Small Resolutions of Closures of K-orbits in Flag Varieties

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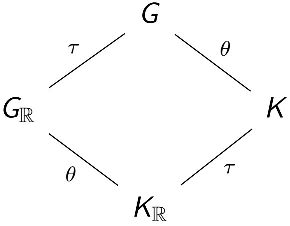
What is K?



$$\theta \circ \tau = \tau \circ \theta$$

- G connected complex reductive algebraic group
- $G_\mathbb{R} = G^\tau$ fixed point subgroup of antiholomorphic involution τ
- $K = G^\theta$ fixed point subgroup of algebraic involution θ
- $K_\mathbb{R} = G_\mathbb{R}^\theta$ is a max'l compact subgroup of $G_\mathbb{R}$

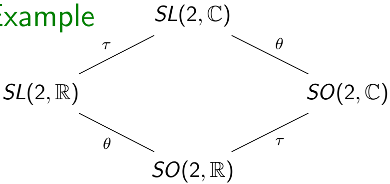
Example

- $G = GL(n, \mathbb{C})$ and K any of $GL(k, \mathbb{C}) \times GL(n-k, \mathbb{C})$, $O(n, \mathbb{C})$, or $Sp(2n, \mathbb{C})$.
- $\theta(g_1, g_2) = (g_2, g_1)$ involution of $G \times G$ gives $K = \Delta G$.

Theorem (Wolf 1969, Matsuki 1979)

Let B be a Borel subgroup and $B \subseteq P \subseteq G$. Then $G_{\mathbb{R}}$ and K act with finitely many orbits on G/P.

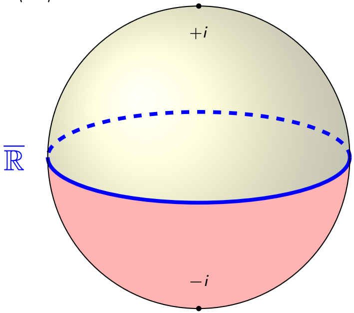
Example



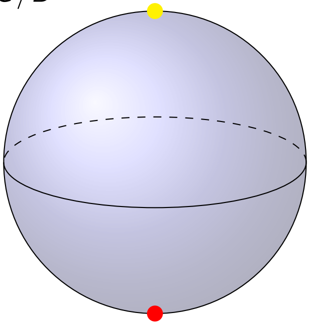
- B upper triangular
- $G/B = \mathbb{P}^1$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az+b \\ cz+d \end{bmatrix}$$

$G_{\mathbb{R}} \backslash G/B =$



$K \backslash G/B =$



Whitney stratification:

$$G/B = \coprod_{v \in V} \mathcal{Q}_v, \qquad V = K \backslash G/B$$

- Compute local polar varieties/multiplicities (too difficult?).
- Determine $(V, \leq)$, where $u \leq v$ means $\mathcal{Q}_u \subseteq \overline{\mathcal{Q}_v}$.
  - Described by, e.g., Richardson-Springer 1994
  - atlas software
- Compute intersection cohomology of $\mathcal{Q}_v$ (and local systems).
  - Solved by Lusztig-Vogan 1983, Vogan 1983
  - atlas software: Kazhdan-Lusztig-Vogan polynomials
- Compute characteristic cycles of intersection cohomology.
  - Solved in certain cases, e.g., highest weight Harish-Chandra modules having regular integral infinitesimal character by Zierau 2018

Example (Schubert varieties)

$\Delta G \subseteq G \times G$ gives $V = W$ the Weyl group, and
$\overline{\mathcal{Q}_v} \cong G \times^B G_w/B$, where $G_w = \overline{B\dot{w}B} \subseteq G$.

Definition

Let $\widetilde{Y}$ and Y be complex algebraic varieties. A *resolution of singularities* of Y is an algebraic morphism $\xi : \widetilde{Y} \to Y$ such that properties (1)-(3) hold:

(1) ξ is proper,

(2) ξ is birational,

(3) $\widetilde{Y}$ is smooth.

A resolution is often required to satisfy :

(4) ξ is an isomorphism over the smooth locus of Y,

which we call *strict*.

Example (Demazure 1974, Hansen 1973)

Let $(s_{i_1}, \ldots, s_{i_\ell})$ be a reduced word for $w \in W$. Then

$$\mu : B \times^B P_{i_1} \times^B \cdots \times^B P_{i_\ell}/B \to G_w/B$$

is a resolution (but rarely strict).

Definition

Let $\xi : \widetilde{Y} \to Y$ be a resolution of singularities. We say that ξ is *small* if for every $r > 0$,

$$\dim(Y) - \dim(Y_r) > 2r,$$

where $Y_r = \{y \in Y \mid \dim(\xi^{-1}(y)) \geq r\}$.

- If ξ is a small resolution then $\xi_* \underline{\mathbb{Q}}^\bullet_{\widetilde{Y}}[\dim(Y)] \cong \mathcal{IC}^\bullet_Y$.
- If ξ is a small resolution of a normal Y then ξ is strict.

Example (Gelfand-MacPherson 1982)

Let $I_0, \ldots, I_m$ be subsets of simple reflections and define

$$\mu : P_{I_0} \times^{R_1} \cdots \times^{R_m} P_{I_m}/R \to G_w/P,$$

where $R \subseteq P_{I_m} \cap P$ and P stabilizes G_w (by right multiplication).

- If $G = GL(n, \mathbb{C})$ and $P \subsetneq G$ is maximal then there exists a small resolution of G_w/P (Zelevinskiĭ 1983).

K-orbits (Barbasch-Evens 1994)

Theorem (Vogan 1983, Chang 1988)

Let $v \in V$. There exists $v_0 \leq v$ and simple reflections $(s_{i_1}, \ldots, s_{i_m})$ such that

$$\mu : G_{v_0} \times^B P_{i_1} \times^B \cdots \times^B P_{i_m}/B \to G_v/B,$$

is a resolution of singularities, where $G_v = \overline{K\dot{v}B} \subseteq G$. Here $G_v/B = \overline{\mathcal{Q}_v} \subseteq G/B$.

Theorem (Barbasch-Evens 1994)

Let $v \in V$ such that P stablizes G_v. If $G = GL(n, \mathbb{C})$ and $P \subsetneq G$ is maximal then there exists $v_0 \leq v$ and $R \subseteq P$ such that

$$\mu : G_{v_0}/R \to G_v/P$$

is a resolution of singularities (any K).

- *If $K = GL(k, \mathbb{C}) \times GL(n-k, \mathbb{C})$ then there exists a small μ.*

$Sp(2n, \mathbb{R})$

- $G = Sp(2n, \mathbb{C})$ (defined by some ω) and $K = GL(n, \mathbb{C})$
- $\mathbb{C}^{2n} = \mathbb{C}^n + \mathbb{C}^{-n}$, $\Lambda^{\pm} : \mathbb{C}^{2n} \to \mathbb{C}^{\pm n}$
- If $1 \leq k \leq n$ then $\mathcal{Q}_{a,b,c} \subseteq \mathrm{Gr}_k^0(\mathbb{C}^{2n})$ (isotropic subspaces) by

  $$\dim(\mathbb{C}^n \cap E^k) = a, \ \dim(\mathbb{C}^{-n} \cap E^k) = b, \ \dim(\mathrm{rad}(\varepsilon)) = c$$

  where $\varepsilon(x, y) = \omega(\Lambda^+(x), \Lambda^-(y))$ symmetric bilinear form on E^k.

Theorem

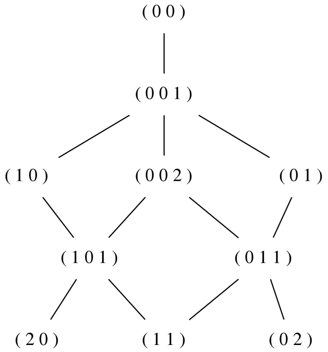
Let $v \in V$ such that P stabilizes G_v. If $P \subsetneq G$ is maximal then there exists $v_0 \leq v$ and $R \subseteq P_{I_1} \cap P$ such that

$$\mu : G_{v_0} \times^{R_1} P_{I_1}/R \to G_v/P$$

is a resolution of singularities.

Example

Let $n = 4$ and $k = 2$. If $c = 0$ then we write $(a, b, c) = (a, b)$. Then μ is small, e.g., for $G_{(a,b)}/P$ when $k \leq \frac{n}{2}$.



- $(0, 0), (2, 0), (1, 1), (0, 2)$ are smooth.
- μ is small for $(1, 0), (0, 1), (0, 0, 2)$.
- μ' is small for $(1, 0, 1)$ and $(0, 1, 1)$.

Main construction

Let $v_0 \in V$ and for $1 \leq i \leq m$, let $w_i \in W$. Suppose

$$\mu : G_{v_0} \times^{R_1} G_{w_1} \times^{R_2} \cdots \times^{R_m} G_{w_m}/B \to G_v/B$$

is a resolution of singularities.

- We write $v = v_0 \star w_1 \star \cdots \star w_m$ (the monoid $(W, \star)$ action).
- For $0 \leq i \leq m$, let $v_i = v_0 \star w_1 \star \cdots \star w_i$. If μ is small then $v_0 < v_1 < \cdots < v_m = v$ all have small resolutions.

Theorem

If W is simply laced then there exists $I_1, \ldots, I_h$ such that

$$\begin{array}{ccc} G_{v_0} \times^{R_1} G_{w_1} \times^{R_2} \cdots \times^{R_m} G_{w_m}/B & \xrightarrow{\mu} & G_v/B \\ \cong \downarrow & \nearrow_{\mu'} & \\ G_{v_0} \times^{R_1} P_{I_1} \times^{R_2'} \cdots \times^{R_h'} P_{I_h}/B & & \end{array}$$

commutes.

Example

If $V = W$ is simply laced and G_w/B is smooth then there exists $I_0, \ldots, I_m$ such that

$$\mu : P_{I_0} \times^{R_1} \cdots \times^{R_m} P_{I_m}/B \to G_w/B$$

is an isomorphism.

Example

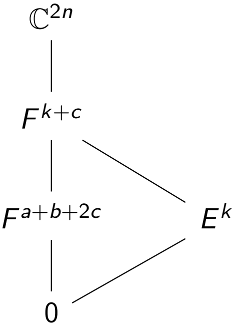
If $G = GL(n, \mathbb{C})$ and $K = GL(k, \mathbb{C}) \times GL(n-k, \mathbb{C})$ then

$(k, n-k)$	1	2	3	4
1	1			
2	1	1		
3	1	.9818	.9767	
4	1	.9583	.9429	.9217

shows ratio of $v \in V$ admitting small resolutions of the form μ.

Resolution of singularities for $(Sp(2n,\mathbb{C}), GL(n,\mathbb{C}))$ revisited

Consider $(a,b,c) \in V_n^{\hat{k}}$.



- $F^\bullet$ isotropic in $\mathbb{C}^{2n}$
- $\dim(\mathbb{C}^n \cap F^{a+b+2c}) = a + c$
- $\dim(\mathbb{C}^{-n} \cap F^{a+b+2c}) = b + c$

Then $\mathrm{pr}(F^{a+b+2c}, F^{k+c}, E^k) = E^k$ projects to $G_{(a,b,c)}/P_{\hat{k}}$ and is isomorphic to the resolution μ.