Atomic decomposition of characters and crystals

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Joint work with Cédric Lecouvey, University of Tours, France;
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Lie algebras and their representations

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Lusztig defined the *t-analogue* $K_{\lambda,\mu}(t)$, i.e., $K_{\lambda,\mu}(1) = K_{\lambda,\mu}$, via

$$\frac{\sum_{w\in W} \operatorname{sgn}(w)\, x^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in R^+}(1 - tx^{-\alpha})} = \sum_{\mu\in P(\lambda)} K_{\lambda,\mu}(t)\, x^{\mu}\,.$$

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We will study another, less understood property: the atomic decomposition (which was only defined in type A by A. Lascoux). Applications and geometric interpretation.

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The dominant part of the t-character:

$$\chi_\lambda^+(t) := \sum_{\mu \in P^+(\lambda)} \widetilde{K}_{\lambda,\mu}(t)\, x^\mu.$$

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Definition. The t-character $\chi_\lambda^+(t)$ (or, equivalently, the Kostka-Foulkes polynomials $K_{\lambda,\nu}(t)$) have a t-atomic decomposition if $A_{\lambda,\mu}(t) \in \mathbb{Z}_{\geq 0}[t]$.

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The irreducible character χ_λ has an atomic decomposition if $A_{\lambda,\mu}(1) \in \mathbb{Z}_{\ge 0}$.

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Define a combinatorial decomposition, based on crystal graphs.

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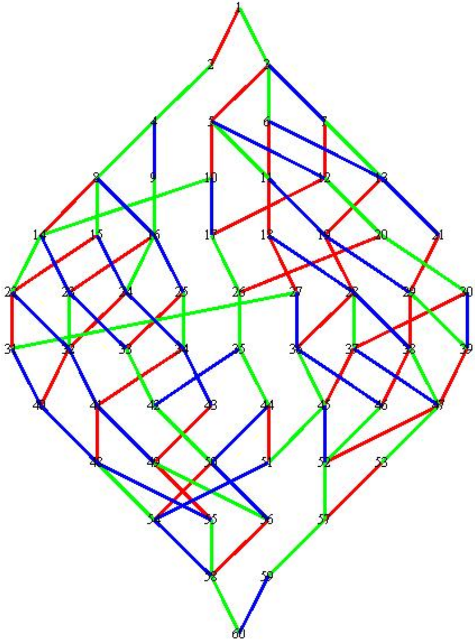
Encode as colored directed graph:

$$f_i(b) = b' \quad \Longleftrightarrow \quad b \xrightarrow{i} b'\,.$$

Example. $\mathfrak{g} = \mathfrak{sl}_4$, $\lambda = (3, 3, 1)$, blue: $\alpha_1 = \varepsilon_1 - \varepsilon_2$, green: $\alpha_2 = \varepsilon_2 - \varepsilon_3$, red: $\alpha_3 = \varepsilon_3 - \varepsilon_4$.



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$$B(\lambda)^+ = \bigsqcup_{h \in H(\lambda)} B(\lambda, h)\,,$$

where $H(\lambda) \subset B(\lambda)^+$, $h \in B(\lambda, h)$ is a distinguished vertex, and $B(\lambda, h)$ contains exactly one vertex of dominant weight ν, for $\nu \leq \mathrm{wt}(h)$.

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Definition. A t-atomic decomposition of $B(\lambda)$ is an atomic decomposition together with a statistic $\mathrm{c} : H(\lambda) \to \mathbb{Z}_{\geq 0}$ such that

$$A_{\lambda,\mu}(t) = \sum_{h \in H(\lambda),\, \mathrm{wt}(h) = \mu} t^{\mathrm{c}(h)}\,.$$

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- various properties of the dominance order – studied by Stembridge, we derive additional structural properties in classical types;
- a modified crystal graph structure on the vertices of $B(\lambda)^+$ and its properties.

Modified crystal structure

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For type B_n, also define similarly

$$\widehat{f}_{w(\alpha_n)} := w f_n w^{-1} \,.$$

Definition. Endow $B(\lambda)^+$ with a modified crystal graph structure, by restricting to those arrows

$$b \to \widehat{f}_\alpha(b) \quad \text{for which } \operatorname{wt}(b) \gtrdot \operatorname{wt}(\widehat{f}_\alpha(b)) \text{ is a cocover}\,.$$

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Theorem. (Lecouvey, L.) We have, under certain conditions:

$$\widehat{f}_\alpha \widehat{f}_\beta(b) = \begin{cases} \widehat{f}_\beta \widehat{f}_\alpha(b) = \widehat{f}_{\alpha+\beta}(b) \neq \mathbf{0} & \text{if } (\alpha,\beta) \in W(\alpha_1,\alpha_2) \\ \widehat{f}_\beta \widehat{f}_\alpha(b) \neq \mathbf{0} & \text{if } (\alpha,\beta) \in W(\alpha_1,\alpha_3)\,. \end{cases}$$

Main result

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- Verify the commutation of the modified crystal operators on these intervals.
- Use this property to iteratively lift the structure of the dominance order to that of the modified crystal poset.

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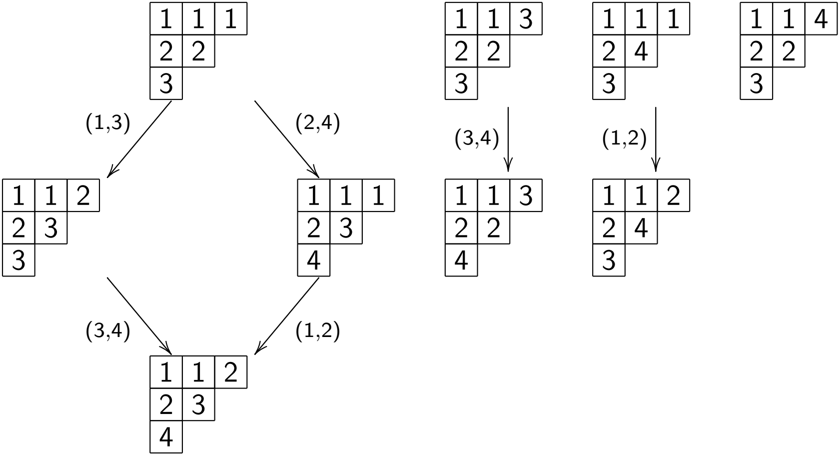

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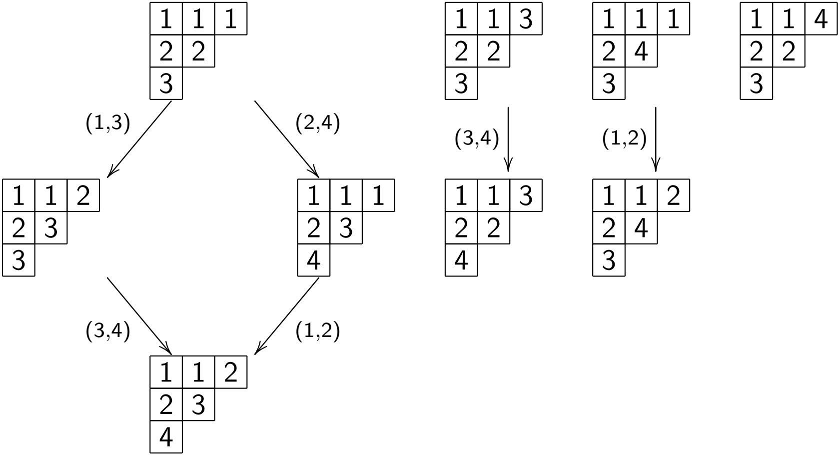

We get the following atomic decomposition of the character:

$$\chi_\lambda^+ = w_{(3,2,1)}^+ + w_{(2,2,2)}^+ + w_{(3,1,1,1)}^+ + w_{(2,2,1,1)}^+ .$$

Geometric interpretation: the geometric Satake correspondence

Given a reductive group G, this gives a geometric realization of $V(\lambda)$ for $G^\vee$, as the intersection cohomology $IH^*(\overline{Gr_\lambda})$ of a Schubert variety in the affine Grassmannian Gr_G.

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says that there is a refinement of the truncation filtration, whose successive quotients are isomorphic to $H^*(\overline{Gr^\mu})$ for $\mu \in P^+(\lambda)$.

Future work

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- Defining on $B(\lambda^+)$ a statistic computing $K_{\lambda,\mu}(t)$. This is constructed recursively on the components, starting from its value on the minimal vertex (determined in previous work with C. Lecouvey).