Spherical and toroidal Schubert Varieties

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Spherical varieties

Spherical varieties

G connected reductive group, B Borel subgroup, X an irreducible G-variety.
X is a spherical G-variety if it is normal and has an open, dense B-orbit.

Spherical $\iff$ single open B-orbit $\implies$ single open G-orbit.
Open G-orbit is of the form G/H, for H an algebraic subgroup.

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Examples

(i) The (partial) flag varieties G/P, where P is a parabolic subgroup of G, for the action of G.

(ii) Any toric variety X for the action of a torus $(\mathbb{C}^*)^{dimX}$.

Classification

The classification of Spherical varieties reduces to two problems.

(1) Classify embeddings of the homogeneous spherical variety G/H into a spherical G-variety X where G/H is the open G-orbit.

(2) Classify homogeneous spherical varieties G/H.

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Motivating Question: What geometric properties can be inferred purely from the colored fan and Luna data?

- Smoothness can be decided. [Camus 2001]
- What about other invariants?

Flag varieties and their Schubert subvarieties

The usual suspects

G is a connected, reductive algebraic group over $\mathbb{C}$
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B is a Borel subgroup containing T
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Parabolic subgroups

Parabolic subgroups are subgroups of G containing a conjugate of B.
For each $I \subset S$ there is an associated standard parabolic subgroup

$$P_I = BW_IB.$$

Have the parabolic decomposition

$$P_I = L \ltimes U_I$$

where U_I is the unipotent radical, L is a reductive group called a Levi subgroup. L is standard if it contains T.

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Spherical varieties

G acts on G/P by left multiplication, and G/P is a spherical G-variety. What about the Schubert varieties?

In general G does not act on $X_P(w)$. The stabilizer $\mathrm{stab}_G(X_P(w))$ is a standard parabolic subgroup of G.

The Levi subgroups of any parabolic subgroup $P \subseteq \mathrm{stab}_G(X_P(w))$ are reductive groups which act on $X_P(w)$.

When are Schubert varieties spherical?

Let $X_P(w) \subseteq G/P$ and $L \subset P \subseteq \mathrm{stab}_G(X_P(w))$.

(1) When is $X_P(w)$ a spherical L-variety?

(2) If $X_P(w)$ is a spherical L-variety, what is its colored fan and Luna data?

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Bringing it all together

Motivating Question: What geometric properties can be inferred purely from the colored fan and Luna data?

A practical method of pursuing our motivating question would be to study the colored fan and Luna data of spherical Schubert varieties since the geometry of Schubert varieties is particularly well understood.

Spherical Schubert varieties in the Grassmannian

The curious case of the Grassmannian

The Grassmannian variety $\mathrm{G}_{d,N}$ is the space of d-dim subspaces of $\mathbb{C}^N$.

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Let $\mathfrak{L}$ be a very ample line bundle on $\mathrm{G}_{d,N}$ (from Plücker embedding).
The homogeneous coordinate ring of $X(w)$ is

$$\mathbb{C}[X(w)] = \bigoplus_{r\geq 0} H^0(X(w), \mathfrak{L}^{\otimes r}|_{X(w)})$$

There is an induced action of L on $\mathbb{C}[X(w)]$.

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Proposition (H-Lakshmibai) The Schubert variety $X(w)$ is a spherical L-variety if and only if $\mathbb{C}[X(w)]$ is a multiplicity free L-module.

The decomposition of the homogeneous coordinate ring: Part 1

Any standard Levi L is of the form

$$L = \mathrm{GL}_{N_1} \times \cdots \times \mathrm{GL}_{N_b}.$$

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Representation theory of L

Polynomial irreducible representations of GL_N are indexed by partitions $\lambda = (a_1, \ldots, a_k)$ of positive integers $a_1 \geq \cdots \geq a_k$ with $k \leq N$.

The irreducible GL_N-representation associated to λ is the Schur-Weyl module $S^\lambda(\mathbb{C}^N)$.

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The skew Schur-Weyl modules are GL_N-representations indexed by skew partitions λ/μ, and denoted $S^{\lambda/\mu}(\mathbb{C}^N)$. Then

$$S^{\lambda_1/\mu_1}(\mathbb{C}^{N_1}) \otimes \cdots \otimes S^{\lambda_b/\mu_b}(\mathbb{C}^{N_b})$$

are certain L-representations. In general, not irreducible!

The decomposition of the homogeneous coordinate ring: Part 1

In 2018, H-Lakshmibai gave an explicit description of the decomposition of $\mathbb{C}[X(w)]$ into irreducible L-modules.

Two sets

$H = \left\{\theta \in W^{P_d} \mid X(\theta) \subseteq X(w) \text{ and } X(\theta) \text{ is } L\text{-stable}\right\}$

$H_r = \{\underline{\theta} = (\theta_1, \ldots, \theta_r) \mid \theta_i \in H \text{ and } X(\theta_1) \subseteq \cdots \subseteq X(\theta_r)\}$

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Theorem. (H-Lakshmibai 2018) We have an isomorphism of L-modules

$$\mathbb{C}[X(w)]_r \cong \bigoplus_{\underline{\theta} \in H_r} \mathbb{W}^*_{\underline{\theta}}$$

where $\mathbb{W}_{\underline{\theta}}$ are certain L-modules of the form $S^{\lambda_1/\mu_1}(\mathbb{C}^{N_1}) \otimes \cdots \otimes S^{\lambda_b/\mu_b}(\mathbb{C}^{N_b})$

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Multiplicity free?

Check two things. Each $\mathbb{W}_{\underline{\theta}}$ is multiplicity free. For $I \subset \mathbb{W}_{\underline{\theta}}$ and $I' \subset \mathbb{W}_{\underline{\theta}'}$; If $I \cong I'$, then $\underline{\theta} = \underline{\theta}'$.

The Classification

Recall $L = \mathrm{GL}_{N_1} \times \cdots \times \mathrm{GL}_{N_b}$

Any $w \in W^{P_d}$ can be represented by $(\ell_1, \ldots, \ell_d)$ with $1 \leq \ell_1 < \cdots < \ell_d \leq N$. Define

$$h_k = |\,\{\ell_j | N_1 + \cdots + N_{k-1} < \ell_j \leq N_1 + \cdots + N_k\}\,|$$

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Theorem (H-Lakshmibai 2019) $C[X(w)]$ is a multiplicity free L-module (equivalently $X(w)$ is a spherical L-variety) if and only if one of the following holds.

(i) $b \leq 2$

(ii) $b = 3$, and at least one of $N_2 = 1$, $h_1 + 1 \geq N_1$, $N_2 = h_2$ with $h_1 + 2 \geq N_1$, $h_2 > 0$ with $h_3 < 2$, $h_2 = 0$ with $h_3 \leq 2$ holds.

(iii) $b \geq 4$, $p_w = 2$ or if $p_w > 2$, then $h_1 + \cdots + h_{p_w - 1} + 1 \geq N_1 + \cdots + N_{p_w - 1}$.

where $1 < p_w < b_L$ is the minimum index such that $h_{p_w+1} + \cdots + h_{b_L} < 2$. Such an index may not exist, if it does not set $p_w = b_L - 1$.

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In our case $G = L$ and $B = B_L$. Want Schubert divisors that are B_L-stable but not L-stable. All Schubert varieties are B_L-stable.

A Schubert divisor will be a color if it is not L-stable.

Theorem. (Can-H-Lakshmibai 2019) Let $X(w)$ be an L_I-spherical Schubert variety in the Grassmannian. Then the Schubert divisor $X(s_k w)$ is a color if and only if $S \setminus I$ contains s_{k-1}. Further, $X(s_k w)$ contains an L-orbit if and only if it contains an L-stable Schubert variety.

Thank you!