

CL-shellable posets with no EL-shellings

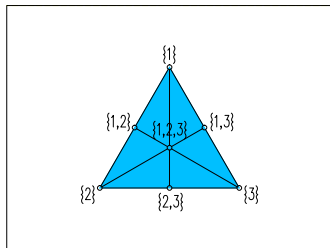
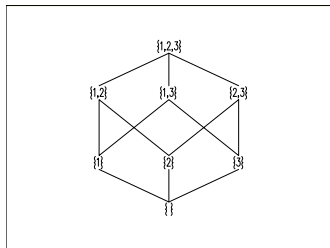
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Order Complex

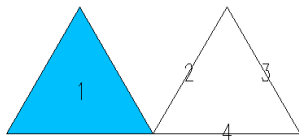
Let P be a finite poset. The order complex of P , denoted by ΔP , is the simplicial complex with $V(\Delta P) = P$ and $F = \{v_1, v_2, \dots, v_k\} \in \Delta P$ if $v_1 < v_2 < \dots < v_k$ in P .



Shellable Complex

Let Δ be a finite simplicial complex. Δ is said to be shellable if its facets can be ordered as F_1, F_2, \dots, F_k such that for each i , $F_i \cap (\cup_{j=1}^{i-1} F_j)$ is pure of codimension 1 in F_i .

We call a poset P shellable if ΔP is shellable.



Shellable Complex

Theorem(Björner-Wachs 96)

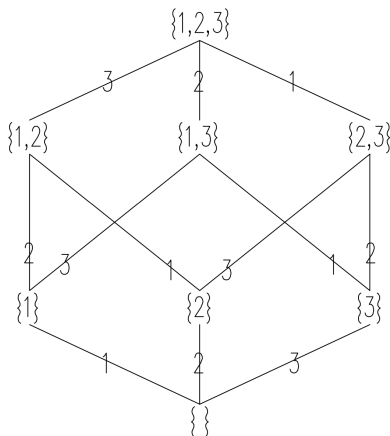
A shellable simplicial complex has the homotopy type of a wedge of spheres (in varying dimensions), where for each i , the number of i -spheres is the number of i -facets whose entire boundary is contained in the union of the earlier facets. Such facets are usually called homology facets.

Corollary

If Δ is shellable then for all i , $\tilde{H}_i(\Delta, \mathbb{Z}) \cong \tilde{H}^i(\Delta, \mathbb{Z}) \cong \mathbb{Z}^{r_i}$, where r_i is the number of homology i -facets in Δ .

EL-Labeling

Let P be a bounded poset. An edge-lexicographical labeling (EL-labeling, for short) of P is an edge labeling such that in each closed interval $[x, y]$ of P , there is a unique weakly increasing maximal chain, which lexicographically precedes all other maximal chains of $[x, y]$.



Theorem(Björner 80, Björner-Wachs 96)

Suppose P is a bounded poset with an EL-labeling. Then the lexicographic order of the maximal chains of P is a shelling of $\Delta(P)$. Moreover, the corresponding order of the maximal chains of \bar{P} is a shelling of $\Delta(\bar{P})$.

Corollary(Björner-Wachs 96)

Suppose P is a poset for which \hat{P} admits an EL-labeling. Then P has the homotopy type of a wedge of spheres, where the number of i -spheres is the number of decreasing maximal $(i+2)$ -chains of \hat{P} . The decreasing maximal $(i+2)$ -chains, with $\hat{0}$ and $\hat{1}$ removed, form a basis for cohomology $\tilde{H}^i(P, \mathbb{Z})$.

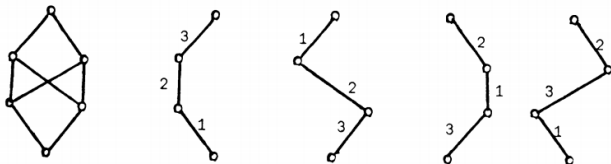
Chain-edge Labeling

For a bounded poset P , let $ME(P)$ be the set of pairs $(c, x \leq y)$ consisting of a maximal chain c and an edge $x \leq y$ along that chain. A chain-edge labeling of P is a map $\lambda : ME(P) \rightarrow \Lambda$, where Λ is some poset, satisfying: If two maximal chains coincide along their bottom d edges, then their labels also coincide along these edges.

It follows from the definition that each maximal chain r of $[\hat{0}, x]$ determines a unique restriction of λ to $ME([x, y])$. This enables one to talk about increasing and decreasing maximal chains and lexicographic order of maximal chains in the rooted interval $[x, y]_r$.

CL-Labeling

Let P be a bounded poset. A chain-lexicographic labeling (CL-labeling, for short) of P is a chain-edge labeling such that in each closed rooted interval $[x, y]_r$ of P , there is a unique weakly increasing maximal chain, which lexicographically precedes all other maximal chains of $[x, y]_r$. A poset that admits a CL-labeling is said to be CL-shellable.



Theorem(Björner-Wachs 96)

Suppose P is a bounded poset with an CL-labeling. Then the lexicographic order of the maximal chains of P is a shelling of $\Delta(P)$. Moreover, the corresponding order of the maximal chains of \bar{P} is a shelling of $\Delta(\bar{P})$.

Corollary

EL-shellability \Rightarrow CL-shellability \Rightarrow Shellability.

Question(Björner-Wachs 83)

Is every CL-shellable poset EL-shellable?

Recursive Atom Ordering

A bounded poset P is said to admit a recursive atom ordering if its length $l(P)$ is 1, or if $l(P) > 1$ and there is an ordering a_1, a_2, \dots, a_t of the atoms of P that satisfies:

- For all $j = 1, 2, \dots, t$ the interval $[a_j, \hat{1}]$ admits a recursive atom ordering in which the atoms of $[a_j, \hat{1}]$ that belong to $[a_i, \hat{1}]$ for some $i < j$ come first.
- For all $i < j$, if $a_i, a_j < y$ then there is a $k < j$ and an atom z of $[a_j, \hat{1}]$ such that $a_k < z \leq y$.

A recursive coatom ordering is a recursive atom ordering of the dual poset P^* .

Example

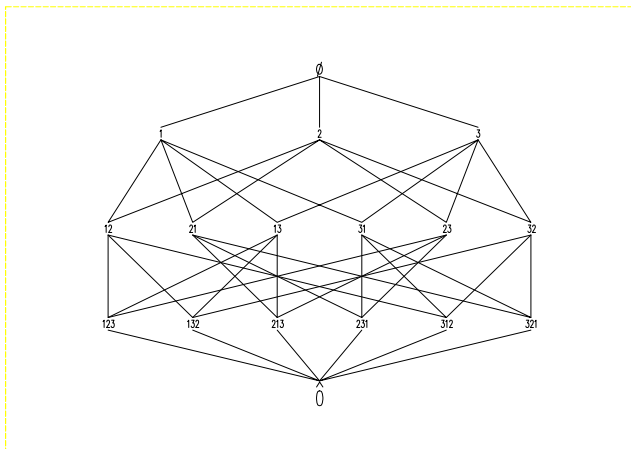


Figure: Injective words poset on $[3]$

Recursive Atom Ordering

Theorem(Björner-Wachs 83, 96)

A bounded poset P is CL-shellable if and only if P admits a recursive atom ordering.

Theorem(Björner-Wachs 83)

An ordering of the facets of a simplicial complex Δ is a shelling if and only if the ordering is a recursive coatom ordering of the face poset $P(\Delta)$.

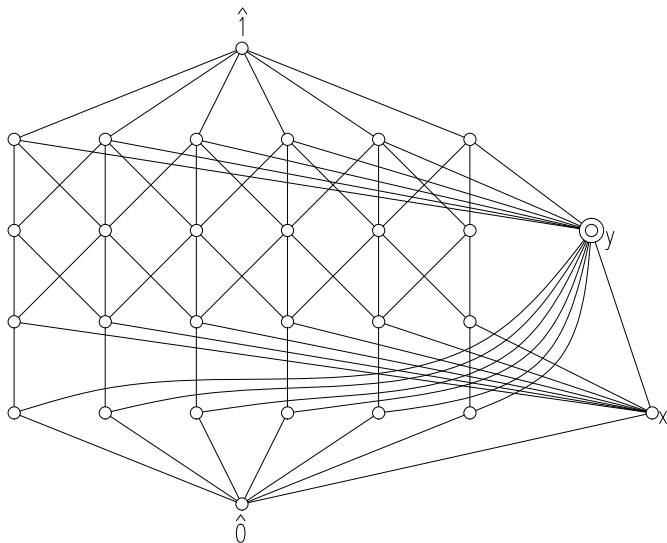
Theorem(Li 19)

A bounded poset P is EL-shellable if and only if P admits a recursive atom ordering such that, for each pair (x, r) , the atom orderings of $[x, \hat{1}]_r$ are the same for all r .

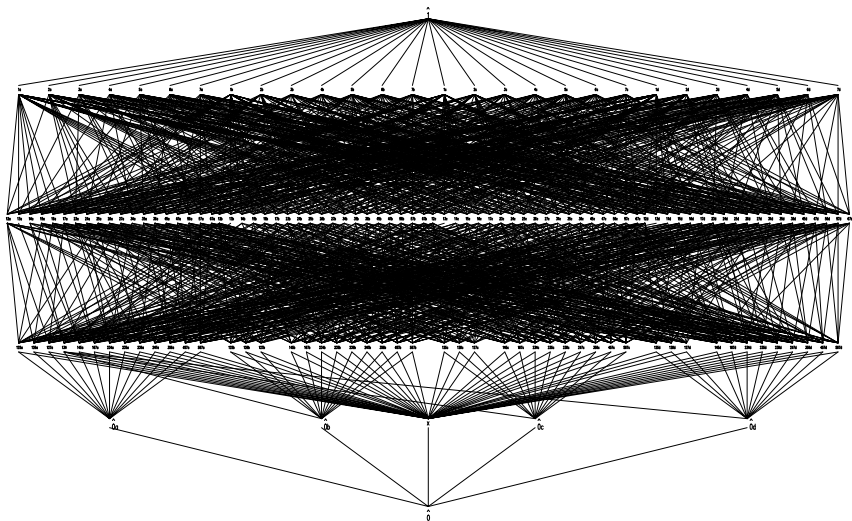
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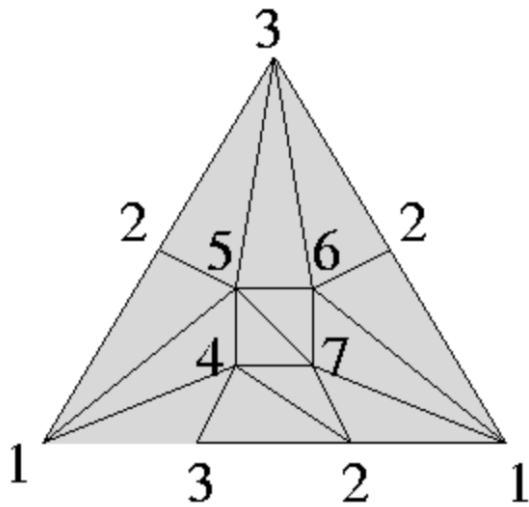
Ungraded Example



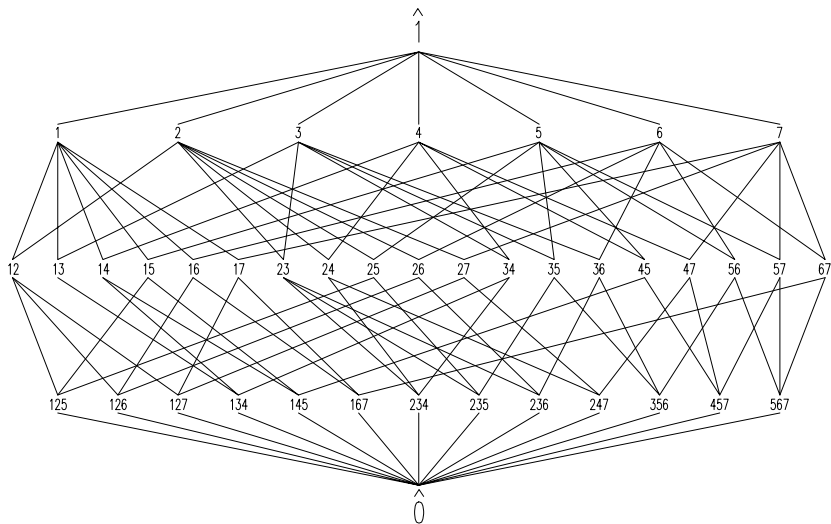
Graded Example



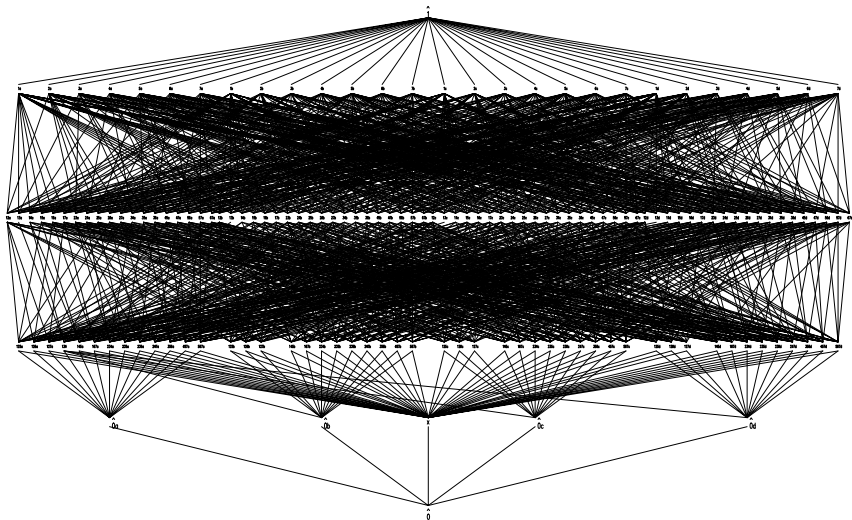
Hachimori's Complex



Dual of $P(H)$



Graded Example



Question

Is there a complete characterization of CL-shellable posets that are not EL-shellable?

Thank you!