Equivariant K-theory and tangent spaces to Schubert varieties

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Flag varieties

Notation

- G = simple algebraic group
- B = Borel subgroup, B^- = opposite Borel subgroup
- T = maximal torus contained in B
- $B = TU$, $B^- = TU^-$
- If V is a representation of T, the set of weights of V is denoted $\Phi(V)$
- $X = G/B$, the flag variety
- $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, $\mathfrak{u}$, $\mathfrak{u}^-$ denote Lie algebras of the corresponding groups.
- W = Weyl group, equipped with Bruhat order
- The T-fixed points of X are xB for $x \in W$.

Tangent spaces to Schubert varieties

There is an open cell in X containing xB :

- Let $U^-(x) = xU^-x^{-1}$ with Lie algebra $\mathfrak{u}^-(x)$
- $U^-(x)xB$ is an open cell C_x containing xB.

Schubert varieties

- $X = G/B$, $X^w = \overline{B^- \cdot wB}$, Schubert variety, codim $\ell(w)$.
- The T-fixed point xB is in X^w if and only if $x \geq w$ in the Bruhat order.
- One would like to understand the singularities of X^w at xB.
- Write T_xX^w for $T_{xB}X^w$.
- More modest goal: Understand the Zariski tangent space T_xX^w, or equivalently, the set of weights $\Phi(T_xX^w)$.
- $\Phi(T_xX^w) \subseteq \Phi(T_xC_x) = x\Phi^-$.

Equivariant K-theory

- For classical groups, $\Phi(T_x X^w)$ has been described.
  - The description is complicated except in type A.
- Goal: obtain some information about $\Phi(T_x X^w)$ from equivariant K-theory.

Motivation

- There are ways to do calculations in equivariant K-theory which are uniform across types.
- One can obtain information about multiplicities from these calculations but some cancellations are required.
- The set of weights $\Phi(T_x X^w)$ is related to these cancellations.

Generalized flag varieties

- Suppose $P = LU_P \supset B$ is a parabolic subgroup.
- $X_P = G/P$ generalized flag variety.
- $X_P^w = \overline{B^- \cdot wP}$, Schubert variety in G/P.
- W^P = minimal coset representatives of W with respect to W_P = Weyl group of L.
- Let $\pi : X \to X_P$. If $w \in W^P$, then $\pi^{-1}(X_P^w) = X^w$.
- Because π is a fiber bundle map, if we understand $\Phi(T_x X_P^w)$ then we can understand $\Phi(T_x X^w)$.

Generalized flag varieties

Remark

Sometimes it is useful to take P to be the largest parabolic subgroup such that w is in W^P, and then study X_P^w.

- The simple roots of the Levi factor L are the α such that $ws_\alpha > w$.

Tangent and normal spaces

- Let $x, w \in W^P$ with $x \geq w$.
- The map $xU_P^- x^{-1} \to X_P$, $y \mapsto y \cdot xP$, gives an isomorphism of $xU_P^- x^{-1}$ with an open cell $C_{x,P}$ in X_P containing xP.
- Let $\Phi_{amb} = \Phi(T_x X_P) = x\Phi(\mathfrak{u}_P^-)$. ("Amb" for "ambient".)
- Let $\Phi_{tan} = \Phi(T_x X_P^w)$.
- Let $\Phi_{nor} = \Phi_{amb} \setminus \Phi_{tan}$.

Equivariant K-theory

- If T acts on a smooth scheme M, $K_T(M)$ denotes the Grothendieck group of T-equivariant coherent sheaves (or vector bundles) on M.
- $K_T(M)$ is a module for $K_T(\text{point})$, which equals the representation ring $R(T)$ of T (spanned by e^λ for $\lambda \in \hat{T}$).
- A T-invariant closed subscheme Y of M has structure sheaf $\mathcal{O}_Y$, which defines a class $[\mathcal{O}_Y] \in K_T(M)$
- If $i_m : \{m\} \hookrightarrow M$ is the inclusion of a T-fixed point, there is a pullback $i_m^* : K_T(M) \to K_T(\{m\}) = R(T)$.

Pullbacks of Schubert classes

If Y is a Schubert variety in a flag variety M, the pullback $i_m^*[\mathcal{O}_Y]$ can be computed.

Notation

- Let $i_x : \{xP\} \to X_P$ denote the inclusion.
- $i_x^*[\mathcal{O}_{X_P^w}]$ denotes the pullback of the Schubert class to xP.
- This is the same as the pullback of $[\mathcal{O}_{X^w}]$ to xB.

The 0-Hecke algebra

The 0-Hecke algebra arises in the formulas for the K-theory pullbacks.

Definition

The 0-Hecke algebra is a free $R(T)$-algebra with basis H_w, for $w \in W$. Multiplication: Let s be a simple reflection.

- $H_s H_w = H_{sw}$ if $l(sw) > l(w)$
- $H_s H_w = H_w$ if $l(sw) < l(w)$
- $H_s^2 = H_s$
- H_1 is the identity element.

Sequences of reflections

Let $\mathbf{s} = (s_1, s_2, \ldots, s_l)$ be a sequence of simple reflections.

Define the Demazure product $\delta(\mathbf{s}) \in W$ by the formula

$$H_{s_1} \cdots H_{s_l} = H_{\delta(\mathbf{s})}.$$

- $\delta(\mathbf{s}) \geq w$ iff $\mathbf{s}$ contains a subexpression multiplying to w (Knutson-Miller).
- In particular, $\delta(\mathbf{s}) \geq s_1 s_2 \cdots s_l$, with equality if $\mathbf{s}$ is reduced.

Subsequences

- Let $w \in W$. Define $T_{w,\mathbf{s}}$ to be the set of sequences $\mathbf{t} = (i_1, \ldots, i_m)$, where $1 \leq i_1 < \cdots < i_m \leq l$, such that $H_{s_{i_1}} \cdots H_{s_{i_m}} = H_w$.
- Define the length $\ell(\mathbf{t}) = m$ and the excess $e(\mathbf{t}) = \ell(\mathbf{t}) - \ell(w)$.

A pullback formula

Reduced expressions and inversion sets

- Let $\mathbf{s} = (s_1, s_2, \ldots, s_l)$ be a reduced expression for x.
- Let $\gamma_i = s_1 \cdots s_{i-1}(\alpha_i)$.
- The inversion set $I(x^{-1}) = \Phi^+ \cap x\Phi^- = \{\gamma_1, \ldots, \gamma_l\}$.

The pullback formula

Theorem (G.-Willems)

Let $x, w \in W^P$, $x \geq w$. Then

$$i_x^*[\mathcal{O}_{X_P^w}] = \sum_{\mathbf{t} \in T_{w,\mathbf{s}}} (-1)^{e(\mathbf{t})} \prod_{i \in \mathbf{t}} (1 - e^{-\gamma_i}).$$

Let $P_{\mathbf{s}}$ denote the right hand side of this expression.

The expression $P_{\mathbf{s}}$

- The expression $P_{\mathbf{s}}$ is a sum of monomials in $1 - e^{-\gamma_1}, \ldots, 1 - e^{-\gamma_l}$.
- There is one monomial for each $\mathbf{t} \in T_{w,\mathbf{s}}$, that is, for each subexpression $\mathbf{t} = (i_1, \ldots, i_m)$ such that $H_{s_{i_1}} \cdots H_{s_{i_m}} = H_w$.
  - That monomial is $\prod_{i \in \mathbf{t}} (1 - e^{-\gamma_i})$ (up to sign).
- We will be interested in the weights γ_i such that $1 - e^{-\gamma_i}$ occurs as a factor in each of these monomials.
- This is equivalent to saying that i lies in every subexpression $\mathbf{t} \in T_{w,\mathbf{s}}$.

Indecomposable elements

Recall that for $x \geq w$ in W^P, we defined

- $\Phi_{amb} = \Phi(T_x X_P) = x\Phi(\mathfrak{u}_P^-)$. ("Amb" for "ambient".)
- $\Phi_{tan} = \Phi(T_x X_P^w)$.
- $\Phi_{nor} = \Phi_{amb} \setminus \Phi_{tan}$.

An element $\alpha \in \Phi_{amb}$ is called indecomposable if α cannot be written as a positive linear combination of other elements of Φ_{amb}.

Weights of the normal space

The main result of the talk is:

Theorem

Let γ_i be indecomposable in Φ_{amb}. Then γ_i is in Φ_{nor} if and only if i lies in every subexpression $\mathbf{t} \in T(w, \mathbf{s})$.

Remark

- If i lies in every subexpression $\mathbf{t} \in T(w, \mathbf{s})$, then $1 - e^{-\gamma_i}$ is a factor of $i_x^*[\mathcal{O}_{X_P^w}]$.
- To motivate why the theorem might be true, we look at the connection between normal spaces and factors of $i_x^*[\mathcal{O}_{X_P^w}]$.

Equivariant K-theory and tangent spaces

By replacing X_P by the cell $C_{x,P}$, which is isomorphic to a vector space V, and X_P^w by its intersection with the cell, we can assume we are in the following model situation:

- V = representation of T such that all weights $\Phi(V)$ lie in an open half-space and all weight spaces are 1-dimensional
- Y = closed T-stable subvariety of V
- The T-fixed point is the origin, and i_x corresponds to $i : \{0\} \hookrightarrow V$.
- In our model situation, i^* is an isomorphism in equivariant K-theory, so we can simply omit the pullbacks to the origin.
- Let

$$\lambda_{-1}(V^*) = \prod_{\alpha \in \Phi(V)} (1 - e^{-\alpha}).$$

Equivariant K-theory and tangent spaces

More definitions

- Let C = tangent cone to Y at 0; then $C \subset V' = T_0 Y$.
- The normal space is V/V'.
- Write $\Phi_{amb} = \Phi(V)$, $\Phi_{tan} = \Phi(V')$, $\Phi_{nor} = \Phi_{amb} \setminus \Phi_{tan}$.

Equivariant K-theory and tangent spaces

- Since $C \subset V'$, we have classes $[\mathcal{O}_C]_{V'} \in K_T(V')$ and $[\mathcal{O}_C]_V \in K_T(V)$.
- We also have $[\mathcal{O}_Y]_V \in K_T(V)$.
  - In our Schubert situation, $[\mathcal{O}_Y]_V$ corresponds to $i_x^*[\mathcal{O}_{X_P^w}] = P_{w,\mathbf{s}}$.
- $[\mathcal{O}_C]_V = [\mathcal{O}_Y]_V$, and $[\mathcal{O}_C]_V = \lambda_{-1}((V/V')^*)[\mathcal{O}_C]_{V'}$.
- Conclude: If $\alpha \in \Phi_{nor}$, then $1 - e^{-\alpha}$ is a factor of $[\mathcal{O}_Y]_V$.
- One can show that if α is indecomposable in Φ_{amb}, then the converse holds: If $1 - e^{-\alpha}$ is a factor of $[\mathcal{O}_Y]_V$ then $\alpha \in \Phi_{nor}$.
- This implies one implication of our main theorem. Suppose γ_i is indecomposable in Φ_{amb}. If i is in each subxpression **t** in $T_{w,\mathbf{s}}$, then $1 - e^{-\gamma_i}$ is a factor of $i_x^*[\mathcal{O}_{X_P^w}] = P_{w,\mathbf{s}}$, so $\gamma_i \in \Phi_{nor}$.

Sketch of the proof of the converse

For the other implication, again suppose γ_i is indecomposable in Φ_{amb}.

- Suppose that there exists some subexpression $\mathbf{t}$ in $T_{w,\mathbf{s}}$ such that i is not in $\mathbf{t}$. We want to show that γ_i is in Φ_{tan}.
- One can describe the set of weights of the coordinate ring $\mathbf{C}[C]$ of the tangent cone in terms of the pullback $i_x^*[\mathcal{O}_{X_P^w}]$.
- The hypothesis that i is not in some $\mathbf{t}$, combined with the formula for $P_{w,\mathbf{s}}$, can be used to show that $-\gamma_i$ is a weight of $\mathbf{C}[C]$.
- Since γ_i is indecomposable, the weight $-\gamma_i$ must occur in the degree 1 component of the graded ring $\mathbf{C}[C]$.
- The weights of this degree 1 component are exactly $-\Phi_{tan}$, so $\gamma_i \in \Phi_{tan}$.