B_{n-1}-orbits on the flag variety II

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General Notation

In this talk $G_i = GL(i)$ for $i = 1, \ldots, n$.

We have chain of inclusions

$G_1 \subset G_2 \subset \ldots \subset G_i \subset G_{i+1} \subset G$.

Let $G_{n-1} = K$ and $G_n = G$

$B_i \subset G_i$=standard upper triangular Borel subgroup.

$Q_K =$ a K-orbit on G/B.

$Q =$ a B_{n-1}-orbit on G/B.

Overview of Talk:

1) Discuss combinatorial model involving partitions for $B_{n-1}\backslash G/B$.

Get e.g.f and explicit formula for $|B_{n-1}\backslash G/B|$.

2) Use (1) to develop explicit set of representatives for B_{n-1}-orbits in terms of flags.

Can use these representatives to study the weak order.

3) **In progress:** Develop second combinatorial model involving Dyck paths for B_{n-1}-orbits using (2) and refined geometric data from first talk.

First combinatorial model of $B_{n-1}\backslash G/B$.

B_{n-1}-orbits on G/B are modeled by PILS.

PILS = partitions into lists.

A list of the set $\{1,\ldots,n\}$ is any ordered non-empty subset.

Notation: $\sigma = (a_1 a_2 \ldots a_k)$.

A PIL of the set $\{1,\ldots,n\}$ is any partition of the set $\{1,\ldots,n\}$ into lists.

Notation: $\Sigma = \{\sigma_1,\ldots,\sigma_\ell\}$.

Examples: For $n = 2$, there are 3 PILS:

$\{(12)\}, \{(21)\}, \{(1), (2)\}$.

For $n = 3$, there are 13 PILS.

6 of form $\{(i_1 i_2 i_3)\}$, 6 of form $\{(i_1 i_2), (i_3)\}$, $\{(1), (2), (3)\}$.

For $n = 4$, there are 73 PILS, and for $n = 5$, there are 501 PILS.

Combinatorial Theorem: There is a one-to-one correspondence:

$$PILS \Leftrightarrow B_{n-1}\backslash G/B.$$

Remarks:

There is a similar correspondence in the orthogonal case involving partitions into signed lists satisfying certain parity conditions depending on whether $G = SO(n)$ is of type B or type D.

Exponential Generating Function for $|B_{n-1}\backslash G/B|$.

Corollary:

Let $a_n = |B_{n-1}\backslash G/B|$.

Then

(1) The e.g.f for the sequence $\{a_n\}_{n=1}^{\infty}$ is

$$e^{\frac{x}{1-x}}.$$

(2)

$$a_n = n! \sum_{i=0}^{n-1} \frac{\binom{n-1}{i}}{(i+1)!}.$$

The correspondence between PILS and B_{n-1}-orbits on G/B is proven using the fibre bundle structure of these orbits discussed in the last talk and structure of K-orbits on G/B.

Notation for Flags and Partial Flags:

Flag:

$$\mathcal{F} := V_1 \subset V_2 \subset \ldots \subset V_i \subset \ldots \subset V_n = \mathbb{C}^n.$$

with $\dim V_i = i$.

Notation: Suppose $V_i = \text{span}\{v_1, \ldots, v_i\}$, then write

$$\mathcal{F} := v_1 \subset v_2 \subset \ldots \subset v_i \subset \ldots \subset v_n.$$

Partial Flag:

$$\mathcal{P} = V_1 \subset V_2 \subset \ldots \subset V_j \subset \ldots \subset V_k = \mathbb{C}^n$$

where $\dim V_j = i_j$.

Notation:

Suppose $V_j = \text{span}\{v_1, \ldots, v_{i_j}\}$ for $j = 1, \ldots, k$.

$\mathcal{P} = \{v_1, \ldots, v_{i_1}\} \subset \{v_{i_1+1}, \ldots, v_{i_2}\} \subset \ldots \subset$

$\subset \{v_{i_1+\ldots+i_{k-1}+1}, \ldots, v_{i_k}\}$

Recall:

Q a B_{n-1}-orbit, $Q \subset Q_K = K \cdot \tilde{\mathfrak{b}}$:

$\exists$ θ-stable parabolic subgroup, $\tilde{B} \subset P \subset G$

such that $\pi : G/B \to G/P$

endows Q with structure of fibre bundle:

"$Q = Q_P \times Q_\ell$".

BASE: $Q_P =$ a B_{n-1}-orbit on partial flag variety $K/(K \cap P) = \pi_{Q_K}$ of K,

FIBRE: $Q_\ell =$ a $B_{\ell-1}$-orbit on G_ℓ/B_ℓ, $\ell \leq n-1$.

Description of K-oribts

Notation:

$\{e_1, \ldots, e_n\}$ = standard basis for $\mathbb{C}^n$.

$\mathbb{C}^{n-1} = \text{span}\{e_1, \ldots, e_{n-1}\}$.

For $i = 1, \ldots, n-1$, $\hat{e}_i = e_i + e_n$.

n-closed K-orbits:

$Q_{i,c}$, $i = 1, \ldots, n$.

In this case, $Q_{i,c} \cong K/B_{n-1}$.

Non-closed orbits:

$Q_{i,j} = K \cdot \mathcal{F}_{i,j}$, $1 \leq i < j \leq n$

$\mathcal{F}_{i,j} :=$

$e_1 \subset \ldots \subset \underbrace{\hat{e}_i}_{i} \subset \ldots \subset e_{j-1} \subset \underbrace{e_n}_{j} \subset e_j \subset \ldots e_{n-1}$.

Suppose: $Q \subset K \cdot \mathcal{F}_{i,j}$

Let $P_{i,j} \subset G$ stabilize partial flag:

$\mathcal{P}_{i,j} = e_1 \subset \ldots \subset \{e_i, \ldots, e_{j-1}, e_n\} \subset e_j \subset \ldots \subset e_{n-1}$.

Note: $\mathcal{F}_{i,j} \subset \mathcal{P}_{i,j}$.

$K/(K \cap P_{i,j}) = K \cdot (\mathcal{P}_{i,j} \cap \mathbb{C}^{n-1})$

$Q_{P_{i,j}} = B_{n-1}$-orbit on $K/(K \cap P_{i,j})$.

$Q_\ell \leftrightarrow B_{\ell-1}$-orbit on G_ℓ/B_ℓ, where $\ell = j - i$.

It follows that:

$Q_{P_{i,j}}$ is determined by an $s \in \mathcal{S}_{n-1}$ with $s(i) < s(i+1) < \ldots < s(j)$.

$$Q_P \leftrightarrow (s(1)\ldots s(i-1)\underbrace{n}_{i}s(j)\ldots s(n-1))\ .$$

By induction

$Q_\ell \leftrightarrow \Sigma_\ell$ where Σ_ℓ is a unique PIL of the set $\{s(i),\ldots,s(j-1)\}$.

Conclusion:

$$Q \leftrightarrow \{(s(1)\ldots s(i-1)\underbrace{n}_{i}s(j)\ldots s(n-1)),\Sigma_\ell\}.$$

Example:

$V = \mathbb{C}^4$ and $G = GL(4)$.

Consider B_3-orbit:

$Q = B_3 \cdot (\hat{e}_3 \subset \hat{e}_1 \subset e_4 \subset e_2)$.

$Q \subset Q_{1,3} = K \cdot (\hat{e}_1 \subset e_2 \subset e_4 \subset e_3.); \ell = 2$

$G/P = G \cdot (\{e_1, e_2, e_4\} \subset e_3) = \mathrm{Gr}(3, \mathbb{C}^4)$.

$K/(K \cap P_{1,3}) = K \cdot (\{e_1, e_2\} \subset e_3) = \mathrm{Gr}(2, \mathbb{C}^3)$.

$Q_{P_{1,3}} = B_3 \cdot (\{e_1, e_3\} \subset e_2) \leftrightarrow s = s_{\epsilon_2 - \epsilon_3}$.

$Q_P \leftrightarrow (42)$.

$Q_2 \leftrightarrow$ is open $B_1 = \mathbb{C}^\times$-orbit on flag variety of $\mathbb{C}^2 = \mathrm{span}\{e_1, e_3\}$.

$Q_2 \leftrightarrow (1)(3)$.

$Q = Q_P \times Q_2 \leftrightarrow \{(42), (1)(3)\}$.

Minimal Elements in the weak order.

Ultimate Goal: Understand strong order (i.e. closure relations) $B_{n-1}\backslash G/B$

As a step in this direction, we prove:

Theorem: Any B_{n-1}-orbit Q which is minimal in the weak order is closed.

Remark: This is not true for orbits of a general spherical H on G/B.

To prove this, we use the theory of PILS to develop a canonical set of representatives for $B_{n-1}\backslash G/B$.

We can then use these representative to understand the Richardson-Springer monoid action.

Standard Form for a flag in G/B

Definition: A flag in $\mathbb{C}^n$

$$\mathcal{F} := v_1 \subset \ldots \subset v_j \subset \ldots \subset v_n.$$

with $v_j = \widehat{e}_{i_j}$ or $v_j = e_{i_j}$ is in *standard form*

if

(1) If $v_k = e_n$, then $v_j = e_{i_j}$ for $j > k$.

(2) If $k < j$ and $v_k = \widehat{e}_{i_k}$ and $v_j = \widehat{e}_{i_j}$, then $i_k > i_j$.

Example:

For $V = \mathbb{C}^5$, the flag

$$e_1 \subset \hat{e}_4 \subset \hat{e}_3 \subset e_5 \subset e_2.$$

is in standard form.

But the flag

$$e_1 \subset \hat{e}_3 \subset \hat{e}_4 \subset e_5 \subset \hat{e}_2$$

is not.

Combinatorial Theorem 2: There is a 1-1 correspondence

$$\{PILS\} \longleftrightarrow \{\text{Flags in standard form}\}.$$

(This is proven purely combinatorially; no geometry involved.)

Using above theorem and an inductive argument using B_{n-1}-orbits on Gr$(\ell, \mathbb{C}^n)$, we can show:

Prop: Every B_{n-1}-orbit Q contains a unique flag in standard form $\mathcal{F}$.

To prove theorem about the weak order:

$Q \subset Q_K$, $Q = Q_P \times Q_\ell$.

Geometry: RS Monoid action is compatible with fibre bundle structure $\Rightarrow$

$Q_c \leq_w Q$,

$Q_c = B_{n-1}$-orbit closed in Q_K.

Combinatorics: An easy computation with standard forms and PILS and induction shows that $Q'_c \leq_w Q_c$,

Q'_c closed in G/B.

A second more refined combinatorial model: Labelled Dyck Paths

Problem: Given two flags $\mathcal{F}$ and $\mathcal{F}'$ in standard form it's hard to tell if the corresponding B_{n-1}-orbits are related in weak (strong) order.

Solution: Connect the combinatorics of the standard form to the geometry of the fibre bundle structure of B_{n-1}-orbits to develop a more sophisticated combinatorial model in terms of labelled Dyck paths.

First step: Iterate fibre bundle characterization of a B_{n-1}-orbit Q to assign to Q the following data:

$$Q \to [(d_0, Q_{i_0,j_0}, s_0), (d_1, Q_{i_1,j_1}, s_1), \ldots, (d_k, Q_{i_k,j_k}, s_k)].$$

with $d_0 = n > d_1 > d_2 > \ldots > d_k$.

$Q_{i_\ell,j_\ell} =$ a $G_{d_\ell - 1}$-orbit on G_{d_ℓ}/B_{d_ℓ}

$s_\ell =$ shortest coset rep of $s_\ell S_{d_{\ell+1}}$ in $S_{d_\ell - 1}/S_{d_{\ell+1}}$.

Key Idea: Data corresponds to a labelled Dyck path of length 2n.

Labels determined by "Weyl group data" $(s_0, \ldots, s_k)$.

Path determined by "K-orbit data" $(Q_{i_0,j_0}, \ldots, Q_{i_k,j_k})$.

How does this work?

Take Q a B_{n-1}-orbit.

$Q \subset Q_K$

If $Q_K = Q_c$ is closed, then Q is a B_{n-1}-orbit on K/B_{n-1} and therefore given by $s_0 \in S_{n-1}$ and the iteration stops.

If $Q_K = Q_{i,j}$ then let $i_0 := i$, $j_0 := j$ and $d_1 = j_0 - i_0$.

Then $Q = Q_{P_{i_0,j_0}} \times Q_{d_1}$,

$Q_{P_{i_0,j_0}}$ = a B_{n-1}-orbit on $K/(K \cap P_{i_0,j_0})$ and so determined by $s_0 \in S_{n-1}/S_{d_1}$.

Q_{d_1}= a B_{d_1-1}-orbit on G_{d_1}/B_{d_1}.

THUS, $Q_{d_1} \subset Q_{i_1,j_1}$, a G_{d_1-1}-orbit on G_{d_1}/B_{d_1}.

Let $d_2 = j_1 - i_1$.

So $Q_{d_1} = Q_{P_{i_1,j_1}} \times Q_{d_2}$.

CONTINUE until reach a G_{d_k-1} -orbit Q_{i_k,j_k} which is closed in G_{d_k}/B_{d_k}.

Current State of Affairs:

We have an easy algorithm to read off "K-orbit data" from unique flag in standard from in B_{n-1}-orbit Q and produce unlabelled Dyck path.

In Progress:

Develop algorithm to read off "Weyl group data" from standard form.

Conjectures:

1) Weyl group data+K-orbit data determines the orbit Q completely.

2) Weak (strong) order on $B_{n-1}\backslash G/B$ can be understood in terms on a natural ordering on labelled Dyck paths.