Linear Orderings: A Sampler

Cynthia Bowers
Honors Thesis
April 26, 1999
advisor: David Lutzer

Preface

What is a typical linearly ordered set? Many mathematicians think of the real line when they picture a linear ordering. However, the real numbers with their usual ordering is not typical. The topology and order structure of linearly ordered sets can be quite different than that of the reals. The goal of this paper is to give an introduction to many interesting linear orderings which is accessible to all mathematicians, including advanced undergraduates.

Much of the material I present is not original research. However, I am bringing together existing mathematics in a new way. I introduce five different types of linear orderings, and, when possible, I relate them to each other. For example, in the first chapter, I examine both the order structure and the topology of the countable ordinals. As a linearly ordered set, it has many properties worth exploring. Even more important, however, is utilizing the set as a tool for constructing or investigating other linear orderings. For example, in Chapter 3, I use the ordinals to index the levels of trees.

Although most of the mathematics in this paper is already known, some of it is not widely published. Therefore, another goal of this paper is to record some of this knowledge in an organized manner. The ultrafilter orderings of Chapter 4 are known to most logicians, but not much has been published on them. Therefore, a lot of this material I had to discover for myself, although it is likely that most of it is already known.

While pulling together these ideas, at times I was able to expand on what was known, and to answer some natural questions which arose. For example, I explore the result of applying the tree construction of section 3.1 to the real numbers. I also give some new characterizations for ultrafilters in Chapters 4 and 5.

Overall, the goal of this paper is expository: to tie together different ideas, and to present them in a complete, thorough, and clear manner. I view this thesis not as a technical paper, merely presenting the ideas, but as an instructional paper, explaining the material in clear detail.

Chapter 1: The Countable Ordinals

The purpose of this chapter is to give an introduction to a strange linearly ordered set – an uncountable well-ordered set with the property that each element has only a countable number of predecessors. This important set is usually called the set of countable ordinals, which we will denote Ω. It will be useful for indexing various uncountable sets throughout this paper. Understanding this set will prove critical for determining the cardinality of sets and for developing cardinal arithmetic. Therefore, taking the time to examine Ω carefully will prove invaluable for our examination of other linear orderings and tree constructions.

It is not immediately obvious that an uncountable well-ordered set could exist. In fact, it seems somewhat counter-intuitive. After all, well-ordering gives a natural sense of counting. Take the whole space, pick its least element. Look at what's left, and take its least element. At first glance, it seems that continuing this process "forever" will give us a countable set. That assumption, however, is false.

To prove that Ω exists, we use Zorn's Lemma. In order to understand Zorn's Lemma, we need some definitions.

1.0.1 Definition: *A set $(P, \leq)$ is a partially ordered set (poset), provided that:*

1) for every $x \in P$, $x \leq x$
2) for every $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$
3) for every $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$.

The set is linearly ordered if it has the additional property that for every distinct $x, y \in P$, either $x < y$ or $y < x$, but not both.

The set is well-ordered if every nonempty set has a first element.

1.0.2 Definition: *A maximal point in a partially ordered set (poset) $(P, \leq)$ is a point m such that no $x \in P$ satisfies $m < x$.*

1.0.3 Definition: *A poset $(P, \leq)$ is inductive provided each chain (linearly ordered subset) in P has an upper bound that belongs to P.*

The logicians give us the following lemma, which is logically equivalent to the Axiom of Choice, and we will accept it without proof.

1.0.4 Zorn's Lemma: *Any nonempty, inductive poset has a maximal element.*

1.0.5 Theorem: *Any nonempty set X can be well ordered.*

Proof: Let P =$\{(S, \leq) : S \subseteq X$ and $\leq$ is a well-ordering of S$\}$. Define $(S_1, \leq_1) \prec (S_2, \leq_2)$ to mean that S_1 is an initial segment of S_2 and $\leq_2$ extends $\leq_1$. The set P is nonempty because you can fix any $x \in X$, let $S_1 = \{x\}$, and define $\leq_1$ as $x \leq_1 x$. Now we must show that P is inductive. Let $C \subseteq P$ be any chain. Let $S_C = \cup\{S : S \in C\}$, and for every $x, y \in S_C$ let $x \leq_C y$ if $x \leq_n y$ for some ordering S_n in the chain. Because C is a chain, $\leq_C$ is well defined, and it is a well-ordering. Thus, $(S_C, \leq_C)$ is an upper bound of C. Therfore, P is inductive. Thus by Zorn's lemma, P has a maximal element, $(S_0, \leq_0)$. We claim that $S_0 = X$ because if there is a $p \in X$ with $p \notin S_0$, define $S_1 = S_0 \cup \{p\}$ and define $\leq_1$ to extend $\leq_0$ with $p \geq_1 s$ for every $s \in S_0$. Then $(S_1, \leq_1)$ would be in P, and S_0 would not be maximal. Therefore, $S_0 = X$, so X has the well ordering $\leq_0$. $\square$

We know that uncountable sets exist (for example, the set of real numbers), and now we know that every set can be well-ordered. Therefore, there exists an uncountable well-ordered set.

1.0.6 Proposition: *There is an uncountable well ordered set Ω such that for each $\alpha \in \Omega$, $\{\beta \in \Omega : \beta < \alpha\}$ is countable, i.e. every element has countably many predecessors.*

Proof: Let S be any uncountable well ordered set with first element 0. Let $T = \{\alpha \in S : [0, \alpha)$ is uncountable$\}$. If $T = \emptyset$, let $\Omega = S$. Otherwise, let ω_1 be the first element of T, and let $\Omega = [0, \omega_1)$. $\square$

It is interesting to note that the particular uncountable well ordered set used in the proof of 1.0.6 does not matter. A Zorn's Lemma argument shows that if the construction of 1.0.6 is applied to another uncountable well ordered set $\hat{S}$ to produce $\hat{\Omega}$, then $\hat{\Omega}$ will be order isomorphic to Ω.

What does this set Ω look like? Traditionally, the first element of Ω is called 0, and the first element greater than 0 is 1, followed by 2, etc. If $\alpha \in \Omega$, the immediate successor of α is the first element of $\{\beta \in \Omega : \alpha < \beta\}$, which exists because Ω is well ordered and $\{\beta \in \Omega : \alpha < \beta\}$ is nonempty (since Ω is uncountable). The immediate successor of α is denoted $\alpha + 1$.

Many elements of Ω obviously have finitely many predecessors (e.g. 1, 2, 3, etc.), but it can also be shown that some elements must have infinitely many predecessors.

1.0.7 Proposition: *Some element of* Ω *has infinitely many predecessors.*

Proof: Suppose, for contradiction, that every element $\alpha \in \Omega$ has a finite number of predecesors. Fix any $\alpha \in \Omega$, and choose any $\gamma \in \Omega$ with $\gamma > \alpha$. Then $\{\beta \in \Omega : \beta < \gamma\}$ is finite, so $\gamma = \alpha + k$ for some integer k. Thus, $\{\gamma \in \Omega : \alpha < \gamma\} = \{\alpha + k : \text{k is a positive integer}\}$, so the set is countable. Therefore, $\Omega = \{\beta \in \Omega : \alpha \geq \beta\} \cup \{\gamma \in \Omega : \alpha < \gamma\}$, but Ω is uncountable and $\{\beta \in \Omega : \alpha \geq \beta\} \cup \{\gamma \in \Omega : \alpha < \gamma\}$ is countable. $\square$

So, Ω has an element with infinitely many predecessors. We will call the first such element ω.

1.0.8 Proposition: ω *is not the immediate successor of any* $\alpha \in \Omega$.

Proof: For contradiction, suppose ω is the immediate successor of some $\alpha \in \Omega$, so $\alpha + 1 = \omega$. The set $\{\beta \in \Omega : \beta \leq \alpha\}$ is finite because ω is the first element with infinitely many predecessors. But $\{\beta \in \Omega : \beta \leq \alpha\}$ is the set of predecessors of ω, and ω has infinitely many predecessors. $\square$

We can think of Ω as beginning with $0, 1, 2, 3, ...,$ and then $\omega, \omega + 1, \omega + 2, ...,$ with $\omega + \omega$ at the top of this list. The list continues with $(\omega + \omega) + 1, (\omega + \omega) + 2, ...,$ all followed by $\omega + \omega + \omega$. This continues forever. For notational convenience, we create an element at the top of Ω called ω_1, so we can write $\Omega = [0, \omega_1)$.

Sometimes an ordinal κ is used to refer to the set $[0, \kappa)$. We will often use this notation in subsequent chapters. Context makes it clear whether κ refers to the ordinal or the interval of ordinals less than κ.

1.0.9 Proposition: *A subset* $S \subseteq \Omega$ *is uncountable if and only if for each* $\alpha \in \Omega$, *there exists a* $\beta \in S$ *with* $\alpha < \beta$.

Proof: Suppose S is uncountable and, for contradiction, there exists $\alpha \in \Omega$ such that for every $\beta \in S$, $\beta \leq \alpha$. But $\{\gamma \in \Omega : \gamma \leq \alpha\}$ is countable, and $S \subseteq \{\gamma \in \Omega : \gamma \leq \alpha\}$, so S is countable. Contradiction.

Conversely, suppose that for each $\alpha \in \Omega$, there exists a $\beta \in S$ with $\alpha < \beta$. For contradiction, assume S is countable. Since $S \subseteq \Omega$, $\{\gamma \in \Omega : \gamma < \beta\}$ is countable for every $\beta \in S$. Thus, $\cup_{\beta \in S}\{\gamma \in \Omega : \gamma < \beta\}$ is countable. Since Ω is uncountable, there exists $\alpha \in \Omega$ such that $\alpha \notin \cup_{\beta \in S}\{\gamma \in \Omega : \gamma < \beta\}$, so there is no $\beta \in S$ with $\alpha < \beta$. Contradiction. $\square$

Note that the uncountable subsets of Ω are precisely the unbounded subsets of Ω, the sets that have points arbitarily far to the right in the interval $[0, \omega_1)$.

1.0.10 Proposition: *There cannot exist an infinite strictly decreasing seqence in Ω.*

Proof: Suppose a strictly decreasing sequence $\{\alpha_n : n \geq 1\}$ exists. Because Ω is well-ordered, $\{\alpha_n : n \geq 1\}$ has a first element, call it β. For some positive integer k, $\beta = \alpha_k$, so for every $l > k$, $\alpha_l < \beta$. But β is the least element of $\{\alpha_n : n \geq 1\}$. Contradiction. $\square$

1.1 The Topology of Ω

Ω has a natural open interval topology $\mathcal{T}$ with respect to its linear order. The point 0 is isolated, and since every element has an immediate successor, basic open neighborhoods of a point $\beta \in \Omega$ are of the form $(\alpha, \beta] = (\alpha, \beta)$. Let $\mathcal{B}$ be the collection of all basic open neighborhoods.

The following lemma will be useful for proving the convergence of nondecreasing sequences in Ω.

1.1.1 Lemma: *Any nondecreasing sequence $\langle \alpha_n \rangle \subseteq \Omega$ converges to its least upper bound.*

Proof: Let $b = lub\{\alpha_{n_k} : n_k \geq 1\}$, which exists because the sequence is countable, so it has an upper bound, and Ω is well-ordered, so b is the least element of the set of upper bounds. For any open set U with $b \in U$, there exists $x, y \in \Omega$ with $b \in (x, y) \subseteq U$ because (x, y) is a basic neighborhood of b. There must exist an $\alpha_{n_0} > x$ because $x < b = lub\{\alpha_n : n \geq 1\}$. For every $m > n_0$, $\alpha_m \geq \alpha_{n_0} > x$, so $\alpha_m \in (x, y) \subseteq U$. So for any open set around b, α_n is eventually in that set. Therefore, $< \alpha_n >$ converges to b. $\square$

1.1.2 Proposition: *Any sequence of points of Ω has a convergent subsequence.*

Proof: Let $\langle \alpha_n \rangle$ be any sequence of points in Ω. If there exists a constant subsequence, we're done, so assume no constant subsequence exists. Since any infinite linearly ordered set has a strictly monotonic sequence, and since Ω has no strictly decreasing sequence, there must be a strictly increasing subsequence of $\langle \alpha_n \rangle$. By the lemma above, this subsequence converges. $\square$

1.1.3 Theorem: Ω *is first countable.*
Proof: We want to show that there is a countable neighborhood base about each point in Ω. For every $\alpha \in \Omega$, let $\mathcal{B}_\alpha = \{(\gamma, \alpha] : \gamma < \alpha\}$. Since

$\{\gamma \in \Omega : \gamma < \alpha\}$ is countable, $\mathcal{B}_\alpha$ is countable. Clearly, $\mathcal{B}_\alpha$ is a neighborhood base at α. Therefore, Ω is first countable. $\square$

1.1.4 Theorem: Ω *is countably compact.*

Proof: Let $\mathcal{U} = \cup\{U_n : n \geq 1\}$ be any countable open cover of Ω. We must show that $\mathcal{U}$ has a finite subcover. Suppose, for contradiction, that $\mathcal{U}$ has no finite subcover. Pick any $c_1 \in U_1$. Since $U_1 \neq \Omega$, there exists $c_2 \in \Omega \setminus U_1$. For any integer n, $\cup\{U_i : 1 \leq i \leq n\} \neq \Omega$, so there exists $c_{n+1} \in \Omega \setminus \cup\{U_i : 1 \leq i \leq n\}$. Thus, $\{c_n : n \geq 1\}$ is an infinite sequence, so it has a subsequence $\langle c_{n_k} \rangle$ which converges to some $d \in \Omega$. There exists an N such that $d \in U_N$, since $\mathcal{U}$ is an open cover. But since $\langle c_{n_k} \rangle$ converges to d, $\langle c_{n_k} \rangle$ is eventually in U_N. This is a contradiction because $\langle c_{n_k} \rangle$ is not eventually in any element of the open cover $\mathcal{U}$. $\square$

1.1.5 Remark: *The proof above shows, more generally, that any space will be countably compact provided each sequence has a convergent subsequence.*

1.1.6 Theorem: Ω *is not Lindelöf.*

Proof: Let $\mathcal{U} = \{[0, \alpha + 1) : \alpha \in \Omega\}$. For any $U \in \mathcal{U}$, U is countable because $\{\beta \in \Omega : \beta < \alpha + 1\}$ is countable. Since Ω is uncountable, and each $U \in \mathcal{U}$ is countable, $\mathcal{U}$ must be uncountable. Similarly, every subcover $\mathcal{V}$ of $\mathcal{U}$ must also be uncountable. Therefore, Ω is not Lindelöf. $\square$

1.1.7 Corollary: Ω *is not second countable, compact, or metrizable.*

Proof: Second countable implies Lindelöf, but Ω is not Lindelöf. Therefore, Ω is not second countable. Compact implies Lindelöf, and Ω is not Lindelöf, so Ω is not compact. In a metric space, compactness is equivalent to seqential compactness, the property that every sequence has a convergent subsequence. Ω is not compact, but it is sequentially compact (by Proposition 1.2.2 above). Therefore, Ω cannot be a metric space. $\square$

1.2 Limit Ordinals of Ω

Many elements of Ω have immediate predecessors. The immediate predecessor of 1 is 0, and the immediate predecessor of 2 is 1. We know that at least one element, namely ω, has no immediate predecessor. The set $L = \{\lambda : \lambda$ has no immediate predecessor in $\Omega\}$ is called the set of limit ordinals.

1.2.1 Proposition: *The set $L = \{\lambda : \lambda$ has no immediate predecessor in $\Omega\}$ is a closed set in Ω.*

Proof: To show that L is closed, we will show that $\Omega \setminus L$ is open. For every $p \in \Omega \setminus L$, p has an immediate predecessor, call it $p-1$. Then $(p-1, p]$ is an open set, and $(p-1, p] = \{p\}$. Thus, $(p-1, p] \subseteq \Omega \setminus L$. Therefore, $\Omega \setminus L$ is open, so L is closed. $\square$

1.2.2 Proposition: *The set $L = \{\lambda : \lambda$ has no immediate predecessor in $\Omega\}$ is an uncountable set.*

Proof: Suppose that L is countable. Let $L_0 = L \cup \{0\}$, a countable set. For every $\alpha \in \Omega \setminus L_0$, $\alpha = \lambda + k$ for some $\lambda \in L_0$ and some integer k. Let $L_1 = \{\lambda + 1 : \lambda \in L_0\}$, and for every integer k, let $L_k = \{\lambda + k : \lambda \in L_0\}$. For $k \geq 0$, L_0 and L_k have the same cardinality because there is a one-to-one onto function $f : L_0 \to L_k$ defined $f(x) = x + k$. Therefore each L_k is countable, so $\cup\{L_k : k \geq 0\}$ is countable. But $\cup\{L_k : k \geq 0\} = \Omega$, and Ω is uncountable. Contradiction. $\square$

1.3 Closed Uncountable Subsets of Ω

In this section, we will examine closed uncountable subsets of Ω. Recall that every uncountable set in Ω is unbounded. For this reason, closed uncountable subsets are often called closed unbounded sets (abbreviated cub).

1.3.1 Proposition: *If C and D are two uncountable closed subsets of Ω, then $C \cap D \neq \emptyset$.*

Proof: To prove this proposition, we will construct interlaced sequences in C and D. Let c_1 be the least element of C. Let d_1 be the least element of $\{d \in D : d > c_1\}$, a nonempty set because D is uncountable. Let c_n be the least element of $\{c \in C : c > d_{n-1}\}$, and let d_n be the least element of $\{d \in D : d > c_n\}$, which exist because C and D are both uncountable. Since $\{c_n, d_n : n \geq 1\}$ is countable, there exists $\beta \in \Omega$ such that $x \leq \beta$ for every $x \in \{c_n, d_n : n \geq 1\}$. So $\{\beta \in \Omega : \beta \geq c_n, d_n$ for every $n \geq 1\}$ is nonempty. Therefore, it has a least element, call it γ. Then $\gamma = lub\{c_n : n \geq 1\} = lub\{d_n : n \geq 1\}$. Thus, $\langle c_n \rangle$ and $\langle d_n \rangle$ converge to γ. But C and D are closed, so they contain their limit points. Since γ is a limit point of C and D, we have $\gamma \in C$ and $\gamma \in D$, so $C \cap D \neq \emptyset$. $\square$

1.3.2 Proposition: *If C and D are two uncountable closed subsets of Ω,* *then $C \cap D$ is an uncountable closed subset of Ω.*

Proof: Since C and D are closed, $C \cap D$ is closed. By the proposition above, $C \cap D \neq \emptyset$. Suppose, for contradiction, that $C \cap D$ is finite. Let γ be the greatest element of $C \cap D$. Consider $C_1 = C \cap [\gamma + 1, \omega_1)$ and $D_1 = D \cap [\gamma+1, \omega_1)$. These are uncountable closed sets in Ω, and $C_1 \cap D_1 = \emptyset$, contradicting the previous proposition. $\square$

The following proposition follows by induction.

1.3.3 Proposition: *If C_1, C_2,..., C_n are uncountable closed subsets of Ω, then $C_1 \cap C_2 \cap ... \cap C_n$ is an uncountable closed subset of Ω.*

It is natural to explore next the intersection of infinitely many uncountable closeds subsets of Ω.

1.3.4 Proposition: *If C_1, C_2, ... are uncountable closed subsets of Ω, then $\cap\{C_n : n \geq 1\}$ is an uncountable closed subset of Ω.*

Proof: Recall that the intersection of any family of closed sets is also closed. To show that $\cap\{C_n : n \geq 1\}$ is uncountable, we will show that for any $\alpha \in \Omega$, some $\beta \in \cap\{C_n : n \geq 1\}$ has $\beta > \alpha$. Fix any $\alpha \in \Omega$. Again we will use the notion of interlaced sequences to find an element of $\cap\{C_n : n \geq 1\}$ which is greater than α.

We may assume that $C_n \supseteq C_{n+1}$ for each n because if it is not, then replace C_n by $C'_n = \cap\{C_j : a \leq j \leq n\}$, which is also an uncountable closed set.

Fix any $\alpha < \omega_1$. Choose any $c_1 \in C_1 \cap [\alpha, \omega_1)$, and recursively choose $c_{n+1} \in C_{n+1} \cap (c_n, \omega_1)$. This is possible because C_m is a closed uncountable set for every m. Then $\alpha \leq c_1 < c_2 < c_3 < ....$ Let $d = \sup\{c_n : n \geq 1\}$. We know that $\alpha < d$, and the sequence $\langle c_n \rangle \to d$. Fix any integer N. Because $\{c_n : n \geq N\} \subseteq C_N$ and C_N is closed, $d \in C_N$. Hence, $d \in \cap\{C_N : N \geq 1\}$, and $d > \alpha$, so $\cap\{C_N : N \geq 1\}$ is a closed uncountable set. $\square$

1.4 Stationary Sets

Recall that the uncountable subsets of Ω are exactly the subsets that have points arbitrarily far to the right in $[0, \omega_1)$. In this section, we will introduce a special kind of uncountable subset, called a stationary set.

1.4.1 Definition: *A subset $S \subseteq \Omega$ is called stationary if for every uncount-*

able closed subset $C \subseteq \Omega$, $S \cap C \neq \emptyset$.

Clearly, if S contains an uncountable closed set, then S is stationary (see 1.3.1). Mary Ellen Rudin showed that that a special kind stationary set exists in Ω. The proof given here is an expanded version of Rudin's proof in [3].

1.4.2 Theorem: *In Ω, there exists a stationary set which contains no uncountable closed set.*

Proof: Consider the following types of subsets which obviously exist in Ω.
(1) sets which contain some uncountable closed set,
(2) sets which are disjoint from some uncountable closed set.

We want to show that the following type of subset exists in Ω:
(3) sets which intersect every uncountable closed set but which contain no uncountable closed set.

Because $|\Omega| \leq |\mathbb{R}|$, there is a one-to-one function f of Ω onto a subset of the real line $\mathbb{R}$. For each positive integer n, let $\mathcal{G}_n$ be a countable collection of intervals with length less than 1/n such that $\mathcal{G}_n$ covers $\mathbb{R}$. We will show that for some n, there exists $G \in \mathcal{G}_n$ such that $f^{-1}[G]$ is a type 3 set.

For contradiction, suppose there is no n and $G \in \mathcal{G}_n$ such that $f^{-1}[G]$ is a type 3 set.

Fix any positive integer n. Suppose, for contradiction, that for every $G \in \mathcal{G}_n$, $f^{-1}[G]$ is type 2. Thus, $f^{-1}[G] \subseteq \Omega \setminus C_G$ for some uncountable closed set C_G. So $C_G \subseteq \Omega \setminus f^{-1}[G]$. Therefore, $\cap\{C_G : G \in \mathcal{G}_n\} \subseteq \cap\{\Omega \setminus f^{-1}[G] : G \in \mathcal{G}_n\}$, which implies that $\cap\{C_G : G \in \mathcal{G}_n\} \subseteq \Omega \setminus \cup\{f^{-1}[G] : G \in \mathcal{G}_n\}$. But $\Omega \setminus \cup\{f^{-1}[G] : G \in \mathcal{G}_n\}$ is empty because $\mathcal{G}_n$ covers $\mathbb{R}$, and $\cap\{C_G : G \in \mathcal{G}_n\}$ is an uncountable closed set because it is the intersection of countably many uncountable closed sets. This is a contradiction, so there does exist a $G \in \mathcal{G}_n$ such that $f^{-1}[G]$ is not type 2. Since we have assumed that no such set is type 3, $f^{-1}[G]$ must be type 1.

For every n, fix $G_n \in \mathcal{G}_n$ such that $f^{-1}[G_n]$ is type 1. There exists $K_n \subseteq f^{-1}[G_n]$ where K_n is an uncountable closed set. Let $L = \cap\{K_n : n \geq 1\}$, which is also an uncountable closed set. Choose any $\alpha, \beta \in L$ with $\alpha \neq \beta$. Because f is a one-to-one function, $f(\alpha) \neq f(\beta)$. Thus, for every n, $\{f(\alpha), f(\beta)\} \subseteq f[K_n] \subseteq G_n$. So $\cap\{G_n : n \geq 1\}$ has at least two elements. But as n increases, the diameter of G_n goes to zero, so either $\cap\{G_n : n \geq 1\}$

is empty or it contains exactly one point. This contradicts our assumption that for every n and $G \in \mathcal{G}_n$, $f^{-1}[G]$ is not type 3. $\square$

1.4.3 Definition: *A set S is called bistationary if both S and $\Omega \setminus S$ are stationary.*

1.4.4 Theorem: *Type 3 subsets of Ω are bistationary.*

Proof: We know type 3 sets are stationary, so we need to examine $\Omega \setminus S$. Suppose $\Omega \setminus S$ is not stationary, then there exists an uncountable closed set C with $C \cap (\Omega \setminus S) \neq \emptyset$. But then C must be contained in S, and S is type 3. Contradiction. $\square$

Now will prove some general results about stationary sets.

1.4.5 Theorem: *Any stationary set S must be uncountable.*

Proof: Suppose S is a countable stationary set. Let C be any uncountable closed set. There exists $\gamma \in C$ with $\gamma > \beta$ for every $\beta \in S$ because C is uncountable while S is countable. The set $C \cap [\gamma, \omega_1)$ is also an uncountable closed set. But every element of $C \cap [\gamma, \omega_1)$ is greater than or equal to γ, and $\gamma > \beta$ for every $\beta \in S$. Therefore, $S \cap (C \cap [\gamma, \omega_1)) = \emptyset$. This is a contradiction because S is stationary. $\square$

1.4.6 Theorem: *If $S = \cup\{S_n : n \geq 1\}$ is stationary, then at least one of the sets S_n is stationary.*

Proof: Suppose $S = \cup\{S_n : n \geq 1\}$ is a stationary set such that for every n, S_n is not stationary. For each n, there exists an uncountable closed set C_n with $S_n \cap C_n = \emptyset$. The set $\cap\{C_n : n \geq 1\}$ is also an uncountable closed set. For every n, since $\cap\{C_n : n \geq 1\} \subseteq C_n$, $S_n \cap (\cap\{C_n : n \geq 1\}) = \emptyset$. Therefore, $(\cup\{S_n : n \geq 1\}) \cap (\cap\{C_n : n \geq 1\}) = \emptyset$. But this is impossible because $S = \cup\{S_n : n \geq 1\}$ is stationary. $\square$

Chapter 2: Lexicographic Orderings

When many mathematicians think of a typical linearly ordered set, they think of a straight line like the real numbers or even the ordinal space ω. The purpose of this chapter is to introduce some linearly ordered sets which are not naturally pictured as a straight line. We will explore some topological properties of these sets, and discover that they are topologically different from the real line.

2.0.1 Definition: *Let κ be a fixed ordinal number. Let F be the set of all functions $f : [0, \kappa) \to \mathbb{R}$. For distinct $f, g \in F$, define $f \preceq g$ to mean that either $f = g$ or $f(\alpha) < g(\alpha)$ where α is the first ordinal at which f and g disagree. The relation $\preceq$ is called the lexicographic ordering.*

2.0.2 Lemma: *The relation $\preceq$ is a linear ordering of the set F.*

Proof: The reflexive property is immediate from the definition. Suppose $f \prec g$ and $g \prec h$ for $f, g, h \in F$. Then $f(\alpha_1) < g(\alpha_1)$ where α_1 is the first ordinal at which f and g disagree, and $g(\alpha_2) < h(\alpha_2)$ where α_2 is the first ordinal at which g and h disagree. Without loss of generality, suppose $\alpha_1 \leq \alpha_2$. Then $g(\alpha_1) \leq h(\alpha_1)$ and $f(\alpha_1) < g(\alpha_1)$, so $f(\alpha_1) < h(\alpha_1)$ and α_1 is the first ordinal at which f and h disagree. Therefore, $f \prec h$.

Finally, we must check anti-symmetry. Fix any $f, g \in F$ with $f \neq g$. Let α be the first ordinal at which f and g disagree. Either $f(\alpha) < g(\alpha)$ or $g(\alpha) < f(\alpha)$, but not both, since $<$ is a linear order. So $f \prec g$ or $g \prec f$, but not both. $\square$

Note that $\prec$ would still be a linear order if any linearly ordered set was used in place of $\mathbb{R}$ in the definiton of F.

Let us examine how the lexicographic ordering works for the simplest example of $\kappa = 2$. Then F is the set of functions $f : [0, 2) \to \mathbb{R}$, so every f can be identified by the ordered pair $(f(0), f(1)) \in \mathbb{R}^2$. In this case, $(a, b) \prec (c, d)$ if and only if $a < c$, or $a = c$ and $b < d$.

We will now examine some lexicographically ordered spaces (with $\kappa = 2$) and their topological structure.

2.1 The Lexicographic Square

2.1.1 Definition: *Points of the lexicographic square X are ordered pairs (x, y) with $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Endow X with the lexicographic ordering and its associated open interval topology.*

2.1.2 Theorem: *The space X is first countable, i.e., each point has a countable neighborhood base.*

Proof: Fix any $z \in X$, and let $z = (x_z, y_z)$ for $x_z, y_z \in [0, 1]$. If $z = (0, 0)$, let $\mathcal{B}_z = \{[(0, 0), (0, 1/n)) : n \in \mathbb{Z}^+\}$. For any open set U with $z \in U$, there exists $b \in X$ with $z \in [0, b) \subseteq U$, so there exists a positive integer n with $[(0, 0), (0, 1/n)) \subseteq [0, b) \subseteq U$. Thus, $\mathcal{B}_z$ is a countable neighborhood base. Similarly, if $z = (1, 1)$, let $\mathcal{B}_z = \{((1, 1-1/n), (1, 1)] : n \in \mathbb{Z}^+\}$. If $z = (x_z, 0)$ for $x_z \neq 0$, let $\mathcal{B}_z = \{((x_z - 1/n, 1), (x_z, 1/n)] : n \in \mathbb{Z}^+\}$. If $z = (x_z, y_z)$ for $x_z \neq 0$ and $y_z \neq 0$, let $\mathcal{B}_z = \{((x_z, y_z - 1/n), (x_z, y_z + 1/n)) : n \in \mathbb{Z}^+\}$. In any of these cases, if U is an open set with $z \in U$, and $z \in (a, b) \subset U$ for $a, b \in X$, there exists $(x, y) \in \mathcal{B}_z$ with $z \in (x, y) \subseteq (a, b) \subseteq U$. So each $\mathcal{B}_z$ is a countable neighborhood base. $\square$

2.1.3 Theorem: *The space X is not separable.*

Proof: Let D be any dense subset of X. We want to show that D must be uncountable. For every $z \in [0, 1]$, let $U_z = ((z, 0), (z, 1))$. Then U_z is an open set. Fix any $z_1, z_2 \in [0, 1]$ with $z_1 \neq z_2$. Then $U_{z_1} \cap U_{z_2} = \emptyset$. For any $z \in [0, 1]$, $U_z \cap D \neq \emptyset$ because D is dense, so D contains at least one element of U_z for every $z \in [0, 1]$. These elements must be distinct since the U_z are disjoint. Therefore, D is uncountable since $[0, 1]$ is uncountable. $\square$

2.1.4 Theorem: *The space X is sequentially compact.*

Proof: Let $\langle a_n \rangle$ be any sequence of points in X, and for each n, let $a_n = (x_n, y_n)$. Since X is a linearly ordered set, $\langle a_n \rangle$ has a monotonic subsequence $\langle a_{n_k} \rangle$. Choosing a subsequence if necessary, assume $\langle a_n \rangle$ is monotonic. Suppose $\langle a_n \rangle$ is non-decreasing (the other case being analogous). If $\langle x_n \rangle$ is eventually constant at some $z_x \in [0, 1]$, choose N_0 such that $\langle x_n \rangle$ is constant for $n \geq N_0$. Let $z_y =$lub$\{y_n : n \geq N_0\}$, which exists because $\{y_n : n \geq N_0\}$ is a subset of $[0, 1]$. Then (z_x, z_y) is the least upper bound of the sequence, and $\langle a_n \rangle$ converges to (z_x, z_y). If $\langle x_n \rangle$ is not eventually constant, let $m =$lub$\{x_n : n \geq 1\}$, which exists because $\{x_n : n \geq 1\}$ is bounded above by 1. We must show that $(m, 0)$ is the least upper bound of $\langle a_n \rangle$. First, we will prove it is an upper bound. Suppose there is some $(x_N, y_N) \in \langle a_n \rangle$ such that $(x_N, y_N) > (m, 0)$. Since m is the least upper bound of $\langle x_n \rangle$ and $(x_N, y_N) > (m, 0)$, then $x_N = m$. But $\langle a_n \rangle$ is increasing, so, for every $n > N$, $x_n = x_N$, which contradicts our assumption that $\langle x_n \rangle$ is not eventually constant. Thus, $(m, 0)$ is an upper bound of $\langle a_n \rangle$. To show that it is the least upper bound, suppose there exists $(x, y) < (m, 0)$ which is also an upper bound of $\langle a_n \rangle$. Since m is the least upper bound of $\langle x_n \rangle$, and $\langle x_n \rangle$ is not eventually constant, we know that $x = m$. But then $y < 0$, which is impossible. Thus, $(m, 0)$ is the least upper bound of $\langle a_n \rangle$, so $\langle a_n \rangle$ converges to m. $\square$

2.1.5 Theorem: *X is not a metric space.*

Proof: For contradiction, suppose X is a metric space. Since X is not separable, it is not second countable and not Lindelöf. Since it is not Lindelöf, it is not compact. However, in a metric space, compactness and sequential compactness are equivalent, and X is sequentially compact. $\square$

2.2 The Double Arrow

2.2.1 Definition: *The points of Y, called the double arrow, are all possible ordered pairs (x, i) where $x \in \mathbb{R}$ and $i \in \{0, 1\}$, and Y carries the lexicographic ordering and usual open interval topology.*

2.2.2 Theorem: *Y is separable.*

Proof: Let $D = \{(q, i) : q \in Q \text{ and } i \in \{0, 1\}\}$. We want to show that D is dense. Fix any open set U. Fix $(x, 0) \in U$ (the case for $(x, 1) \in U$ is analogous). There is a basic open set $V = ((y, i), (x, 1))$ with $(x, 0) \in V \subseteq U$. Then $y < x$, so choose some rational q with $y < q < x$. Thus, $(q, 0) \in D \cap V \subset D \cap U$. Since $D \cap U \neq \emptyset$ for every open set U, the set D is dense. $\square$

2.2.3 Theorem: *Y is not second countable.*

Proof: Let $\mathcal{B}$ be any base for Y. We want to show that $\mathcal{B}$ is uncountable. For any nonzero $x \in \mathbb{R}$, examine the nonempty open set $((0, 0), (x, 1)) = ((0, 0), (x, 0)]$. There exists $B(x) \in \mathcal{B}$ with $(x, 0) \in B(x) \subseteq ((0, 0), (x, 0)]$, so $B(x) = (b_x, (x, 0)]$ for some $b_x \in ((0, 0), (x, 0))$. For every $y, z \in \mathbb{R}$ with

$y \neq z$, $B_y \neq B_z$ because $(y, 0) \neq (z, 0)$. Therefore, $B : \mathbb{R} \to \mathcal{B}$ is one-to-one. Since $\mathbb{R}$ is uncountable, so is $\mathcal{B}$. Therefore, Y has no countable base. $\square$

2.2.4 Theorem: *Y is not a metric space.*

Proof: Y is not second countable, but it is separable. In a metric space, these two are equivalent, so Y is not a metric space. $\square$

2.3 A Lexicographic Ordering with $\kappa = \omega$

In this section, we will let $\kappa = \omega$ and consider the space F of all functions from $[0, \omega)$ into the set $\mathbb{Z}$ of all integers. There are two natural topologies of F, the lexicographic open interval topology $\mathcal{L}$ and the product topology $\mathcal{P}$. Using the two lemmas below, we will show that these two topologies are equivalent.

The basic open sets of the product topology $\mathcal{P}$ have the form $P = \{f_0\} \times \{f_1\} \times ... \times \{f_n\} \times \mathbb{Z} \times \mathbb{Z} \times ...$ where f_i are any integers. The set P consists of all $g \in F$ with the property that $g(i) = f_i$ for $0 \leq i \leq n$ (note that there is no restriciton on $g(i)$ for $i > n$).

2.3.1 Lemma: $\mathcal{P} \subseteq \mathcal{L}$.

Proof: We will prove that $\mathcal{P} \subseteq \mathcal{L}$ by showing that if P is a basic open set in the topology $\mathcal{P}$ and $f \in P$, then there exist $u, v \in F$ such that $f \in (u, v) \subseteq P$. Let $P = \{f_0\} \times \{f_1\} \times ... \times \{f_n\} \times \mathbb{Z} \times \mathbb{Z} \times ...$ for some integers f_i, and fix any $f \in P$. Let $u \in F$ be any function with $u(i) = f_i$ for $0 \leq i \leq n$, and $u(n+1) = f(n+1) - 1$. Since $n+1$ is the first ordinal at which u and f disagree, and $u(n+1) < f(n+1)$, $u \prec f$. Let $v \in F$ be any function with $v(i) = f_i$ for $0 \leq i \leq n$, and $v(n+1) = f(n+1) + 1$, so $f \prec v$. Therefore, $f \in (u, v)$.

All that remains is to show that $(u, v) \subseteq P$. Fix any $g \in (u, v)$. Since $u(i) = f_i = v(i)$ for $0 \leq i \leq n$, and $u \prec g \prec v$, $g(i) = f_i$ for $0 \leq i \leq n$. Therefore, $g \in P$ for every $g \in (u, v)$. So $f \in (u, v) \subseteq P$. Thus, for any $U \in \mathcal{P}$ and for any $f \in U$, there exists P basic open set in $\mathcal{P}$ with $f \in P \subseteq U$, and there exists $(u, v) \in \mathcal{L}$ with $f \in (u, v) \subset P \subset U$, so $U \in \mathcal{L}$. So $\mathcal{P} \subseteq \mathcal{L}$. $\square$

2.3.2 Lemma: $\mathcal{L} \subseteq \mathcal{P}$.

Proof: Here we will show that every basic neighborhood of $\mathcal{L}$ contains a basic neighborhood of $\mathcal{P}$. Let (u,v) be any basic open set in $\mathcal{L}$, and let f be an element of (u,v). For any $g,h \in F$, let $n(g,h)$ =min$\{i \in \mathbb{Z}^+ : g(i) \neq h(i)\}$.

Let $P_1 = \{f(0)\} \times \{f(1)\} \times ... \times \{f(n(u,f)-1)\} \times \mathbb{Z} \times \mathbb{Z} \times ....$ It is clear that $f \in P_1$. For every $p \in P_1$, $u \prec p$ because $p(i) = f(i) = u(i)$ for $0 \leq i \leq n(u,f)-1$ and $p(n(u,f)) = f(n(u,f)) > u(n(u,f))$. Therefore, $P_1 \subseteq (u,\rightarrow)$.

Let $P_2 = \{f(0)\} \times \{f(1)\} \times ... \times \{f(n(v,f)-1)\} \times \mathbb{Z} \times \mathbb{Z} \times ....$ Note that $f \in P_2$. For every $p \in P_2$, $p \prec v$ because $p(i) = f(i) = v(i)$ for $0 \leq i \leq n(v,f)-1$ and $p(n(v,f)) = f(n(v,f)) < v(n(v,f))$. Therefore, $P_2 \subseteq (\leftarrow, v)$.

Since $P_1 \subseteq (u,\rightarrow)$ and $P_2 \subseteq (\leftarrow, v)$, $P_1 \cap P_2 \subseteq (u,v)$. Since P_1, P_2 are the basic sets and $f \in P_1 \cap P_2$, there exists a basic open set $P \in \mathcal{P}$ with $f \in P \subseteq P_1 \cap P_2 \subseteq (u,v)$. Therefore, for any $(u,v) \in \mathcal{L}$ and $f \in (u,v)$, there is a basic open set $P \in \mathcal{P}$ with $f \in P \subseteq (u,v)$, so $\mathcal{L} \subseteq \mathcal{P}$.$\square$

Therefore, we have shown that $\mathcal{P} \subseteq \mathcal{L}$ and $\mathcal{L} \subseteq \mathcal{P}$. So $\mathcal{P} = \mathcal{L}$; the lexicographic order topology is equivalent to the product topology. We will use this knowledge to learn about the lexicographic product space $(F, \mathcal{L})$.

2.3.3 Theorem: *The topological space $(F,\mathcal{L})$ is metric space.*

Proof: The set F of all functions $f : [0,\omega) \rightarrow \mathbb{Z}$ can be viewed as the set of functions $f : [0,\omega) \rightarrow \cup_{n \geq 0} Z_n : f(n) \in Z_n$ for every $n \in [0,\omega)\}$ for each $Z_n = \mathbb{Z}$. Thus, $F = \Pi\{Z_n : n \in [0,\omega)\}$. Since the product of countably many metric spaces is also a metric space, and $\mathbb{Z}$ is a metric space, $\Pi\{Z_n : n \in [0,\omega)\} = \mathbb{Z} \times \mathbb{Z} \times ...$ is a metric space with its product topology. Since $\mathcal{L}$ is equivalent to the product topology, $(F,\mathcal{L})$ is a metric space. $\square$

2.3.4 Theorem: *The space $(F,\mathcal{L})$ is separable.*

Proof: We will show that the countable collection $\mathcal{B} = \{\{i_0\} \times \{i_1\} \times ... \times \{i_n\} \times \mathbb{Z} \times \mathbb{Z} \times ...$, for $i_j \in \mathbb{Z}$ and $n < \omega$, is a base for $(F,\mathcal{L}) = (F,\mathcal{P})$. Let U be open in $(F,\mathcal{P})$ and $f \in U$. There is a basic open set $B = B_0 \times B_1 \times ... \times B_n \times \mathbb{Z} \times \mathbb{Z} \times ...$ with $f \in B \subseteq U$. Then $f(i) \in B_i$ for $1 \leq i \leq n$. Then $\{f(0)\} \times \{f(1)\} \times ... \times \{f(n)\} \times \mathbb{Z} \times \mathbb{Z} \times ... \subseteq B_0 \times B_1 \times ... \times B_n \times \mathbb{Z} \times \mathbb{Z} \times ... \subseteq U$. Thus, $\mathcal{B}$ is a countable base for $(F,\mathcal{P}) = (F,\mathcal{L})$.$\square$

Chapter 3: Trees and Lines

In this chapter, we will examine partially ordered sets called trees. We will study methods for constructing trees from lines and constructing lines from trees. Much of the material in this section is based on [1].

3.0.1 Definition: *A tree is a partially ordered set (poset) $\langle T, \leq \rangle$ such that for every $x \in T$, the set $\{y \in T : y < x\}$ is well ordered by $\leq$.*

We must consider the following definitions before we begin our study of trees. Let T be any tree.

3.0.2 Definition: *For $x \in T$, the height of x, denoted ht(x), is the cardinality of $\{y \in T : y < x\}$. The α'th level of T is the set $T_\alpha = \{x \in T : ht(x) = \alpha\}$. The successors of $x \in T$ are $T^x = \{y \in T : x \leq y\}$.*

3.0.3 Definition: *A chain is a linearly ordered subset of T. A branch is a chain which is $\leq$-closed (i.e. if x is in the branch, and $y \leq x$, then y is in the branch). An antichain is a set of pairwise incomparable elements of T.*

Note that a branch may, or may not, have a largest point, so we will define the height of a branch b to be $\sup\{h(x) : x \in b\}$.

We will also define a special kind of tree.

3.0.5 Definition: *A Souslin tree is a tree with cardinality ω_1 such that every chain and antichain is countable.*

3.0.6 Definition: *A tree T of height ω_1 is normal if:*
(i) there is a unique least point, called the root of T,
(ii) each point has successors at each greater level $< \omega_1$,
(iii) each point x has at least two immediate successors,
(iv) for any limit ordinal β, each branch with height β has at most one immediate successor, and
(v) each level T_ν is countable.

3.1 The Souslin Problem

The material in this section is a representation of material from [1].

There are many different "tree to line" and "line to tree" constructions. The motivation behind one of them is the Souslin problem. First we must set the stage for these constructions by introducing the Souslin hypothesis.

3.1.1 Definition: *A linearly ordered set S is densely ordered if $a, b \in S$ and $a < b$ implies $(a, b) \neq \emptyset$. A linearly ordered set S is complete if each nonempty subset that is bounded above has a least upper bound in S. A set S is an ordered continuum if S is densely ordered, complete, and has no endpoints.*

It is known that every separable ordered continuum S is order isomorphic to $\mathbb{R}$ with the usual ordering. In 1920, M. Souslin asked whether the same result could be attained if separability was replaced by the following weaker assumption [4].

3.1.2 The Souslin property: *Every family of pairwise disjoint open intervals in S is countable.*

3.1.3 Souslin's hypothesis: *Every ordered continuum with the Souslin property is order isomorphic to $\mathbb{R}$.*

Souslin's hypothesis is equivalent to the statement "every ordered continuum with the Souslin property is separable."

3.1.4 Definition: *An ordered continuum is a Souslin line if it has the Souslin property, but it is not separable.*

Note that the existence of a Souslin line is equivalent to the negation of Souslin's hypothesis. A Souslin line exists if and only if Souslin's hypothesis is false. This problem was studied for 50 years before being "solved." It is now known that it is impossible to prove or disprove Souslin's hypothesis under the normal axioms of set theory (ZFC). However, what is relevant here is the connection between a Souslin line and a Souslin tree. We will use the following lemmas in our construction.

3.1.5 Lemma: *If a tree T has the property that every non-maximal point has at least two immediate successors, and T has no uncountable antichains, then T has no uncountable chains.*

Proof: Suppose $C \subseteq T$ is an uncountable chain. For any $b_\alpha \in C$, let $b_{\alpha+1}$ be the successor of b_α that is in C, and let $\hat{b}_{\alpha+1}$ be a successor of b_α that is not in C. We will show that, for any $\alpha, \beta < \omega_1$ with $\alpha < \beta$, $\hat{b}_{\alpha+1}$ and $\hat{b}_{\beta+1}$ are incomparable. First note that it is impossible for $\hat{b}_{\beta+1} \leq \hat{b}_{\alpha+1}$ because it would imply $\beta + 1 \leq \alpha + 1$. For contradiction, consider the other case that $\hat{b}_{\alpha+1} \leq \hat{b}_{\beta+1}$. Since b_β and $b_{\alpha+1}$ are in C, $b_{\alpha+1}$ is a predecessor (or equal to) b_β. Since b_β is a predecessor of $\hat{b}_{\beta+1}$, $b_{\alpha+1}$ is a predecessor of $\hat{b}_{\beta+1}$. But because $\hat{b}_{\alpha+1} \leq \hat{b}_{\beta+1}$, $\hat{b}_{\alpha+1}$ is a predecessor of $\hat{b}_{\beta+1}$. But it is impossible for both $b_{\alpha+1}$ and $\hat{b}_{\alpha+1}$ to be predecessors of $\hat{b}_{\beta+1}$ because they lie in the same level $\alpha + 1$ and are distinct, so they are not comparable. Therefore, $\hat{b}_{\alpha+1}$ and $\hat{b}_{\beta+1}$ are incomparable, so an uncountable chain leads to an uncountable antichain. $\square$

The next series of lemmas will show that, given any Souslin tree, one can construct a normal Souslin tree.

3.1.6 Lemma: *Let T be any Souslin tree. Then there exists a $T' \subset T$ such that*

1) T' is a Souslin tree, and

2) for each $x \in T'$, the set $\{y \in T' : x \leq y\}$ is uncountable.

Proof: For each $\alpha < \omega_1$, some $x \in$ level α of T has the property that $|T^x| = \omega_1$. Otherwise, we could write $T = \{x : ht(x) < \alpha\} \cup (\cup\{T^x : ht(x) = \alpha\}) \cup T_\alpha$, showing that the uncountable set T is a countable union of countable sets, and that is impossible.

Let $T' = \{x : |T^x| = \omega_1\}$. Note that T' is a Souslin tree because it is uncoutable, but it has no uncountable chains or antichains. Fix $x \in T'$. Because T^x is uncountable, for each $\alpha > ht(x)$, some point $y_\alpha \in T^x \cap T_\alpha$ has $|T^{y_\alpha}| = \omega_1$, so $y_\alpha \in T'$. But then $\{y_\alpha : ht(x) < \alpha < \omega_1\} \subseteq \{y \in T' : x \leq y\}$ showing that $(T')^x$ is uncountable for each $x \in T'$. Therefore, T' is a Souslin tree with the property that for each $x \in T'$, the set $\{y \in T' : x \leq y\}$ is uncountable. $\square$

3.1.7 Lemma *Let T be a Souslin tree in which each element has ω_1 successors. Then there is a $T' \subseteq T$ such that*

1) T' is Souslin,

2) if $x \in T'$ has level α in T', then x has at least two successors at level $\alpha + 1$ of T', and

3) each $x \in T'$ has $|(T')^x| = \omega_1$.

Proof: Denote the root of T by 0. Fix $x \in T$. Because $|T^x| = \omega_1$ and each level of T is countable, $T^x \cap T_\alpha \neq \emptyset$ for each $\alpha < \omega_1$. If some $x \in T$ had $|T^x \cap T_\alpha| = 1$ for each $\alpha > ht(x)$, then T^x would be an uncountable chain in T. Hence for each $x \in T$, there is some ordinal $\lambda(x) > ht(x)$ such that $|T^x \cap T_{\lambda(x)}| \geq 2$.

Let $\mu(0) = \lambda(0)$. The set $\cup\{T_\alpha : 0 \leq \alpha < \lambda(0)\}$ is countable, so the ordinal $\mu(1) = \sup\{\lambda(x) : x \in \cup\{T_\alpha : 0 \leq \alpha < \lambda(0)\}\}$ is countable. Observe that each $x \in \cup\{T_\alpha : 0 \leq \alpha < \lambda(0)\}$ has at least two successors at level $\mu(1)$. The set $\cup\{T_\alpha : 0 \leq \alpha < \mu(1)\}$ is countable, so that $\mu(2) = \sup\{\lambda(x) : x \in \cup\{T_\alpha : 0 \leq \alpha < \mu(1)\}\}$ is countable. Observe that each element of $\cup\{T_\alpha : 0 \leq \alpha < \mu(1)\}$ has at least two successors at level $\mu(2)$. For induction, suppose $\gamma < \omega_1$ and for each $\alpha < \gamma$, we have found $\mu(\alpha)$ such that if $\alpha < \beta < \gamma$, then

a) $\mu(\alpha) < \mu(\beta)$, and

b) each point of $\cup\{T_\delta : 0 \leq \delta < \mu(\alpha)\}$ has at least two successors at level $\mu(\beta)$ of T.

The set $\cup\{T_\delta : 0 \leq \delta < \mu(\alpha)$ for some $\alpha < \gamma\}$ is countable, so the ordinal $\mu(\alpha) = \sup\{\lambda(x) : x \in \cup\{T_\delta : 0 \leq \delta < \mu(\alpha)$ for some $\alpha < \gamma\}\}$ is countable, and for each $\alpha < \gamma$, if $x \in \cup\{T_\delta : 0 \leq \delta < \mu(\alpha)\}$, then x has at least two successors at level $\mu(\gamma)$.

This recursion produces a strictly increasing transfinite sequence $\{\mu(\alpha) : 0 \leq \alpha < \omega_1\}$ with the property that each x belonging to level $\mu(\alpha)$ of T has at least two successors at level $\mu(\alpha + 1)$ of T, and hence has at least two successors at level $\mu(\beta)$ of T whenever $\beta > \alpha$.

Define a new tree $T' = \cup\{T_{\mu(\alpha)} : 0 \leq \alpha < \omega_1\} \cup \{0\}$, and order T' by restricting the order of T to its subset. Then the $0th$ level of T' is the same as the $0th$ level of T, the first level of T' is the $\mu(1)$-level of T, etc. If $x \in T'$, then x has at least two successors that belong to level $ht(x) + 1$ of T'. Furthermore, T' is uncountable and contains no uncountable chains or antichains, so T' is Souslin.

Fix any $x \in T'$. Then the level of x in T is $\mu(\alpha)$ for some $\alpha < \omega_1$. For each $\beta > \alpha$, x has a successor in level $\mu(\beta)$ of T, which is a subset of T'. Thus x has successors in uncountably many levels of T', so x has uncountably many successors in T'. $\square$

We will use the following definition in our next lemma.

3.1.8 Definition: *Let L be a chain of any tree T. If $ht(L) = \lambda$ is a limit ordinal, we will say that L is continuable in T if for some $x \in T_\lambda$, $L \subseteq \{y \in T : y \leq x\}$.*

3.1.9 Lemma: *Suppose T is a Souslin tree such that every $x \in T$ has uncountably many successors, and each $x \in T$ has at least two successors at level $ht(x) + 1$. Then there exists a $T' \subset T$ such that*

1) T' is a Souslin tree,

2) for each $x \in T'$, the set $\{y \in T' : x \leq y\}$ is uncountable,

3) each $x \in T'$ has at least two successors at level $ht(x) + 1$ of T' where $ht(x)$ is the height of x in T', and

4) if b is a branch of T' with height λ where λ is limit ordinal, and if b is countinuable in T', then there is exactly one $x \in T'$ with $b \subseteq \{y \in T' : y \leq x\}$.

Proof: Suppose b is a branch of T with limit ordinal height λ. For each continuable λ-branch b of T, let $T(\lambda, b) = \{x \in T_\lambda : b \cup \{x\}$ is a chain$\}$. Observe that if b, c are distinct continuable λ-branches of T, then $T(\lambda, b) \cap T(\lambda, c) = \emptyset$. For each continuable λ-branch b of T, choose a point $x_b \in T(\lambda, b)$. Let $T' = \{x_b : b$ is a continuable λ-branch of T where $\lambda < \omega_1$ is a limit ordinal$\}$, and let T' inherit its ordering from T.

Claim 1: For each $x_b \in T'$, the set $(T')^{x_b} = \{x_c \in T' : x_b \leq x_c\}$ is uncountable.

Proof: Fix x_b and let λ be the height of x_b in T. Because x_b has uncountably many successors in T, for each limit ordinal $\mu > \lambda$, there is a point $y(x_b, \mu) \in T_\mu$ with $x_b < y(x_b, \mu)$. Then $c(\mu) = \{z \in T : z < y(x_b, \mu)\}$ is a branch of T and is continuable in T because $y(x_b, \mu) \in T$. Hence $x_{c(\mu)}$ exists and belongs to T'. Observe that the set of predecessors of $x_{c(\mu)}$ in T is precisely the set $c(\mu)$, and becuase $x_b < y(x_b, \mu)$, we know that $x_b \in c(\mu)$. Hence, $x_b < x_{c(\mu)}$ so that $x_c(\mu) \in (T')^{x_b}$. Observe that if $\mu < \nu$ are both limit ordinals greater than the height of x_b in T, then $x_{c(\mu)}$ and $x_{c(\nu)}$ belong to different levels of T and are therefore distinct. Hence $(T')^{x_b}$ contains the uncountable set $\{x_{c(\mu)} : \mu >$ height of x_b in T and μ is a limit ordinal$\}$, so $(T')^{x_b}$ is uncountable.

Claim 2: If $x_b \in T'$ and the height of x_b in T' is α, then there are elements $x_c \neq x_d$ in T' with $x_b < x_c$ and $x_b < x_d$ such that $\alpha + 1$ is the height of x_c and x_d in T'.
Proof: Let λ be the level of x_b in T. We know that x_b has at least two successors in level $\lambda + 1$, say u and v. We also know that u has a successor y at level $\mu = \lambda + \omega$, and v has a successor z at level μ. Let $c = \{w \in T : w < y\}$, and let $d = \{w \in T : w < z\}$. Then c and d are distinct branches of T, each with height μ, and each continuable in T. Therefore, x_c and x_d are distinct members of T' with height $\alpha + 1$ in T', and $x_b < x_c$ and $x_b < x_d$. Therefore, each $x_b \in T'$ has at least two successors in T' at the next level.

Claim 3: If b is any continuable λ-branch of T', where λ is a limit ordinal, then b has exactly one continuation in T'.
Proof: Let $d = \{w \in T :$ for some $x \in b, w \leq x\}$. Then d is a branch of T whose height is some limit ordinal μ. It two distinct points x_b and x_c of T' were both continuations of b, then x_b and x_c would both be continuations of the μ-branch d belonging to T_μ, and that is impossible because of the way the points of T' were chosen. Therefore, each continuable λ-branch of T' has exactly one continuation in T'. $\square$

3.1.10 Lemma: *Every Souslin tree has a subset which is a normal Souslin tree.*

Proof: Follows from Lemmas 3.1.6, 3.1.7, and 3.1.9. $\square$

3.1.11 Theorem: *There exists a Souslin line if and only if there exists a Souslin tree.*

Proof: First we will prove that a Souslin line gives rise to a Souslin tree. Suppose S is a Souslin line, i.e. every family of pairwise disjoint open intervals of S is countable, but S is not separable. We will repeatedly apply the following construction: let D be a countable subset of S. Since S is not separable, D is not dense, so $S \setminus \overline{D} \neq \emptyset$ where $\overline{D}$ denotes the closure of D. Since $\overline{D}$ is closed, $S \setminus \overline{D}$ is open, so $S \setminus \overline{D}$ is the union of a nonempty collection $I(D)$ of pairwise disjoint open intervals. The collection $I(D)$ is countable since S has the Souslin property.

Next, we recursively define countable sets D_α and I_α. Let D_0 consist of any fixed point from S. By induction, assume D_β is defined for $\beta < \alpha$, and set $I_\alpha = I(\cup_{\beta<\alpha} D_\beta)$. Let D_α consist of one point from each interval in I_α. Let $T = \cup_{\alpha<\omega_1} I_\alpha$ and partially order T by reverse inclusion (so $\leq$ means $\supseteq$). We want to show that $\langle T, \supseteq \rangle$ is a Souslin tree.

Every antichain in T is just a family of pairwise disjoint open intervals in S, so the antichain is countable since S has the Souslin property. The tree T also has the property that each nonmaximal point x has at least two immediate successors. Therefore, by Lemma 3.1.5, an uncountable chain implies an uncountable antichain. But every antichain is countable, so every chain is countable. The tree T must have uncountably many levels because, if it did not, some countable level, say I_α, of T would be empty. Thus, $\overline{\cup_{\beta<\alpha} D_\beta} = S$, but that is impossible because $\cup_{\beta<\alpha} D_\beta$ is countable and S is not separable. Therefore, the cardinality of T is ω_1. So T is a Souslin tree.

Next we need to show that the existence of a Souslin tree implies the existence of a Souslin line. By Lemmas 3.1.7 and 3.1.9, if there exists a Souslin tree, then there exists a normal Souslin tree. Suppose $\langle T, \leq \rangle$ is a normal Souslin tree. We can assume it satisfies the property that every point in T has ω immediate successors, because we can form such a tree from a normal Souslin tree by throwing out all nodes not on limit ordinal levels.

Claim 1: For $\alpha > 0$, there is a linear ordering $\leq_\alpha$ of T_α of order type of the rationals satisfying the following property: for ordinals α and β, if $\alpha < \beta$, and if $x, y \in T_\beta$ and $x', y' \in T_\alpha$ with $x' < x$ and $y' < y$ and $x' <_\alpha y'$, then $x <_\beta y$.

Proof: Since every element of T has ω immediate successors, each level higher than T_0 is countably infinite. Give T_1 an arbitrary ordering of order type of the rationals. At level T_2, if $a, b \in T_2$ have distinct predecessors $a', b' \in T_1$, respectively, and $a' <_1 b'$, then let $a <_2 b$. For all $a, b \in T_2$ which have the same predecessor in T_1, arbitrarily order them with order type of the rationals, which is possible because there are ω successors of each element of T_1. Then $<_2$ is a linear ordering.

Suppose $\beta < \omega_1$ and we have defined orderings $<_\alpha$ of L_α for each $\alpha < \beta$ in such a way that the compatibility restrictions in the claim hold. If β is not a limit ordinal, then $\beta = \alpha + 1$ for some α and we proceed to define T_β from T_α just as we defined T_2 from T_1. If β is a limit ordinal, then each $a \in L_\beta$ is the unique successor of the branch $\{y \in T : y < a\}$. Hence if $a \neq b$ belong to T_β then there is some $\alpha < \beta$ such that some $a' \neq b' \in T_\alpha$ have $a' < a$

and $b' < b$. Then either $a' <_\alpha b'$ or $b' <_\alpha a'$, and we define the relation $<_\beta$ between a and b to be the same as the relation $<_\alpha$ between a' and b'. So $<_\beta$ is a linear ordering, and $<_\beta$ is of order type of rationals.

Let S be the set of maximal branches of T, which exist by Zorn's Lemma. When b is a branch of T, let b_α be the point in b at height α. Order S by $\leq$ as follows: let b, d be distinct points of S. Then there is an α with $b_\alpha \neq d_\alpha$. Choose the least such α. Then for all $\beta < \alpha$, we have $b_\beta = d_\beta$, and because level α of the tree is linearly ordered by $<_\alpha$, either $b_\alpha <_\alpha d_\alpha$ or vice versa. If $b_\alpha <_\alpha d_\alpha$, define $b < d$ in S. Otherwise, define $d < b$ in S. Note that S is densely ordered because the ordering of T_α for every $\alpha < \omega_1$ is of order type of the rationals.

We will now show that S has the Souslin property. For every $x \in T$, let $I_x = \{b \in S : x \in b\}$.

Claim 2: For every $x \in T$, I_x is open in $(S, \leq)$.

Proof: Fix any $x \in T$, and fix $\alpha \in \omega_1$ such that $x \in T_\alpha$. Let $\beta = \alpha + 1$. Choose any $b \in I_x$, and consider b_β. Since x has ω immediate successors at level β, and the level has order type of the rationals, there exists $a_\beta, c_\beta \in T_\beta$ with $a_\beta <_\beta b_\beta <_\beta c_\beta$, and $x < a_\beta$ and $x < c_\beta$ in $\langle T, \leq \rangle$. By Zorn's Lemma, there exist maximal branches $a, c \in I_x$ with $a_\beta \in a$ and $c_\beta \in c$. Since $x = a_\alpha = b_\alpha = c_\alpha$, a, b, and c agree at every level less than α (otherwise, some element of the tree would have two immediate predecessors). So β is the first level at which a, b, c disagree, so $a < b < c$ in $(S, \leq)$. Therefore, $b \in (a, c) \subseteq I_x$, so I_x is open.

Claim 3: For any open interval J in the set $(S, <)$, there exists $x_J \in T$ such that $I_{x_J} \subseteq J$.

Proof: Choose $b < d$ in J. For α the first ordinal with $b_\alpha \neq d_\alpha$ and $b_\beta = d_\beta$ for every $\beta < \alpha$, fix any $x_J \in T_\alpha$ such that $b_\alpha <_\alpha x_J <_\alpha d_\alpha$. Such an x_J exists because T_α is ordered like the rationals. Every $y \in I_{x_J}$ has $y_\alpha = x_J$, so by the compatibility property of the ordering of levels, $b_\gamma <_\gamma y_\gamma$ at every level γ where b and y disagree. Similarly, $y_\gamma <_\gamma d_\gamma$ at every level γ where y and d disagree. We will now show that $b_\gamma = y_\gamma = d_\gamma$ for every $\gamma < \alpha$. If $y_\gamma < b_\gamma = d_\gamma$, then by the compatibility property, $y_\alpha < b_\alpha$, but that is impossible. Similarly, if $y_\gamma > b_\gamma = d_\gamma$, then by compatibility, $y_\alpha > d_\alpha$. Therefore, α is the first level at which b, y, and d disagree. Therefore, $b < y < d$ in $(S, <)$. Therefore, $I_{x_J} \subset J$.

We must show that when open intervals K and L in $(S, \leq)$ are disjoint, x_K and x_L are incomparable in T. If $x_K \leq x_L$ in T, then there would exist a branch b_0 containing x_K and x_L, so $b_0 \in I_{x_K}$ and $b_0 \in I_{x_L}$. But this is impossible since K and L are disjoint, so x_K and x_L are incomparable in T. Therefore, if there exists an uncountable family of disjoint open intervals in $(S, \leq)$, then there exists an uncountable collection of incomparable elements in T, i.e. an uncountable antichain. Since a Souslin tree has no uncountable antichain, every family of disjoint open intervals in $(S, \leq)$ must be countable.

All that remains is to show that S is not separable. Suppose there exists countable dense set $D \subseteq S$. Let $\alpha = \sup\{ht(b) : b \in D\}$. We know that $\alpha < \omega_1$ because there are no uncountable chains in S. So if we take $x \in T_\alpha$, I_x contains an open interval disjoint from D, which is impossible since D is dense. Therefore, S is not separable. So S is a densely ordered set with the Souslin property which is not order isomorphic to $\mathbb{R}$ since it is not separable.$\square$

3.2 Applying the Line-to-Tree Construction to $\mathbb{R}$

Following the basic steps of the line-to-tree construction of the proof of Theorem 1.2.7, one can create a tree from the real line. What would this tree look like? Since the real line is separable, at some level $\alpha \leq \omega_1$, it seems that $I_\alpha = I(\cup_{\beta<\alpha} D_\beta)$ may be the empty set because $\cup_{\beta<\alpha} D_\beta$ could be dense, so $\mathbb{R} \setminus \overline{\cup_{\beta<\alpha} D_\beta} = \emptyset$. In fact, we will prove that any tree resulting from this construction will be countable. We will make use of the following lemma.

3.2.1 Lemma: *Suppose $\{x_\alpha : \alpha < \omega_1\}$ is a set of real numbers such that if $\alpha < \beta < \omega_1$, then $x_\alpha \leq x_\beta$. Then there is an $\alpha < \omega_1$ such that $x_\alpha = x_\beta$ for each $\alpha \leq \beta < \omega_1$.*

Proof: If not, one can recursively define ordinals α_β for $\beta < \omega_1$ such that if $\beta < \gamma < \omega_1$, then $x_{\alpha_\beta} < x_{\alpha_\gamma}$. Then the collection $\{(x_{\alpha_\beta}, x_{\alpha_{\beta+1}}) : \beta < \omega_1\}$ is an uncountable pairwise disjoint collection of open intervals in $\mathbb{R}$. But that is impossible because $\mathbb{R}$ is separable. $\square$

3.2.2 Theorem: *Suppose the tree T consists of open intervals of $\mathbb{R}$, partially ordered by reverse inclusion, and has the property that for each $I, J \in T$, either I and J are comparable in the ordering of T, or else $I \cap J = \emptyset$. Then T is countable.*

Proof: For contradiction, suppose T is uncountable. For each rational number q, let $T(q) = \{J \in T : q \in J\}$. Since the rationals are dense, $T = \cup\{T(q) : q \in \mathbb{Q}\}$. Thus, for some $q_0 \in \mathbb{Q}$, $T(q_0)$ must be uncountable.

Because $q_0 \in I \cap J$ for each $I, J \in T(q_0)$, the property that overlapping intervals are comparable implies that $I \subseteq J$ or $J \subseteq I$ for each $I, J \in T(q_0)$. Therefore, $T(q_0)$ is a chain. Let α_0 be the first level of the tree such that $T(q_0) \cap T_{\alpha_0} \neq \emptyset$, i.e. the first level which $T(q_0)$ intersects. Let $J(\alpha_0)$ be the member of $T(q_0) \cap T_{\alpha_0} \neq \emptyset$, which is unique because otherwise two elements of $T(q_0)$ would be on the same level, and thus not comparable. Recursively choose ordinals α_β and intervals $J(\alpha_\beta) \in T(q_0) \cap T_{\alpha_\beta}$ so that $\alpha_\beta < \alpha_\gamma$ whenever $\beta < \gamma$. Then $J(\alpha_\gamma) \subset J(\alpha_\beta)$ and $J(\alpha_\gamma) \neq J(\alpha_\beta)$ whenever $\beta < \gamma < \omega_1$. For each α_β, let $l(\alpha_\beta)$ be the left endpoint of $J(\alpha_\beta)$, and let $r(\alpha_\beta)$ be the right endpoint of $J(\alpha_\beta)$. Thus, $\beta < \gamma < \omega_1$ implies that $l(\alpha_\beta) \leq l(\alpha_\gamma)$ and $r(\alpha_\beta) \geq r(\alpha_\gamma)$. By the lemma above, there exists β_1 such that $\gamma \geq \beta_1$ implies $l(\alpha_{\beta_1}) = l(\alpha_\gamma)$. Analogously, there exists β_2 such that $\gamma \geq \beta_2$ implies $r(\alpha_{\beta_2}) = r(\alpha_\gamma)$. So for any $\delta, \gamma \geq$max(β_1, β_2), $(l(\alpha_\delta), r(\alpha_\delta)) = (l(\alpha_\gamma), r(\alpha_\gamma))$, i.e. $J(\alpha_\delta) = J(\alpha_\gamma)$ for all $\delta, \gamma \geq$max(β_1, β_2). Therefore, $T(q_0) = \{J_\alpha : \alpha < \omega_1\}$ is countable, which contradicts our assumption. $\square$

Chapter 4: Ultrafilter Orderings

In this chapter, we will define a special collection of sets called an ultrafilter. This construction will then be used to define and linearly order equivalence classes on the set of functions from ω to ω (denoted ω^ω). We will then study the properties of this linearly ordered set.

4.1 Filters and Ultrafilters

4.1.1 Definition: *A filter on a set X is a nonempty collection $\mathcal{F}$ of subsets of X satisfying*

a) $\emptyset \notin \mathcal{F}$
b) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$
c) if $A \subset B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

Let us consider an example of a filter. For any infinite set X, the collection $\{S \subseteq X : S \text{ is co-finite}\}$ is a filter on X. It is obvious that this collection has properties a) and b). Property c) follows because if $A \subset B$ and $A \in \mathcal{F}$, then $(X \setminus A)$ is finite and $(X \setminus B) \subseteq (X \setminus A)$. Thus, $(X \setminus B)$ is finite, so $B \in \mathcal{F}$.

Next we will prove, using Zorn's Lemma, that a maximal filter exists.

4.1.2 Theorem: *Given any filter $\mathcal{F}$ on a set X, there is a filter $\mathcal{U}$ on X such that*
a) $\mathcal{F} \subseteq \mathcal{U}$
b) no filter on X strictly contains $\mathcal{U}$.

Proof: Let $\mathcal{P} = \{\mathcal{W} : \mathcal{W} \text{ is a filter and } \mathcal{F} \subseteq \mathcal{W}\}$. Order $\mathcal{P}$ by inclusion. We want to use Zorn's Lemma to prove that $\mathcal{P}$ has a maximal element. The collection $\mathcal{P}$ is nonempty because $\mathcal{F} \in \mathcal{P}$. Let $\mathcal{C} \subseteq \mathcal{P}$ be any chain. Let $\mathcal{W}_0 = \cup\{\mathcal{W} : \mathcal{W} \in \mathcal{C}\}$. The set $\mathcal{W}_0$ is an upper bound of $\mathcal{C}$, and $\mathcal{W}_0 \in \mathcal{P}$. Therefore, $\mathcal{P}$ is inductive. Thus, by Zorn's Lemma, $\mathcal{P}$ has a maximal element, which is the filter $\mathcal{U}$ above. $\square$

4.1.3 Definition: *An ultrafilter on X is any maximal filter on X.*

We have some easy ways to recognize ultrafilters. The following theorem states properties which characterize ultrafilters. We will present two additional characterizations in 4.3 and 5.5.

4.1.4 Theorem: *Let $\mathcal{F}$ be a filter on X. The following are equivalent:*

1) $\mathcal{F}$ is an ultrafilter

2) if $A \subseteq X$ and $A \notin \mathcal{F}$, then for some $B \in \mathcal{F}$, $A \cap B = \emptyset$

3) if $A_1 \cup A_2 \in \mathcal{F}$, then $A_1 \in \mathcal{F}$ or $A_2 \in \mathcal{F}$

4) if $A \subseteq X$, then either $A \in \mathcal{F}$ or $(X - A) \in \mathcal{F}$.

Proof: First, we will show 1) $\Rightarrow$ 2).
Let $\mathcal{U}$ be an ultrafilter. Fix any $A \subseteq X$ with $A \notin \mathcal{U}$. Suppose for contradiction that for every $B \in \mathcal{U}$, $A \cap B \neq \emptyset$. For each $B \in \mathcal{U}$, let $C_B = A \cap B$. Consider $\mathcal{V} = \mathcal{U} \cup \{C_B : B \in \mathcal{U}\}$. Let $\mathcal{W}$ be the collection of all sets which contain an element of $\mathcal{V}$.

We will show that $\mathcal{W}$ is a filter. First we will show that the intersection of two elements in $\mathcal{W}$ is also in $\mathcal{W}$. Since $\mathcal{W}$ is the collection of sets containing an element of $\mathcal{V}$, we just need to show that the intersection of two elements elements $D, E \in \mathcal{V}$ is also in $\mathcal{V}$. Obviously, if $D, E \in \mathcal{U}$, then $D \cap E \in \mathcal{V}$. If $D \in \mathcal{U}$ and $E = C_{B_0}$ for some $B_0 \in \mathcal{U}$, then $D \cap E = D \cap C_{B_0} = D \cap (A \cap B_0) = A \cap (B_0 \cap D)$. But $B_0 \cap D \in \mathcal{U}$, so $A \cap (B_0 \cap D) = C_{B_0 \cap D} \in \mathcal{V}$. Similarly, if $D = C_{B_1}$ and $E = C_{B_2}$ for some $B_1, B_2 \in \mathcal{U}$, then $D \cap E = (A \cap B_1) \cap (A \cap B_2) = A \cap (B_1 \cap B_2)$. But $(B_1 \cap B_2) \in \mathcal{U}$, so $D \cap E = C_{B_1 \cap B_2} \in \mathcal{V}$.

Clearly, every set which contains an element of $\mathcal{W}$ is also in $\mathcal{W}$. Therefore $\mathcal{W}$ is a filter. But this contradicts the maximality of $\mathcal{U}$ because $A \in \mathcal{W} \setminus \mathcal{U}$. Thus 1) $\Rightarrow$ 2).

Next we will show 2) $\Rightarrow$ 3).
Suppose that $A_1 \cup A_2 \in \mathcal{U}$ and $A_1 \notin \mathcal{U}$. There exists $B \in \mathcal{U}$ with $A_1 \cap B = \emptyset$. Since $A_1 \cup A_2 \in \mathcal{U}$, we know that $B \cap (A_1 \cup A_2) \in \mathcal{U}$. Therefore, $B \cap (A_1 \cup A_2) = (B \cap A_1) \cup (B \cap A_2) = B \cap A_2 \in \mathcal{U}$. But $(B \cap A_2) \subset A_2$, so $A_2 \in \mathcal{U}$. Thus, 2) $\Rightarrow$ 3).

To show 3) $\Rightarrow$ 4), note that $X \in \mathcal{F}$ since X contains an element of the filter. Since $X = A \cup (X \setminus A)$, either $A \in \mathcal{F}$ or $(X - A) \in \mathcal{F}$.

Finally, we will show 4) $\Rightarrow$ 1).
Suppose $\mathcal{G}$ is a filter strictly containing $\mathcal{F}$. Let $G \in \mathcal{G} \setminus \mathcal{F}$. Since $G \notin \mathcal{F}$ and $\mathcal{F}$ has property 4), $X \setminus G \in \mathcal{F}$. But $\mathcal{F} \subset \mathcal{G}$, $X \setminus G \in \mathcal{G}$. This is a contraction because G and $X \setminus G$ cannot both be in a filter (since their intersection is empty).$\square$

4.1.5 Corollary: *Suppose $\mathcal{U}$ is an ultrafilter. If A_1 and A_2 are disjoint and*

$A_1 \cup A_2 \in \mathcal{U}$, then exactly one of them is in $\mathcal{U}$. If $B_1 \cup B_2 \cup ... \cup B_n \in \mathcal{U}$, then some B_i is in $\mathcal{U}$.

Proof: The statements follow directly from property 3) above. $\square$

As an example of an ultrafilter, consider $\{S \subseteq X : p \in S\}$ for some infinite set X and some $p \in X$. This collection is clearly a filter, and it satisfies property 4) above, so it is an ultrafilter.

4.1.6 Corollary: *If $\mathcal{U}$ is an ultrafilter on X, then $\cap\{S : S \in \mathcal{U}\}$ has at most one point.*

Proof: Suppose $\cap\{S : S \in \mathcal{U}\}$ has more than one point. Let x_1, x_2 be distinct points in $\cap\{S : S \in \mathcal{U}\}$. For every $S \in \mathcal{U}$, $\{x_1, x_2\} \subseteq S$. Therefore, $\{x_1\} \notin \mathcal{U}$ and $\{x_2\} \notin \mathcal{U}$. By property 3 of the previous theorem, $\{x_1, x_2\} \notin \mathcal{U}$. But this implies that $X \setminus \{x_1, x_2\} \in \mathcal{U}$, which is impossible because $\{x_1, x_2\}$ is a subset of each member of $\mathcal{U}$. Contradiction. $\square$

Corollary 4.1.6 suggests that we define two types of ultrafilers, one with empty intersection and one with intersection consisting of a single point.

4.1.7 Definition: *A fixed ultrafilter is an ultrafilter $\mathcal{U}$ with $\cap\{S : S \in \mathcal{U}\}$ consisting of a single point.*

4.1.8 Definition: *A free ultrafilter is an ultrafilter $\mathcal{U}$ with $\cap\{S : S \in \mathcal{U}\} = \emptyset$.*

Our previous example of $\{S \subseteq X : p \in S\}$ for some infinite set X is clearly a fixed ultrafilter.

Finding an example of a free ultrafilter is not as straightfoward. Consider the sets $[0, \omega)$, $[1, \omega)$, $[2, \omega)$, ..., and consider the filter generated by these sets (i.e. the collection of all sets containing one of the intervals). As in 4.1.2, use Zorn's Lemma to extend it to an ultrafilter. This ultrafilter contains $[0, \omega)$, $[1, \omega)$, $[2, \omega)$, ..., so the intersection of all members of this ultrafilter must be empty.

4.1.9 Theorem: *Let X be any set. Then X is infinite if and only if there is a free ultrafilter on X.*

Proof: Suppose X is infinite. Let $\mathcal{U} = \{\text{co-finite subsets of } X\}$. Note that $\mathcal{U}$ is a filter. Apply Theorem 4.1.2 to find an ultrafilter $\mathcal{W}$ containing $\mathcal{U}$. The set $\mathcal{U}$ has empty intersection, so $\mathcal{W}$ is an ultrafilter with empty intersection.

Thus, there is a free ultrafilter on X.

Conversely, suppose X is any set and $\mathcal{U}$ is an free ultrafilter on X. For contradiction, assume X is finite. Since X is finite, $\mathcal{U}$ is finite (because $\mathcal{U} \subseteq$ power set of X). So $\cap\{U : U \in \mathcal{U}\}$ is a finite intersection of members of $\mathcal{U}$, so $\cap\{U : U \in \mathcal{U}\} \in \mathcal{U}$. But $\cap\{U : U \in \mathcal{U}\} = \emptyset$ and $\emptyset \notin \mathcal{U}$. Therefore, X must be infinite. $\square$

4.2 Equivalence Classes on ω^ω

In this section, we will use a free ultrafilter to define equivalence classes on ω^ω, and then define a linear ordering on the family of equivalence classes.

Throughout this section, we let $\mathcal{U}$ be a free ultrafilter on ω.

4.2.1 Definition: *Let f, g be functions from ω to ω. Define $f \equiv g$ to mean $\{x \in \omega : f(x) = g(x)\} \in \mathcal{U}$.*

In other words, we will say that two functions are equivalent if they agree on some member of the ultrafilter.

4.2.2 Proposition: *The relation $\equiv$ is an equivalence relation on ω^ω.*

Proof: The relation is obviously reflexive and symmetric, so only transitivity needs to be checked. Let f, g, h be elements of ω^ω. Suppose $f \equiv g$ and $g \equiv h$, then $\{x \in \omega : f(x) = g(x)\} \in \mathcal{U}$, and $\{x \in \omega : g(x) = h(x)\} \in \mathcal{U}$. Since $\mathcal{U}$ is a filter, $\{x \in \omega : f(x) = g(x)\} \cap \{x \in \omega : g(x) = h(x)\} = \{x \in \omega : f(x) = g(x) = h(x)\} \in \mathcal{U}$. Thus, $\{x \in \omega : f(x) = h(x)\} \in \mathcal{U}$ because it contains $\{x \in \omega : f(x) = g(x) = h(x)\}$. Therefore, $f \equiv h$. $\square$

4.2.3 Notation: *We will denote the equivalence class to which f belongs by [f].*

4.2.4 Definition: *Let $\Psi = \{[f] : f \in \omega^\omega\}$.*

4.2.5 Definition: *For $[f], [g] \in \Psi$, define $[f] \prec [g]$ to mean that $\{x \in \omega : f(x) < g(x)\} \in \mathcal{U}$.*

4.2.6 Lemma: *The relation $\prec$ is well defined, i.e. if $f \equiv f'$ and $g \equiv g'$ and $\{x \in \omega : f(x) < g(x)\} \in \mathcal{U}$, then $\{x \in \omega : f'(x) < g'(x)\} \in \mathcal{U}$.*

Proof: Since $f \equiv f'$ and $g \equiv g'$, $\{x \in \omega : f(x) = f'(x)\} \in \mathcal{U}$ and $\{x \in \omega : g(x) = g'(x)\} \in \mathcal{U}$. Since $\{x \in \omega : f(x) < g(x)\} \in \mathcal{U}$, we know $\{x \in \omega :$

$f(x) < g(x)\} \cap \{x \in \omega : f(x) = f'(x)\} = \{x \in \omega : f(x) = f'(x) < g(x)\} \in \mathcal{U}$. So $\{x \in \omega : f'(x) < g(x)\} \in \mathcal{U}$ since it contains $\{x \in \omega : f(x) = f'(x) < g(x)\}$. Similarly, since $g \equiv g'$, the set $\{x \in \omega : f'(x) < g(x)\} \cap \{x \in \omega : g(x) = g'(x)\} \in \mathcal{U}$. So $\{x \in \omega : f'(x) < g(x) = g'(x)\} \in \mathcal{U}$, and thus $\{x \in \omega : f'(x) < g'(x)\} \in \mathcal{U}$. $\square$

We will write $[f] \preceq [g]$ to mean either $[f] \prec [g]$ or $[f] = [g]$.

4.2.7 Lemma: *$\preceq$ is a linear ordering.*

Proof: The reflexive property is automatically satisfied. Suppose $[f] \prec [g]$ and $[g] \prec [h]$. Then $\{x \in \omega : f(x) < g(x)\} \cap \{x \in \omega : g(x) < h(x)\} = \{x \in \omega : f(x) < g(x) < h(x)\} \in \mathcal{U}$. Thus, $\{x \in \omega : f(x) < h(x)\} \in \mathcal{U}$, since it contains $\{x \in \omega : f(x) < g(x) < h(x)\} \in \mathcal{U}$. Therefore, $[f] \prec [h]$, so $\prec$ is transitive.

Suppose $[f], [g] \in \Psi$. We want that either $[f] \prec [g]$ or $[g] \preceq [f]$, but not both. The set $\{x \in \omega : f(x) < g(x)\} \cup \{x \in \omega : g(x) \leq f(x)\} = \omega$. Since $\mathcal{U}$ is an ultrafilter, exactly one of $\{x \in \omega : f(x) < g(x)\}$ and $\{x \in \omega : g(x) \leq f(x)\}$ is in $\mathcal{U}$, by 4.1.4. $\square$

Now that we have defined a linear ordering on Ψ, we can consider the structure of the linearly ordered space.

4.2.8 Theorem: *Every $[f] \in \Psi$ has an immediate successor.*

Proof: Fix any $[f] \in \Psi$ and any $U_0 \in \mathcal{U}$. Let $g(x) = f(x) + 1$ for every $x \in U_0$, and let $g(x) = 0$ for every $x \notin U_0$. Since $f(x) < g(x)$ on a member of the ultrafilter, $[f] \prec [g]$. If there exists $[h] \in \Psi$ with $[f] \prec [h] \prec [g]$, then $\{x \in \omega : f(x) < h(x)\} \in \mathcal{U}$, and $\{x \in \omega : h(x) < g(x)\} \in \mathcal{U}$. Therefore, $\{x \in \omega : f(x) < h(x) < g(x)\} \in \mathcal{U}$, so $\{x \in \omega : f(x) < h(x) < g(x)\} \cap U_0 \neq \emptyset$. Thus, there exists an $x \in U_0$ with $f(x) < h(x) < g(x)$. But there is no $\alpha \in \omega$ with $f(x) < \alpha < g(x)$ because $g(x) = f(x) + 1$ for $x \in U_0$. $\square$

4.2.9 Notation: *For any $\alpha \in \omega$, let $\overline{\alpha}$ denote the function that is constantly α.*

4.2.10 Theorem: *Every $[f] \in \Psi \setminus \{\overline{0}\}$ has an immediate predecessor.*

Proof: Fix any $[f] \neq [\overline{0}]$. Then there is some $U \in \mathcal{U}$ with $f(x) \neq 0$ for every $x \in U$. (Such a U exists because f is not equal to $\overline{0}$ on any element of the ultrafilter. For example, $\omega \setminus \{x \in \omega : f(x) = 0\} \in \mathcal{U}$ since $\{x \in \omega : f(x) = 0\} \notin \mathcal{U}$.) Let $g(x) = f(x) - 1$ for every $x \in U$, and let $g(x) = 0$ for every $x \notin U$. Since $g(x) < f(x)$ on a member of the ultrafilter, $[g] \prec [f]$. Suppose there exists $[h] \in \Psi$ with $[g] \prec [h] \prec [f]$, then $\{x \in \omega : g(x) < h(x) < f(x)\} \in \mathcal{U}$. Thus, $\{x \in \omega : g(x) < h(x) < f(x)\} \cap U \neq \emptyset$, so there exists $x \in U$ with $g(x) < h(x) < f(x)$. But, as in 4.2.8, there is no $\alpha \in \omega$ with $g(x) < \alpha < f(x)$ for $x \in U$. Therefore, $[g]$ is the immediate predecessor of $[f]$. $\square$

4.2.11 Theorem: *The set Ψ is uncountable.*

Proof: Choose any sequence of $U_n \in \mathcal{U}$ with $U_{n+1} \subset U_n$ and $\cap\{U_n : n \geq 1\} = \emptyset$, e.g., $U_n = \omega \setminus \{0, 1, ..., n\}$. Suppose Ψ is countable. Call its elements $[f_1], [f_2], [f_3], ...$ For every $x \in U_1 \setminus U_2$, let $g(x) = f_1(x) + 1$, and for every $x \in U_2 \setminus U_3$, let $g(x) = max\{f_1(x), f_2(x)\} + 1$. For any integer n and $x \in U_n \setminus U_{n+1}$, let $g(x) = max\{f_1(x), f_2(x), ..., f_n(x)\} + 1$. For $x \in \omega \setminus U_1$, let $g(x) = 0$. Note that we have defined g at every $x \in \omega$ because $\cap_{n \geq 1} U_n = \emptyset$. Thus, $f_1 < g$ on U_1, $f_2 < g$ on $U_2, ...,$ so $[f_n] \prec [g]$ for every $n \geq 1$. Then $[g] \in \Psi \setminus \{[f_n] : n \geq 1\}$, a contradiction. $\square$

Note that the proof above also shows that Ψ has uncountable cofinality. For any countable subset of Ψ, there is an element of Ψ greater than every element of that subset.

Together, Theorems 4.2.8 and 4.2.10 show that Ψ has the discrete metric topology. Thus, this linearly ordered space is not interesting topologically. However, it appears that Ψ may have an interesting *order* structure nonetheless. We will begin investigating the order structure of Ψ by determining the number of predecessors of each element.

In order to examine the number of predecessors of elements of Ψ, we will use the notion of *essentially bounded* and *essentially unbounded*.

4.2.12 Definition: *A function $f \in \omega^\omega$ is essentially unbounded if it is unbounded on every member of the ultrafilter. Otherwise, f is essentially bounded.*

4.2.13 Lemma: *If $f \in \omega^\omega$ is essentially bounded, then [f] has finitely many predecessors.*

Proof: It is easy to see that for any $n \in \omega$, $[\overline{n}]$ has finitely many predecessors (because its immediate predecessor is $[\overline{n-1}]$, preceded by $[\overline{n-2}]$, etc). We will show that for every essentially bounded function f, $[f] = [\overline{n}]$ for some $n \in \omega$. Choose $U_0 \in \mathcal{U}$ such that f is bounded on U_0. Since $\{f(x) : x \in U_0\} \subseteq \omega$ and is bounded, it has a greatest element, call it m. Note that $\{x \in U_0 : f(x) = m\} \cup \{x \in U_0 : f(x) = m-1\} \cup ... \cup \{x \in U_0 : f(x) = 0\} = U_0 \in \mathcal{U}$, so $\{x \in U_0 : f(x) = n\} \in \mathcal{U}$ for some $n \leq m$ by Corollary 4.1.5. Therefore, $[f] = [n]$ for some $n \in \omega$. $\square$

4.2.14 Lemma: *If f is essentially unbounded, then $[f]$ has uncountably many predecessors.*

Proof: For every $n \geq 1$, let $U_n = \{x \in \omega : f(x) > n\}$. Note that $U_{n+1} \subseteq U_n$. Each $U_n \in \mathcal{U}$ because if not, $\{x \in \omega : f(x) \leq n\} \in \mathcal{U}$ for some n. That is impossible because f is essentially unbounded. Let $[g_1], [g_2], ...$ be countably many predecessors of $[f]$. We will find a predecessor $[h]$ of $[f]$ that is not among the $[g_n]$. For every $x \in U_1 \setminus U_2$, choose $h(x) < f(x)$ and $h(x) \neq g_1(x)$. Such a choice for $h(x)$ is possible because $x \in U_1$, so $f(x) \geq 2$. For every $x \in U_2 \setminus U_3$, choose $h(x) \in \{0, 1, ..., f(x) - 1\} \setminus \{g_1(x), g_2(x)\}$. For any n and any $x \in U_n \setminus U_{n+1}$, choose $h(x) \in \{0, 1, ..., f(x) - 1\} \setminus \{g_1(x), g_2(x), ..., g_n(x)\}$. Such a choice is possible because for $x \in U_n$, the set $\{0, 1, ..., f(x) - 1\}$ has at least $n+1$ elements. To complete the definition of h, let $h(x) = 0$ for every $x \notin U_1$. We know that $h \neq g_i$ on U_i for any $i \geq 1$, and $h < f$ on U_1. Thus, $[h] \neq [g_i]$ for any i, but $[h] \prec [f]$. So, for any countable set of predecessors of $[f]$, one can find another predecessor of $[f]$ not in that set. Therefore, $[f]$ has uncountably many predecessors. $\square$

Thus there are two kinds of elements of Ψ: essentially bounded, which have finitely many predecessors, and essentially unbounded, which have uncountably many predecessors. We can visualize this property of Ψ by considering the lexicographic product

$$(\{0\} \times (\mathbb{Z}^+ \cup \{0\})) \cup (\mathbb{P}^+ \times \mathbb{Z}),$$

where $\mathbb{Z}$ denotes the set integers, $\mathbb{Z}^+$ denotes the set of positive integers, and $\mathbb{P}^+$ denotes the set of positive irrationals. Elements of $(\{0\} \times (\mathbb{Z}^+ \cup \{0\}))$ have finitely many predecessors, and elements of $\mathbb{P}^+ \times \mathbb{Z}$ have uncountably many predecessors. Every element of $(\{0\} \times (\mathbb{Z}^+ \cup \{0\})) \cup (\mathbb{P}^+ \times \mathbb{Z})$ has an immediate successor, and every element except $(0, 0)$ has an immediate predecessor.

We do not know whether Ψ is order isomorphic to the set described above, or to a set like it. In other words, is there a subset $S \subseteq (0, \infty)$ such that Ψ is order isomorphic to the lexicographic product

$$(\{0\} \times (\mathbb{Z}^+ \cup \{0\})) \cup (S \times \mathbb{Z})?$$

4.3 A New Characterization of Ultrafilters

It is also possible to consider an ordering defined by a filter rather than an ultrafilter. If $\mathcal{F}$ is a filter, we can define an equivalence relation on ω^ω just as in 4.2.1, and an order $\preceq$ on the set of all those equivalence classes, just as in 4.2.5. Starting with only a filter, we obtain a partial order. As we proved in this chapter, if the filter is a maximal filter, then the partial order is linear. The converse is also true.

4.3.1 Theorem: *Let $\mathcal{F}$ be a free filter on ω and let $\equiv$ be the equivalence relation on ω^ω defined by $f \equiv g$ if $\{x \in \omega : f(x) = g(x)\} \in \mathcal{F}$. Let Φ be the set of equivalence classes $[f]$ defined by $\equiv$. Define $[f] \prec [g]$ to mean that $\{x \in \omega : f(x) < g(x)\} \in \mathcal{F}$. Then $\prec$ is a partial ordering of Φ and $\prec$ is a linear order if and only if $\mathcal{F}$ is an ultrafilter.*

Proof: Section 4.2 shows that if $\mathcal{F}$ is an ultrafilter, then $\prec$ is a linear ordering. We prove the converse. Fix $A \subseteq \omega$. We will show that either $A \in \mathcal{F}$ or $\omega \setminus A \in \mathcal{F}$, and then apply 4.1.4.

For any $B \subseteq \omega$, define $\chi_B : \omega \to \omega$ by $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ if $x \in \omega \setminus B$. Consider the functions χ_A and $\chi_{\omega\setminus A}$. Since $\prec$ is a linear ordering, either $\chi_A \prec \chi_{\omega\setminus A}$ or vice versa. If $\chi_A \prec \chi_{\omega\setminus A}$, then the set $\{x \in \omega : \chi_A(x) < \chi_{\omega\setminus A}(x)\}$ belongs to $\mathcal{F}$. But $\{x \in \omega : \chi_A(x) < \chi_{\omega\setminus A}(x)\} = \{x \in \omega : \chi_A(x) = 0 \text{ and } \chi_{\omega\setminus A}(x) = 1\} = \omega \setminus A$. If $\chi_{\omega\setminus A} \prec \chi_A$, then the set $\{x \in \omega : \chi_{\omega\setminus A}(x) < \chi_A(x)\}$ belongs to $\mathcal{F}$. But $\{x \in \omega : \chi_{\omega\setminus A}(x) < \chi_A(x)\} = \{x \in \omega : \chi_{\omega\setminus A}(x) = 0 \text{ and } \chi_A(x) = 1\} = A$. Thus, either $A \in \mathcal{F}$ or $\omega \setminus A \in \mathcal{F}$, so $\mathcal{F}$ is an ultrafilter. $\square$

Therefore, we can add the following characterization of ultrafilter to Theorem 4.1.4.

5) The order $\prec$ defined by $\mathcal{F}$ is linear.

Chapter 5: Nonarchimedean Fields

In this chapter, we will use ultrafilters to give an example of an ordered field with some interesting properties which distinguish it from usual fields like the real numbers.

Let $\mathbb{R}^\omega$ be the set of all functions from $[0, \omega)$ to the field $\mathbb{R}$ of real numbers. Define operations on $\mathbb{R}^\omega$ pointwise, so for every $f, g \in \mathbb{R}^\omega$ and for every $x \in \omega$,

$(f + g)(x) = f(x) + g(x)$, and

$(f * g)(x) = f(x) * g(x)$.

Note that $(\mathbb{R}^\omega, +, *)$ is a ring with unity. A basic result of ring theory says that every ring with unity has a maximal ideal (the proof is straightforward using Zorn's Lemma). We will examine one maximal ideal of $\mathbb{R}^\omega$.

5.1 A Maximal Ideal and Quotient Ring

5.1.1 Lemma: *Fix any free ultrafilter $\mathcal{U}$ on ω. Let $M = \{f \in \mathbb{R}^\omega : z(f) \in \mathcal{U}\}$, where $z(f) = \{x \in \omega : f(x) = 0\}$. Then M is a maximal ideal of $\mathbb{R}^\omega$.*

Proof: First we will show that M is an ideal. The set M is closed under $+$. For any $f, g \in M$, $z(f), z(g) \in \mathcal{U}$, so $z(f) \cap z(g) \in \mathcal{U}$. Since $z(f) \cap z(g) \subseteq z(f + g)$, $f + g \in M$. The set M is also closed under multiplication by any element of $\mathbb{R}^\omega$. For any $g \in \mathbb{R}^\omega$ and for any $f \in M$, $z(g * f) = \{x \in \omega : g(x) * f(x) = 0\}$. We know that $z(f) \in \mathcal{U}$ and $z(f) \subseteq z(g * f)$, so $z(g * f) \in \mathcal{U}$, so $g * f \in M$. Therefore, M is an ideal.

Now we must show that M is maximal. Let N be ideal with $M \subset N$ and $M \neq N$. Fix any $f \in N \setminus M$. Since $f \notin M$, $z(f) \notin \mathcal{U}$, so $\omega \setminus z(f) \in \mathcal{U}$. Let $g(x) = 1$ for every $x \in z(f)$, and let $g(x) = 0$ for every $x \in \omega \setminus z(f)$. Note that $g \in M \subseteq N$. Let $h(x) = (f(x))^{-1}$ for every $x \in \omega \setminus z(f)$, and let $h(x) = 0$ for every $x \in z(f)$. For every $x \in \omega$, $h(x) * f(x) + g(x) = 1$. Since $f, g \in N$ and N is ideal, $\overline{1} \in N$. Therefore, $N = \mathbb{R}^\omega$, so M is a maximal ideal. $\square$

Since M is a maximal ideal, the quotient ring R^ω/M is a field. We will denote the elements of $\mathbb{R}^\omega/M$ by $\{x + M\} = \{x + m : m \in M\}$, the cosets of M. We will say that $x \equiv y$ if $\{x + M\} = \{y + M\}$, i.e. if $x - y \in M$.

5.1.2 Lemma: *Let $\pi : \mathbb{R}^\omega \to \mathbb{R}^\omega/M$ be the natural projection, given by $\pi(x) = \{x + M\}$. Then*

a) if $r \neq s$ in $\mathbb{R}$, then $\overline{r} \neq \overline{s}$ in $\mathbb{R}^\omega$, and $\pi(\overline{r}) \neq \pi(\overline{s})$ in $\mathbb{R}^\omega/M$

b) the cardinality of $\mathbb{R}^\omega/M$ is 2^ω.

c) the subset $\{\{\overline{r} + M\} : r \in \mathbb{R}\}$ of $\mathbb{R}^\omega/M$ is a subfield of $\mathbb{R}^\omega/M$, and is field isomorphic to $\mathbb{R}$.

d) the subset $\{\{\overline{r} + M\} : r \in \mathbb{R}$ of $\mathbb{R}^\omega/M\}$ is a proper subset of $\mathbb{R}^\omega/M$.

Proof: a) If $r \neq s$ in $\mathbb{R}$, $\overline{r}$ and $\overline{s}$ are distinct functions in $\mathbb{R}^\omega$. Thus, $\overline{r} - \overline{s} = \overline{t}$ for some $t \in \mathbb{R}$, $t \neq 0$. So $z(\overline{r} - \overline{s}) = \emptyset \notin \mathcal{U}$, so $\overline{r} - \overline{s} \notin M$. Therefore, $\{\overline{r} + M\} \neq \{\overline{s} + M\}$, so $\pi(\overline{r}) \neq \pi(\overline{s})$.

b) The cardinality of $\mathbb{R}^\omega/M$ is less than or equal to the cardinality of $\mathbb{R}^\omega$ because $\mathbb{R}^\omega/M$ is a collection of equivalence classes of $\mathbb{R}^\omega$, and the cardinality of $\mathbb{R}^\omega$ is $(2^\omega)^\omega = 2^{\omega * \omega} = 2^\omega$. So $|\mathbb{R}^\omega/M| \leq 2^\omega$. By part a) above, $|\mathbb{R}^\omega/M| \geq |\mathbb{R}| = 2^\omega$. Thus, $|\mathbb{R}^\omega/M| = 2^\omega$.

c) The natural projection $\pi(\overline{r}) = \{\overline{r} + M\}$ gives an isomorphism between the subfield $\{\{\overline{r} + M\} : r \in \mathbb{R}\}$ of $\mathbb{R}^\omega/M$ and $\mathbb{R}$.

d) Consider $f \in \mathbb{R}^\omega$ given by $f(x) = x$ for every $x \in \omega$. We will show that for any $r \in \mathbb{R}$, $\{f + M\} \neq \{\overline{r} + M\}$. If $\{f + M\} = \{\overline{r} + M\}$, then $f = \overline{r} + m$ for some $m \in M$. Since $m = f - \overline{r}$ and $f(x) = x$ for every $x \in \omega$, $z(m) = \{r\}$. But $\mathcal{U}$ is a free ultrafilter, so $\{r\} \notin \mathcal{U}$, so $m \notin M$. Therefore, for any $r \in \mathbb{R}$, $\{f + M\} \neq \{\overline{r} + M\}$, so $\{\{\overline{r} + M\} : r \in \mathbb{R}\}$ is a proper subset of $\mathbb{R}^\omega/M$. $\square$

5.2 An Ordering on $\mathbb{R}^\omega/M$

5.2.1 Definition: *For $f, g \in \mathbb{R}^\omega$, $\{f + M\} \preceq \{g + M\}$ if there exists $p \in \mathbb{R}^\omega$ with $p(x) \geq 0$ for all $x \in \omega$, and $\{g + M\} = \{f + M\} + \{p + M\}$.*

Note that this definition is equivalent to "$\{f + M\} \preceq \{g + M\}$ if there exists $p \in \mathbb{R}^\omega$ with $p(x) \geq 0$ for all $x \in \omega$, and $g - (f + p) \in M$."

5.2.2 Lemma: *The order $\preceq$ is well defined.*

Proof: Suppose for $f, g \in \mathbb{R}^\omega$, there exists $p \in \mathbb{R}^\omega$ with $p(x) \geq 0$ for every $x \in \omega$, and $g - (f + p) \in M$. Suppose $f' \equiv f$ and $g' \equiv f$, then $f' - f \in M$ and $g' - g \in M$. So $(g' - g) - (f' - f) = m_1$ for some $m_1 \in M$, so $(g' - f') + (f - g) = m_1$. But $g - (f + p) = m_2$ for some $m_2 \in M$, so $-(m_2 + p) = (f - g)$, so $(g' - f') - (m_2 + p) = m_1$. Thus, $g' - f' - p = m_1 + m_2$, so $g' - (f' + p) \in M$, so $g' \equiv f' + p$. $\square$

5.2.3 Lemma: *$\preceq$ is a linear ordering on $\mathbb{R}^\omega/M$.*

Proof: The reflexive property is satisfied because $f - f = \overline{0} \in M$. Suppose $\{f+M\} \preceq \{g+M\}$ and $\{g+M\} \preceq \{h+M\}$. Then there exists $p_f, p_g \in \mathbb{R}^\omega$ with $p_f(x), p_g(x) \geq 0$ for every $x \in \omega$, and $g-(f+p_f) \in M$, and $h-(g+p_h) \in M$. Thus, $g-(f+p_f)+h-(g+p_h) = h-(f+(p_f+p_h)) \in M$. Since p_f and p_h are always greater than or equal to zero, then $(p_f + p_h)$ is always greater than or equal to zero. Therefore, $\{h + M\} \preceq \{f + M\}$.

To show the antisymmetric property, we will show that for any $g \in \mathbb{R}^\omega$, either $\{g+M\} \preceq \{\overline{0}+M\}$, or $\{\overline{0}+M\} \preceq \{g+M\}$, but not both. This is sufficient to prove the antisymmetric property because if $\{(f-g)+M\} \preceq \{\overline{0}+M\}$, then $\{f+M\} \preceq \{g+M\}$.

Consider $U_0 = \{j \in \omega : g(j) \geq 0\}$. If $U_0 \in \mathcal{U}$, then let $p(x) = g(x)$ for every $x \in U_0$, and let $p(x) = 0$ for every $x \notin U_0$. Then $p(x) \geq 0$ for every $x \in \omega$, and $g - (\overline{0} + p) \in M$, so $\{\overline{0} + M\} \preceq \{g + M\}$. If $U_0 \notin \mathcal{U}$, then $\omega \setminus U_0 \in \mathcal{U}$, so $\{j \in \omega : g(x) < 0\} \in \mathcal{U}$. So let $p(x) = -g(x)$ for every $x \in \omega \setminus U_0$, and let $p(x) = 0$ otherwise. Thus, $p(x) \geq 0$ for every $x \in \omega$, and $\overline{0} - (g+p) \in M$, so $\{g + M\} \preceq \{\overline{0} + M\}$. $\square$

5.2.4 Lemma: *The linear ordering $\preceq$ on $\mathbb{R}^\omega/M$ is compatible with the field operations of $\mathbb{R}^\omega/M$, i.e.*

a) if $\{f+M\} \preceq \{g+M\}$, then $\{f+M\}+\{h+M\} \preceq \{g+M\}+\{h+M\}$ for any $h \in \mathbb{R}^\omega$.

b) if $\{\overline{0}+M\} \preceq \{f+M\}$ and $\{\overline{0}+M\} \preceq \{g+M\}$, then $\{\overline{0}+M\} \preceq \{f+M\} * \{g+M\}$.

Proof: a) Suppose $\{f + M\} \preceq \{g + M\}$. Then there exists $l \in \mathbb{R}^\omega/M$ with $l(x) \geq 0$ for every $x \in \omega$ and $g-(f+l) \in M$. Thus, $(g+h)-(f+h+l) = g-(f+l) \in M$, so $\{(f+h)+M\} \preceq \{(g+h)+M\}$.

b) Suppose $\{\overline{0} + M\} \preceq \{f + M\}$ and $\{\overline{0} + M\} \preceq \{g + M\}$. Then there exists $h \in \mathbb{R}^\omega$ with $h(x) \geq 0$ for every $x \in \omega$, and $f - (\overline{0}+h) \in M$. Since M is an ideal, $g * [f - (\overline{0}+h)] = gf - (\overline{0}+gh) \in M$. So $z(fg - gh) \in \mathcal{U}$. Let $U_0 = z(fg-gh) \in \mathcal{U}$. Since $\{\overline{0} + M\} \preceq \{g + M\}$, there exists some $U_1 \in \mathcal{U}$ with $g(x) \geq 0$ for every $x \in U_1$. Let k be the function defined by $k(x) = gh(x)$ for every $x \in U_0 \cap U_1$, and $k(x) = 0$ otherwise. For every $x \in U_0 \cap U_1$, $[fg - (\overline{0}+k)](x) = 0$. So $U_0 \cap U_1 \subseteq z(fg-(\overline{0}+k))$, so $z(fg-(\overline{0}+k)) \in \mathcal{U}$.

Therefore, $fg - (\overline{0} + k) \in M$, so $\{\overline{0} + M\} \preceq \{f + M\} * \{g + M\}$. $\square$

5.3 A Nonarchimedean Field

In this section, we show that the linearly ordered field $\mathbb{R}^\omega/M$ is very different from the familiar field $\mathbb{R}$. First we show that $\mathbb{R}^\omega/M$ is not isomorphic to any subfield of $\mathbb{R}$.

5.3.1 Theorem: *For any subfield $S \subseteq \mathbb{R}$, there is no field isomorphism $f : \mathbb{R}^\omega/M \to S$.*

Proof: The subfield $\{\{\overline{r} + M\} : r \in \mathbb{R}\}$ is isomorphic to $\mathbb{R}$. Let $g(r) = \{\overline{r}+M\}$ be the isomorphism from $\mathbb{R}$ into $\mathbb{R}^\omega/M$. Suppose, for contradiction, that there exists a field isomorphism $f : \mathbb{R}^\omega/M \to S$ for some $S \subseteq \mathbb{R}$. Then $f \circ g : \mathbb{R} \to \mathbb{R}$ is a field isomorphism with $f \circ g[\mathbb{R}] \subset \mathbb{R}$. We will show that $f \circ g[\mathbb{R}] \neq \mathbb{R}$. Define $i(x) = x$ for every $x \in \mathbb{R}$. We know $\{i+M\} \neq \{\overline{r}+M\}$ for any $r \in \mathbb{R}$, so $f(\{i+M\}) \notin f[g[\mathbb{R}]]$. But $f(\{i+M\}) \in \mathbb{R}$, so $f \circ g[\mathbb{R}] \neq \mathbb{R}$. Therefore, $f \circ g : \mathbb{R} \to \mathbb{R}$ is a field isomorphism with $f \circ g[\mathbb{R}] \subset \mathbb{R}$ and $f \circ g[\mathbb{R}] \neq \mathbb{R}$. But this is impossible because of the following lemma. $\square$

5.3.2 Lemma: *The only field isomorphism $h : \mathbb{R} \to S$ for $S \subseteq \mathbb{R}$ is $h(x) = x$.*

Proof: First we will show that $h(n) = n$ for every positive integer n. Since h is one-to-one, $h(1) \neq 0$, so $h(1) = 1$. For any positive integer n, $h(n) = h(\underbrace{1 + 1 + ... + 1}_{n\ times}) = \underbrace{h(1) + h(1) + ... + h(1)}_{n\ times} = \underbrace{1 + 1 + ... + 1}_{n\ times} = n$. For any negative integer $-n$, $h(-n) = h(-1) * h(n) = -1 * n = -n$. For every integer n, $1 = h(n * (1/n)) = h(n) * h(1/n)$, so $h(1/n) = 1/n$. For each rational m/n, $h(m/n) = h(m * 1/n) = h(m) * h(1/n) = m * (1/n)$.

Next we will show that if $x > 0$, then $h(x) > 0$. Fix any $x > 0$. There exists y with $y^2 = x$, so $h(x) = h(y^2) = [h(y)]^2$, so $h(x) \geq 0$. Then h is order preserving. Thus, for any x and rational p, q with $p < x < q$, we get $p = h(p) < h(x) < h(q) = q$. So if $h(x) < x$, choose a rational p with $h(x) < p < x$. But since h is order preserving, $p = h(p) < h(x)$, so $h(x)$ could not be less than x. Analogously, $h(x)$ cannot be greater than x. Therefore, for any x, $h(x) = x$. $\square$

We have concluded that $\mathbb{R}^\omega/M$ is not isomorphic to any subset of the real numbers. We will now explore some special properties of $\mathbb{R}^\omega/M$ which distinguish it from $\mathbb{R}$.

The following is a familiar property of the real numbers.

5.3.3 Definition: *The archimedean property of the real numbers is: for each $x \in \mathbb{R}$, some positive integer n has $x < n$. In other words, for each $x \in \mathbb{R}$ with $x > 0$, there exists a positive integer n with $0 < 1/n < x$.*

We will now show that $\mathbb{R}^\omega/M$ does not have an analogous property.

5.3.4 Theorem: *There exists $\{i + M\} \in \mathbb{R}^\omega/M$ with $\{\overline{n} + M\} \preceq \{i + M\}$ for every positive integer n. There exists some $\{j + M\} \in \mathbb{R}^\omega/M$ with $\{\overline{0} + M\} \prec \{j + M\} \prec \{\overline{1/n} + M\}$ for each n.*

Proof: Let $i(x) = x$ for every $x \in \mathbb{R}$. Suppose there exists a positive integer n with $\{i + M\} \preceq \{\overline{n} + M\}$. Then there exists $p \in \mathbb{R}^\omega$ with $p(x) \geq 0$ for every $x \in \mathbb{R}$, and $\overline{n} - (i + p) \in M$. Let $U_0 = z(\overline{n} - (i + p)) \in \mathcal{U}$. For every $x \in U_0$, $n - (i(x) + p(x)) = 0$, so $p(x) = n - i(x) = n - x$. Since $p(x) \geq 0$, $n > x$ for every $x \in U_0$, so U_0 is finite. But $U_0 \in \mathcal{U}$, and no member of a free ultrafilter is finite. Therefore, $\{\overline{n} + M\} \preceq \{i + M\}$ for every positive integer n. Similarly, for $j = 1/i$, $\{\overline{0} + M\} \prec \{j + M\} \prec \{\overline{1/n} + M\}$ for every n. $\square$

In other words, there exists elements of $\mathbb{R}^\omega/M$ which are "infinitely large" or "infinitetesimal (infinitely small)" elements. Such a field is said to be nonarchimedean.

5.4 An η_1-set

In this section, we show that the nonarchimedean field $\mathbb{R}^\omega/M$ is an η_1-set. Throughout the section we will use $[f]$ to denote $\{f + M\}$.

5.4.1 Definition: *A linearly ordered set $(X, <)$ is an η_1-set if, given any countable sets $A, B \subset X$ with $a < b$ for every $a \in A$ and $b \in B$, there exists an $x \in X$ with $A \subset (\leftarrow, x)$ and $B \subset (x, \rightarrow)$.*

It is easy to see that $\mathbb{R}$ is not an η_1-set. Consider $A = \{-1/n : n \in \mathbb{Z}^+\} \cup \{0\}$, and $B = \{1/n : n \in \mathbb{Z}^+\}$. If there were an $x \in \mathbb{R}$ strictly between A and B, then x would be less than $1/n$ for every positive integer n. No such x exists in $\mathbb{R}$. However, we will show that $\mathbb{R}^\omega/M$ is an η_1-set. The remaining proofs in this section are taken from [2].

First we need to prove the following lemmas.

5.4.2 Lemma: *Suppose $f, g \in \mathbb{R}^\omega$ have the property that $[f] \prec [g]$ in $\mathbb{R}^\omega/M$. Then there is a $g' \in \mathbb{R}^\omega$ such that*

a) $[g'] = [g]$

b) $f(x) \leq g'(x)$ for each $x \in \omega$.

Proof: To say $[f] \prec [g]$ in $\mathbb{R}^\omega/M$ means that there is a function $p \in \mathbb{R}^\omega$ with $p(x) \geq 0$ for all $x \in \omega$ and $[f+p] = [g]$. Hence there is a member $U_0 \in \mathcal{U}$ with $f(x)+p(x)-g(x) = 0$ for all $x \in U_0$. Define $g'(x) = f(x)+p(x)$ for all $x \in \omega$. Then $g'(x) - g(x) = 0$ for all $x \in U_0$ so that $[g'] = [g]$. Also note that $p(x) \geq 0$ for all $x \in \omega$ yields $f(x) \leq f(x)+p(x) = g'(x)$ as required. □

5.4.3 Lemma *For any function $g, h \in \mathbb{R}^\omega$, if $f(x) =max(g(x), h(x))$ for each $x \in \omega$, then $[f] =max([g], [h])$.*

Proof: Without loss of generality, assume $[g] \prec [h]$ in $\mathbb{R}^\omega/M$. Then there is some $U_0 \in \mathcal{U}$ such that $g(x) < h(x)$ for each $x \in U_0$. Then $f(x) = h(x)$ for each $x \in U_0$, so $[f] = [h]$. □

5.4.4 Lemma: *For any functions $g, h \in \mathbb{R}^\omega$, if $f(x) =min(g(x), h(x))$ for each $x \in \omega$, then $[f] =min([g], [h])$.*

Proof: Analogous to Lemma 5.4.3 above. □

5.4.5 Lemma: *Suppose $f(x) \leq h(x)$ for all $x \in \omega$, and suppose $g \in \mathbb{R}^\omega$ has $[f] \prec [g] \prec [h]$ in $\mathbb{R}^\omega$. Then there is some $g' \in \mathbb{R}^\omega$ with*

a) $f(x) \leq g'(x) \leq h(x)$ for all $x \in \omega$, and

b) $[g'] = [g]$.

Proof: Following [2, Lemma 13.5], we define $g'(x) =\min(\max(f(x), g(x)), h(x))$ for each $x \in \omega$. We claim that for each $x \in \omega$, we have $f(x) \leq g'(x)$. Fix $x \in \omega$. If $g'(x) = h(x)$, then the assumption $f(x) \leq h(x)$ implies $f(x) \leq g'(x)$. If $g'(x) =\max(f(x), g(x))$, then $f(x) \leq g'(x)$. Next we claim $g'(x) \leq h(x)$ for each $x \in \omega$. If $g'(x) = h(x)$ there is nothing to prove. If $g'(x) =\max(f(x), g(x))$, then $\max(f(x), g(x))$ must be less than or equal to $h(x)$, so $g'(x) \leq h(x)$. Finally, we claim that $[g'] = [g]$. This is true because, by Lemma 5.4.4, $[g'] =\min([\max(f, g)], [h])$, and by Lemma 5.4.3, $[\max(f, g)] =\max([f], [g])$. But we know that $[f] \prec [g] \prec [h]$, so $\max([f], [g]) = [g]$, and $\min([g], [h]) = [g]$. Therefore, $[g'] = [g]$, as required. □

5.4.6 Lemma: *Let C be any countable subset of $\mathbb{R}^\omega$. For each $g \in C$, there is a $g' \in \mathbb{R}^\omega$ such that*

a) $[g'] = [g]$ in $\mathbb{R}^\omega$, and

b) if $g_1, g_2 \in C$ have $[g_1] \prec [g_2]$ in $\mathbb{R}^\omega/M$, then $g_1'(x) \leq g_2'(x)$ for each $x \in \omega$.

Proof: Index the countable set $C = \{g_n : n \geq 1\}$. Let $g_1' = g_1$. Consider g_2. If $[g_2] = [g_1]$, let $g_2' = g_2$. If $[g_2] \neq [g_1]$, then apply Lemma 5.4.2 to find g_2'. Suppose $n \geq 3$ and suppose for $j < n$ we have chosen g_j' in such a way that a) and b) hold when applied to members of $\{g_1, g_2, ..., g_{n-1}\}$. Consider g_n. There are four cases:

i) if $[g_n] = [g_j]$ for some $j < n$, let $g_n' = g_j'$,

ii) if $[g_n] \prec [g_j]$ where $[g_j]$ =min$\{[g_1], ..., [g_{n-1}]\}$, then apply Lemma 5.4.3 to find g_n' with $g_n'(x) \leq g_j'(x)$ for all x,

iii) if $[g_n] \prec [g_j]$ where $[g_j]$ is the largest of $[g_1], ..., [g_{n-1}]$ in the linearly ordered set $\mathbb{R}^\omega/M$, apply Lemma 5.4.4 to find g_n'.

iv) if none of the above cases hold, then $[g_n]$ lies between two members of $\{[g_1], ..., [g_{n-1}]\}$. Let $[g_i]$ and $[g_j]$ be the closest neighbors of $[g_n]$ in that set with $[g_i] \prec [g_n] \prec [g_j]$. Use Lemma 5.4.5 to choose g_n' with $[g_n'] = [g_n]$ and $g_i'(x) \leq g_n'(x) \leq g_j'(x)$ for each $x \in \omega$.

This induction constructs g_n' for each n in such a way that a) and b) are satisfied. □

5.4.7 Lemma: *Suppose A and B are countable subsets of $\mathbb{R}^\omega/M$ with the property that $[f] \prec [g]$ for each $[f] \in A$ and $[g] \in B$. Then there is a function $h \in \mathbb{R}^\omega$ such that for every $[f] \in A$ and $[g] \in B$, we have $[f] \preceq [h] \preceq [g]$.*

Proof: Because A is a countable linearly ordered set, we may choose $[f_n] \in A$ with the property that $[f_1] \prec [f_2] \prec [f_3] \prec ...$, and for every $[f] \in A$ some $[f_n]$ has $[f] \preceq [f_n]$. Similarly, we may choose $[g_n] \in B$ with $[g_{n+1}] \preceq [g]$ for all n, and such that for any $[g] \in B$, we have $[g_n] \prec [g]$ for some n. Then use Lemma 5.4.6 to choose functions $f_n', g_n' \in \mathbb{R}^\omega$ such that $[f_n'] = [f_n]$, $[g_n'] = [g_n]$ and $f_m(x) \leq f_{m+1}(x) \leq g_{n+1}(x) \leq g_n(x)$ for every $x \in \omega$. For each fixed $x \in \omega$, we define $h(x) = \sup\{f_n(x) : n \geq 1\}$, which exists because $\langle f_n(x) \rangle$ is an increasing sequence of real numbers bounded by $g_1(x)$. Then $f_m(x) \leq h(x) \leq g_n(x)$ for each $m, n \geq 1$ and each $x \in \omega$ so that for each $m, n \geq 1$ we have $[f_n] \preceq [h] \preceq [g_n]$. For any $[f] \in A$ and $[g] \in B$, we can find

$m, n \geq 1$ with $[f] \preceq [f_m] \preceq [h] \preceq [g_n] \preceq [g]$ as required. $\square$

5.4.8 Lemma: *Suppose B is a countable subset of $\mathbb{R}^\omega/M$ with the property that $[0] \prec [g]$ for each $[g] \in B$. Then there is a function $h \in \mathbb{R}^\omega$ such that*

a) $[0] \prec [h]$, and

b) $[h] \prec [g]$ for each $[g] \in B$.

Proof: As in the proof of Lemma 5.4.7, we may choose a monotonic decreasing sequence $\langle g_n \rangle \subseteq B$ with the property that for each $[g] \in B$, some $[g_n]$ has $[g_n] \preceq [g]$.

Because $\mathcal{U}$ is a free ultrafilter on ω, we may choose sets $U_n \in \mathcal{U}$ with $U_0 = \omega$, $U_{n+1} \subset U_n$, and $\cap\{U_n : n \geq 1\} = \emptyset$. For every n, because $[0] \prec [g_n]$, there is a member $V_n \in \mathcal{U}$ such that $g_n(x) > 0$ for each $x \in V_n$. Define $W_n = U_n \cap (\cap\{V_j : 1 \leq j \leq n\})$. Then we have $W_n \in \mathcal{U}$, $g_n(x) > 0$ for each $x \in W_n$, and $\cap\{W_n : n \geq 1\} = \emptyset$. Define a function $h : \omega \to \mathbb{R}$ by the rule that $h(x) = 1$ if $x \in \omega \setminus W_1$, and $h(x) = (1/2)*\min\{g_1(x), g_2(x), ..., g_n(x)\}$ if $x \in W_n \setminus W_{n-1}$ for some $n \geq 2$. Note that h is defined for all x because $\cap\{W_n : n \geq 1\} = \emptyset$, and $h(x) > 0$ for all x, and $h(x) < g_n(x)$ for all $x \in W_n$. Therefore, $[h]$ satisfies a) and b) as required. $\square$

5.4.9 Theorem: $\mathbb{R}^\omega/M$ *is an $\eta_1$set.*

Proof: Suppose A and B are countable subsets of $\mathbb{R}^\omega/M$ with $[f] \prec [g]$ for each $[f] \in A$ and $[g] \in B$. There are four cases:

i) Suppose A has no largest element, and B has no smallest element. Use Lemma 5.4.7 to find $h \in \mathbb{R}^\omega$ with the property that for each $[f] \in A$ and $[g] \in B$, $[f] \preceq [h] \preceq [g]$. Because A has no largest point, we must have $[f] \prec [h]$ for each $[f] \in A$. Similarly, $[h] \prec [g]$ for each $[g] \in B$.

ii) Suppose A has a largest element $[f_0]$ and B has a smallest element $[g_0]$. Then let $h = (f_0 + g_0)/2$, so $[f] \prec [h] \prec [g]$ for every $[f] \in A$ and $[g] \in B$.

iii) Suppose A has a largest element $[f_0]$ and B has no smallest element. Let $B^* = \{[g - f_0] : [g] \in B\}$. Then Lemma 5.4.8 can be applied to B^* to find a function h^* with $[0] \prec [h^*] \prec [g - f_0]$ for every $[g] \in B$. Let $h = h^* + f_0$, so $[f] \prec [h] \prec [g]$ for every $[f] \in A$ and $[g] \in B$.

iv) The case when A has no largest element and B has a smallest element $[g_0]$ is analogous to case iii).

Therefore, $\mathbb{R}^\omega/M$ is an η_1-set. $\square$

The fact that $\mathbb{R}^\omega/M$ is an η_1-set has interesting consequences for the subfield $\mathbb{R}$ of $\mathbb{R}^\omega/M$ (where $\mathbb{R}$ is viewed as $\{[\overline{r}] : r \in \mathbb{R}\}$). Fix $r \in \mathbb{R}$. Find functions, h and k in $\mathbb{R}^\omega$ such that for each $n \geq 1$, $[\overline{r-1/n}] \prec [h] \prec [\overline{r}] \prec [k] \prec [\overline{r+1/n}]$. Consider the open interval $([h],[k])$ in $\mathbb{R}^\omega/M$. We have $([h],[k]) \cap \mathbb{R} = \{[\overline{r}]\}$. Therefore, if $\mathbb{R}^\omega/M$ is equipped with the usual open interval topology of $\prec$, then every $[\overline{r}] \in \mathbb{R}$ is isolated in the relative topology that $\mathbb{R}$ inherits from $\mathbb{R}^\omega/M$. Therefore, the usual space $\mathbb{R}$ is not a topological subspace of $\mathbb{R}^\omega/M$ even though the usual field $\mathbb{R}$ is a subfield of $\mathbb{R}^\omega/M$.

5.5 Another Characterization of Ultrafilters

The ideas in this chapter lead to one more characterization for ultrafilters.

5.5.1 Theorem: *Suppose $\mathcal{F}$ is a filter. Let $I = \{f \in \mathbb{R}^\omega : z(f) \in \mathcal{F}\}$, which is an ideal. Then $\mathcal{F}$ is an ultrafilter if and only if I is a maximal ideal.*

Proof: Note that, for any filter $\mathcal{F}$, $I = \{f \in \mathbb{R}^\omega : z(f) \in \mathcal{F}\}$ is an ideal (see proof of Lemma 5.1.1). Lemma 5.1.1 proved that $\mathcal{F}$ an ultrafilter implies that I is a maximal ideal. We want to show the converse.

Suppose I is a maximal ideal. Let $S \subseteq \omega$. We want to show that either $S \in \mathcal{F}$ or $\omega \setminus S \in \mathcal{F}$. Suppose $S \notin \mathcal{F}$. For any set A, let $\chi_A(x) = 1$ for every $x \in A$, and $\chi_A(x) = 0$ otherwise. The zero set $z(\chi_{\omega\setminus S}) = S$, so $\chi_{\omega\setminus S} \notin I$. Therefore, by the maximality of I, the ideal generated by I and $\chi_{\omega\setminus S}$, denoted $\langle I \cup \{\chi_{\omega\setminus S}\}\rangle$, must be the whole ring $\mathbb{R}^\omega$. Therefore, $\langle I \cup \{\chi_{\omega\setminus S}\}\rangle = \{j + g\chi_{\omega\setminus S} : g \in \mathbb{R}^\omega, j \in I\} = \mathbb{R}^\omega$. So, $\overline{1} \in \langle I \cup \{\chi_{\omega\setminus S}\}\rangle$, i.e. $\overline{1} = j + g\chi_{\omega\setminus S}$ for some $j \in I$ and $g \in \mathbb{R}^\omega$. For every $x \in S$, $\chi_{\omega\setminus S}(x) = 0$, so $1 = j(x) + 0$. Thus, for every $x \in S$, $i(x) = 1$, so $z(j) \subseteq \omega \setminus S$. Since $j \in I$, $z(j) \in \mathcal{F}$, so $\omega \setminus S \in \mathcal{F}$. Therefore, $\mathcal{F}$ has the property that either $S \in \mathcal{F}$ or $\omega \setminus S \in \mathcal{F}$, so $\mathcal{F}$ is an ultrafilter. $\square$

This characterization can be added the Theorem 4.1.4 as:
(6) The ideal $I = \{f \in \mathbb{R}^\omega : z(f) \in \mathcal{F}\}$ is maximal.

References

[1] K.J. Devlin and H. Johnsbråten, *The Souslin Problem*, Lecture Notes in Mathematics 405, Springer-Verlag, New York, 1974.

[2] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand Reinhold, New York, 1960.

[3] M.E. Rudin, *A Subset of the Countable Ordinals*, American Mathematical Monthly, 64 (1957), 351.

[4] M. Souslin, Problème 3, Fundamenta Mathematica 1 (1920), 223.