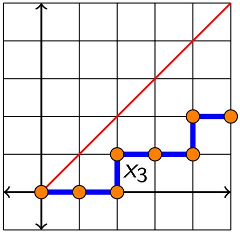

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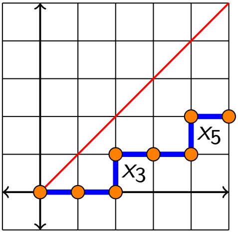

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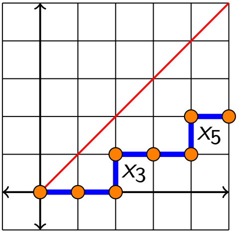

Figure: A $(5,2)$-subdiagonal path P.

The monomial $\mathbf{x}_P$ attached to P is $x_3 x_5$.

Schuberts modulo quasisymmetrics

Example

$$\begin{aligned}\mathfrak{S}_{321} &= x_1^2 x_2 = F_{21}(x_1, x_2)\\ &= F_{21}(\mathbf{x}_4) - F_2(\mathbf{x}_4)F_1(x_3, x_4) + F_1(\mathbf{x}_4)F_2(x_3, x_4) - F_{21}(x_3, x_4).\\ &\equiv -x_3^2 x_4 \mod J_4\end{aligned}$$

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Example

$$\begin{aligned}\mathfrak{S}_{1\times 321} &= x_2^2 x_3 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2 x_3 + x_1 x_2^2 \\ &= F_{21}(x_1, x_2, x_3) + F_{12}(x_1, x_2) \\ &\equiv -x_3 x_4^2 \mod J_4\end{aligned}$$

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Thus $\langle \mathfrak{S}_{1\times 321} \rangle_4 = \langle \mathfrak{S}_{321} \rangle_4 = 1$.

Schuberts modulo quasisymmetrics

Conjecture (Nadeau-T.'19)

For $w \in S_n$ satisfying $\ell(w) \leq n-1$, the polynomial $(-1)^{\ell(w)}\mathfrak{S}_w$ reduced modulo J_n expands positively in the ABB basis.

Schuberts modulo quasisymmetrics: stable limits

Fix permutation w, let $N := \ell(w) + 1$. Consider the sequence of polynomials obtained by reducing $\mathfrak{S}_w$ modulo J_m for $m \geq N$.

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Example

Set $w = 2413$. For $4 \leq m \leq 7$, we have the following representatives for $\mathfrak{S}_w$ mod J_m

$$\begin{aligned}
&-x_3^2x_4^1 - x_3^1x_4^2,\\
&-x_3^2x_4^1 - x_3^1x_4^2 - x_3^2x_5 - 2x_3x_4x_5 - x_4^2x_5 - x_3x_5^2 - x_4x_5^2,\\
&-F_{12}(x_3, x_4, x_5, x_6) - F_{21}(x_3, x_4, x_5, x_6)\\
&-F_{12}(x_3, x_4, x_5, x_6, x_7) - F_{21}(x_3, x_4, x_5, x_6, x_7).
\end{aligned}$$

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$$-x_3^2 x_4^1 - x_3^1 x_4^2 - x_3^2 x_5 - 2x_3 x_4 x_5 - x_4^2 x_5 - x_3 x_5^2 - x_4 x_5^2,$$
$$- F_{12}(x_3, x_4, x_5, x_6) - F_{21}(x_3, x_4, x_5, x_6)$$
$$- F_{12}(x_3, x_4, x_5, x_6, x_7) - F_{21}(x_3, x_4, x_5, x_6, x_7).$$

From the viewpoint of DS, only the first expansion is pertinent. That said, maybe the limit object has a nicer description and we can truncate to compute the relevant DS.

Schuberts modulo QSym: infinitely many variables

Consider the quotient $\mathbb{Q}[\mathbf{x}]/J_\infty$. By work of Aval-Bergeron, this has a basis indexed by subdiagonal paths.

Conjecture (Nadeau-T.'19)

For a permutation w, the polynomial $(-1)^{\ell(w)}\mathfrak{S}_w$ reduced modulo J_∞ expands positively in terms of backstable limits of summands in the BJS formula.

DS and Peterson

How does divided symmetrization of Schubert polynomials show up in the Anderson-Tymoczko class of the Peterson variety?

- Suppose $\prod_{j-i>1}(x_i - x_j) = \sum_{w \in S_n} a_w \mathfrak{S}_w + G$ where $G \in I_n$.
- Infer that

$$\frac{\mathfrak{S}_{uw_0}\Delta(x_1,\ldots,x_n)}{\prod_{1\le i\le n-1}(x_i - x_{i+1})} = \sum_{w\in S_n} a_w \mathfrak{S}_{uw_0}\mathfrak{S}_w + G\mathfrak{S}_{uw_0}.$$

- Antisymmetrize both sides to conclude that

$$\langle \mathfrak{S}_{uw_0} \rangle_n = a_u.$$