Lecture 3: A Combinatorial Parameterization of Nilpotent Orbits

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Setup

- $\mathfrak{g}$: a semisimple Lie algebra over $\mathbb{C}$
- $\mathfrak{h}$: a Cartan subalgebra of $\mathfrak{g}$
  $\Delta = \Delta(\mathfrak{h};\mathfrak{g})$: the roots of $\mathfrak{h}$ in $\mathfrak{g}$
  $\Pi = \Pi(\mathfrak{h};\mathfrak{g})$: a set of simple roots of Δ
- G : the adjoint group of $\mathfrak{g}$ (a complex algebraic group)
- $\mathcal{N} = \mathcal{N}_{\mathfrak{g}}$: the cone of nilpotent elements in $\mathfrak{g}$

$$\mathcal{N} = \coprod_{\mathcal{O}\in G\backslash\mathcal{N}} \mathcal{O}$$

- Our problem: parameterizing $G\backslash\mathcal{N}$, the (finite) set of nilpotent orbits

- $x \in \mathcal{O} \in G\backslash\mathcal{N}$
- x extends to a Jacobson-Morozov standard triple $\{x, h, y\}$

$$[x, y] = h \quad , \quad [h, x] = 2x \quad , \quad [h, y] = -2y$$

- h can be conjugated to element in dominant Weyl chamber

Theorem (Kostant, 1959)

Suppose $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. *A nilpotent orbit* $\mathcal{O}$ *is completely determined by the values* $[\alpha_1(h), \alpha_2(h), \ldots, \alpha_n(h)]$. *Moreover, the only possible values of* $\alpha_i(h)$ *are* 0, 1, *and* 2.

Weighted Dynkin diagrams

Upshot

- Each nilpotent orbits corresponds to a certain labeling of the nodes of Dynkin diagram of $\mathfrak{g}$ by one of $\{0,1,2\}$.
- Such a labeled Dynkin diagram is called a *weighted Dynkin diagram* (or *WDD*).

Example

WDDs for $\mathfrak{sl}(4)$

$$\begin{array}{ccccc} 2 & & 2 & & 2 \\ \bullet & - & \bullet & - & \bullet \end{array} \qquad \begin{array}{ccccc} 0 & & 2 & & 0 \\ \bullet & - & \bullet & - & \bullet \end{array}$$

$$\begin{array}{ccccc} 2 & & 0 & & 2 \\ \bullet & - & \bullet & - & \bullet \end{array} \qquad \begin{array}{ccccc} 1 & & 0 & & 1 \\ \bullet & - & \bullet & - & \bullet \end{array}$$

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \bullet & - & \bullet & - & \bullet \end{array}$$

Remarks

- Relatively few of the $3^{rank(\mathfrak{g})}$ possible diagrams are actually realized as WDDs.
- No algorithm for predicting which diagrams occur.
- More accurate to say WDDs provide a **unique labeling** (rather than a parameterization) of nilpotent orbits.
- Nevertheless, some general characteristics of an orbit can be read off its WDD.

Partition-type classifications

When $\mathfrak{g}$ is of classical type ($\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, or $\mathfrak{so}_{2n}$) the nilpotent orbits of $\mathfrak{g}$ can be parameterized by certain families of partitions.

Let N denote the dimension of the standard representation of $\mathfrak{g}$.

$$G \backslash \mathcal{N} \ni \mathcal{O} \ni x \quad \longrightarrow \mathbf{X} \in M(N, N)$$

The Jordan normal form of $\mathbf{X}$ completely determines $\mathcal{O}$.

The partition $\mathbf{p}_{\mathcal{O}}$ corresponding to $\mathcal{O}$ is the list of sizes of the irreducible Jordan blocks that occur in the Jordan normal form of $\mathbf{X}$.

$$[3,2,2,1] \longleftrightarrow \left(\begin{array}{ccc|cc|cc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \in \mathfrak{sl}_8$$

Unlike WDD labeling, the partitions $\mathbf{p}_{\mathcal{O}}$ that do occur can be determined by simple rules.

- The nilpotent orbits of $\mathfrak{sl}_n$ are in a one-to-one correspondence with the set $\mathcal{P}_A = \mathcal{P}(n)$ of partitions of n.
- The nilpotent orbits of $\mathfrak{so}_{2n+1}$ are in a one-to-one correspondence with the set $\mathcal{P}_B(2n+1)$ consisting of partitions $2n+1$ such that even parts only occur with even multiplicity.
- The nilpotent orbits of $\mathfrak{sp}_{2n}$ are in a one-to-one correspondence with the set $\mathcal{P}_C(2n)$ consisting of partitions $2n$ such that odd parts only occur with even multiplicity.
- The nilpotent orbits of $\mathfrak{so}_{2n}$ are in a *nearly* one-to-one correspondence with the set $\mathcal{P}_D(2n)$ consisting of partitions $2n$ such that even parts only occur with even multiplicity. Partitions in $\mathcal{P}_D(2n)$ which consist *only* of even parts (necessarily each with even multiplicity) are called *very even* partitions. To each very even partition there corresponds *two* distinct nilpotent orbits.

Remarks

- Given a partition $\mathbf{p} \in \mathcal{P}_G$ it is easy to write down a representative matrix $\mathbf{X}$ using recipes found, e.g., in Collingwood and McGovern.
- The closure relations amongst the nilpotent orbits can be inferred directly from the dominance partial ordering of partitions

$$\mathcal{O}_{\mathbf{p}} \subseteq \overline{\mathcal{O}_{\mathbf{p}'}} \quad \Longleftrightarrow \quad \mathbf{p} \leq_{dom} \mathbf{p}' \equiv \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} p_j' \quad \forall \; i$$

- There exist nice algorithms for computing dimensions of orbits, induced orbits, Spaltenstein duals, etc. using the partition parameterizations.
- But partition classification schemes apply **only** to Lie algebras of classical type.

Digression: Standard Levis and Standard Parabolics

Definition

Let
$\Gamma \equiv$ a subset of the simple roots Π of $\mathfrak{g}$,
$\Delta_\Gamma \equiv$ the subset of Δ generated by the simple roots $\alpha \in \Gamma$,
$\Delta(\mathfrak{u}_\Gamma) \equiv \Delta^+ - \Delta_\Gamma$.
The *standard Levi subalgebra* corresponding to Γ is the reductive subalgebra $\mathfrak{l}_\Gamma$ of $\mathfrak{g}$ given by

$$\mathfrak{l}_\Gamma = \mathfrak{h} + \sum_{\alpha \in \Delta_\Gamma} \mathfrak{g}_\alpha$$

The *standard parabolic subalgebra* corresponding to Γ is the parabolic subalgebra $\mathfrak{p}_\Gamma$ of $\mathfrak{g}$ given by

$$\mathfrak{p}_\Gamma = \mathfrak{l}_\Gamma + \sum_{\alpha \in \Delta(\mathfrak{u}_\Gamma)} \mathfrak{g}_\alpha$$

There are two fundamental ways of lifting a nilpotent orbit $\mathcal{O}_{\mathfrak{l}}$ of a Levi subalgebra $\mathfrak{l}$ to a nilpotent orbit in $\mathfrak{g}$.

- **Bala-Carter inclusion:**

$$inc_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}}\right) := G \cdot \mathcal{O}_{\mathfrak{l}} = G\text{-saturation of } \mathcal{O}_{\mathfrak{l}} \text{ in } \mathfrak{g}$$

- **induction:** Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ be any extension of $\mathfrak{l}$ to a parabolic subalgebra of $\mathfrak{g}$. Then

$$ind_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}}\right) := \text{unique dense orbit in } G \cdot \left(\mathcal{O}_{\mathfrak{l}} + \mathfrak{u}\right)$$

Theorem (Lusztig-Spaltenstein) $ind_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}}\right)$ exists and is independent of the choice of the nilradical $\mathfrak{u}$.

Definition

An element $x \in \mathcal{N}$ is **distinguished** in $\mathfrak{g}$ if it is not contained in any proper Levi subalgebra of $\mathfrak{g}$.

- If x is distinguished, then so is every element of $G \cdot x$.
  It thus makes sense to speak of *distinguished orbits*.
- The principal orbit (i.e. the maximal nilpotent orbit) is always distinguished.
- For $\mathfrak{sl}_n$ the principal orbit is the only distinguished orbit.
- E_8, on the other hand, has 11 different distinguished orbits

Bala-Carter classification: version I

Theorem

(Bala-Carter, 1976) Every nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ is the Bala-Carter inclusion of a distinguished orbit $\mathcal{O}_{\mathfrak{l}}$ in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. In fact, the correspondence

$$G\backslash\mathcal{N} \longleftrightarrow \{G\text{-conjugacy classes of distinguished orbits of Levi subalgebras}\}$$

is one-to-one.

There are a number of facts that go into the proof of the above theorem:

- $\{$minimal Levi subalgebras containing $x\}$
  $\longleftrightarrow \quad \{$maximal toral subalgebras of $\mathfrak{g}^x\}$
- $\Longrightarrow$ minimal Levis containing x are all conjugate
- x is distinguished if and only if $\mathfrak{g}^x$ has no semisimple elements

Let $x \longrightarrow \{x, h, y\}$: J-M triple

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k \quad ; \quad [h, z] = kz \text{ if } z \in \mathfrak{g}_k$$

- If x is distinguished, then $\dim \mathfrak{g}_0 = \dim \mathfrak{g}_2$
- If x is distinguished, then $\mathcal{O}_x$ is even. $(\mathfrak{g}_{2j+1} = 0)$

B-C idea: inclusion of distinguished orbits provides an inductive construction of nilpotent orbits.

To get this inductive scheme going one still needs a manageable way of classifying the distinguished orbits of a reductive Lie group.

Distinguished parabolics and a construction of distinguished orbits

Definition

A parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$ is called *distinguished* if

$$\dim \mathfrak{l} = \dim \left(\mathfrak{u} / \left[\mathfrak{u}, \mathfrak{u} \right] \right) \quad .$$

Theorem

Every distinguished orbit of $\mathfrak{l}$ is obtained by parabolic induction of the trivial orbit of a distinguished parabolic algebra $\mathfrak{p} = \mathfrak{l}' + \mathfrak{u}$ of $\mathfrak{l}$.

$$\mathcal{O} \text{ distinguishe} \Rightarrow \mathcal{O} = ind_{\mathfrak{l}}^{\mathfrak{g}} \left(\mathbf{0} \right)$$

for some distinguished parabolic $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$.

Bala-Carter classification, version II

Theorem

There is a natural one-to-one correspondence between nilpotent orbits of $\mathfrak{g}$ and G-conjugacy classes of pairs $(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}})$ where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_{\mathfrak{l}} = \mathfrak{l}' + \mathfrak{u}$ is a distinguished parabolic subalgebra of $\mathfrak{l}$. The correspondence is given by

$$(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}}) \to inc_{\mathfrak{l}}^{\mathfrak{g}} \left(ind_{\mathfrak{l}'}^{\mathfrak{l}} \left(\mathbf{0} \right) \right)$$

Definition

A nilpotent orbit obtained by parabolic induction from the trivial orbit of a Levi subalgebra is called a **Richardson orbit**:

$$\mathcal{O} \text{ is Richardson} \Longleftrightarrow \mathcal{O} = ind_{\mathfrak{l}}^{\mathfrak{g}} \left(\mathbf{0} \right) \quad \text{for some Levi subalgebra } \mathfrak{l}$$

Classifying distinguished parabolics

Theorem

The conjugacy classes of parabolic subalgebras of a semisimple Lie algebra are in a one-to-one correspondence with the subsets $\Gamma \in 2^{\Pi}$*. The correspondence is given by*

$$\Gamma \longrightarrow G\text{-conj class of } \mathfrak{p}_{\Gamma}$$

where $\mathfrak{p}_{\Gamma}$ *is the standard parabolic corresponding to* $\Gamma \subset \Pi$

Recall

Definition

A parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$ is called *distinguished* if

$$\dim \mathfrak{l} = \dim \left(\mathfrak{u} / \left[\mathfrak{u}, \mathfrak{u} \right] \right) \quad .$$

Canonical grading of a standard parabolic

Recall, given $\Gamma \subset \Pi$,

$$\begin{aligned} \Delta_\Gamma &= \Delta \cap span_{\mathbb{Z}} \{\alpha \mid \alpha \in \Gamma\} = \Delta(\mathfrak{h}; \mathfrak{l}_\Gamma) \\ \Delta(\mathfrak{u}_\Gamma) &= \Delta^+ - \Delta_\Gamma \end{aligned}$$

Definition

$\mathfrak{p}_{\Gamma,i} :=$ direct sum of the root spaces $\mathfrak{g}_\alpha$ such that α is a sum of exactly i simple roots in $\Delta(\mathfrak{u}_\Gamma)$ plus a (possibly trivial) sum of roots in $\mathfrak{l}_\Gamma$

Lemma

Let $\mathfrak{p}_\Gamma = \mathfrak{l}_\Gamma + \mathfrak{u}_\Gamma$ be a standard parabolic. Then

$$\dim(\mathfrak{u}/[\mathfrak{u},\mathfrak{u}]) = \dim \mathfrak{p}_{\Gamma,1}$$

A simple criterion for distinguished parabolics

Theorem

Let $\Gamma \subset \Pi$. Then $\mathfrak{p}_\gamma$ is distinguished in $\mathfrak{g}$ **iff**

$$|\Delta(\mathfrak{l}_\Gamma)| + rnk(\mathfrak{g}) = \#\ \textit{of roots with exactly one component in}\ \Pi - \Gamma$$

Definition

A **distinguished subset** of simple roots is a subset $\Gamma \subset \Pi$ such that

$$|\Delta(\mathfrak{l}_\Gamma)| + rnk(\mathfrak{g}) = \#\ \text{of roots with exactly one component in}\ \Pi - \Gamma$$

distinguished Γ's $\leftrightarrow$ conj classess of distinguished parabolics

Classifying conjugacy classes of Levi subalgebras

The correspondence

$$\Pi \supset \Gamma \longrightarrow \text{conjugacy class of } \mathfrak{l}_\Gamma$$

is many-to-one in general.

Theorem

(Dynkin, 1957) The conjugacy classes of Levi subalgebras of $\mathfrak{g}$ are in a one-to-one correspondence with the W-conjugacy classes of subsets of simple roots.

Definition

Let $\widetilde{\Gamma}$ be a conjugacy class of subsets $\Gamma \subset \Pi$. The **standard Gamma** of $\widetilde{\Gamma}$ is the first in the natural lexicographical ordering of $\widetilde{\Gamma}$.

Obviously, $\Gamma \longrightarrow \mathfrak{l}_\Gamma$ provides a one-to-one correspondence between standard Gammas and conjugacy classes of Levi subalgebras.

Putting this all together

Theorem

The nilpotent orbits of $\mathfrak{g}$ *are in a one-to-one correspondence with pairs* $[\Gamma, \gamma]$ *where* $\Gamma \subset \Pi$ *is a standard Gamma for* $\mathfrak{g}$ *and* $\gamma \subset \Gamma$ *is a distinguished subset of* Γ*.*
The correspondence is given by

$$[\Gamma, \gamma] \to inc_{\mathfrak{l}_\Gamma}^{\mathfrak{g}} \left(ind_{\mathfrak{l}_\gamma}^{\mathfrak{l}_\Gamma} (\mathbf{0}) \right)$$

Definition

A **combinatorial Bala-Carter parameter** (CBCP) is a pair $[\Gamma, \gamma]$ where $\Gamma \subset \Pi$ is a standard Gamma for $\mathfrak{g}$ and $\gamma \subset \Gamma$ is a distinguished subset of Γ.

CBC diagrams

Definition

The **CBC diagram** corresponding to a CBCP $[\Gamma, \gamma]$ for $\mathfrak{g}$ is the Dynkin diagram of $\mathfrak{g}$ with the nodes shaded as follows:

- The nodes corresponding to simple roots in $\Pi - \Gamma$ are represented as open circles.
- The nodes corresponding to simple roots in $\Gamma - \gamma$ are represented by a filled circles.
- The nodes corresponding to simple roots in γ are represented by asterixes.

BC label	$[\Gamma,\gamma]$	*CBC diagram*
$\mathbf{0}$	$[[],[]]$	$\circ \Lleftarrow \circ$
A_1	$[[2],[]]$	$\circ \Lleftarrow \bullet$
$\widetilde{A_1}$	$[[1],[]]$	$\bullet \Lleftarrow \circ$
$G_2(a_1)$	$[[1,2],[1]]$	$* \Lleftarrow \bullet$
G_2	$[[1,2],[]]$	$\bullet \Lleftarrow \bullet$

Advantages of CBCPs

- Easy to compute (using e.g. John Stembridge's Coxeter package)
- Parameterization is unambiguous and uniform across all reductive Lie algebras.
  - Classical and exceptional cases are treated exactly the same.
  - No ambiguities in labeling (unlike the even-even partitions for $\mathfrak{so}(2n)$ or the three pairs of non-conjugate isomorphic Levi's for E_7).
- Once one knows how to write down representatives for dististinguished orbits (relatively easy), one immediately can immediately construct a representative x for any nilpotent orbit $\mathcal{O}_{[\Gamma,\gamma]}$.
- Easy to read off the usual Bala-Carter labels for the exceptional Lie algebras.
- Very simple to figure out the partition corresponding to the CBCP of a classical Lie algebra and easy to figure out an orbit's CBCP from its partition. (Next.)

All Levis of $\mathfrak{sl}_n$ are sums of $\mathfrak{gl}_k$'s

For $\mathfrak{sl}(k)$ distinguished orbit $\Rightarrow$ principal orbit

$\Rightarrow$ all CBCPs are of the form $[\Gamma, []]$, with Γ some subset of $\{1, 2, 3, \ldots, n-1\}$.

Recipe:

- Form list $\mathbf{l} = \{\ell_1, \ell_2, \ldots, \ell_k\}$ of the lengths of the maximal strictly consecutive subsequences of Γ. E.g., in $\mathfrak{sl}_{15}$,

$$\Gamma = [1, 2, 3, 5, 6, 8, 9, 11, 13] \quad \Longrightarrow \mathbf{l} = [3, 2, 2, 1, 1]$$

- Add 1 to each entry in $\mathbf{l}$. E.g.,

$$[4, 3, 3, 2, 2]$$

- Add as many additional 1's to the tail of $\mathbf{l}$ as necessary to convert $\mathbf{l}$ into a partition of n. E.g.,

$$[4, 3, 3, 2, 2, 1] \in \mathcal{P}(15)$$

- The resulting partition of n is the partition corresponding to the orbit $\mathcal{O}_{[\Gamma, []]}$.

Partitions to Γ's for $\mathfrak{sl}(n)$

Going back from partitions to Γ's is done by

- subtracting 1 from each part to get a list **l** of lengths of strictly consecutive subsequences of simple roots.
- Reconstruct a Γ from **l** in the obvious fashion.
  (N.B. the resulting Γ will automatically be a *standard Gamma* for $\mathfrak{sl}(n)$.)

Ex.

$$\mathbf{p} = [3,2,2,1,1,1] \longrightarrow \mathbf{l} = [2,1,1,0,0,0] \longrightarrow \{1,2,4,6\}$$

$$CBCP = [\{1,2,4,6\},\{\}]$$

Definition

The **dismemberment** of a CBC diagram is the CBC obtained by removing all the circled nodes.
The G-**tail** of a CBC diagram is the connected component of the dismembered CBC diagram containing the last simple root of Π (w.r.t. Bourbaki ordering).
The A-**head** of a CBC diagram is what remains of the dismembered CBC diagram after the G-tail is removed.

Example

$\bullet - \bullet - \circ - \bullet - \circ - \bullet - * \Longleftarrow \bullet$ $\quad \nearrow$ $\bullet - * \Longleftarrow \bullet$ *G*-tail

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \searrow$ $\bullet - \bullet \quad \bullet$ *A*-head

N.B. The *G*-tail is the CBC diagram of a distinguished orbit of a simple factor of $\mathfrak{l}_\Gamma$ of the same Cartan type as $\mathfrak{g}$.

N.B. The *A*-head is consists of the Dynkin diagrams of the factors of $\mathfrak{l}_\Gamma$ that are of Cartan type *A*.

Recipe for $\mathfrak{so}(2n+1)$, $\mathfrak{sp}(2n)$, and $\mathfrak{so}(2n)$

To determine the partition in $\mathcal{P}_G$ corresponding the orbit with CBCP $[\Gamma, \gamma]$

- Determine G-tail and A-head of CBC diagram for $[\Gamma, \gamma]$.
- The G-tail prescribes a Richardson orbit for a classical Lie algebra of the same Cartan type as $\mathfrak{g}$. Use the method described in my December lectures to compute the partition $\mathbf{p}_G$ corresponding to this Richardson orbit.
  (Or look the partition up in the tables I provided in Appendix B.)
- Construct the partition $\mathbf{p}_A$ corresponding to the A-head in exactly the same way as we did for Γ's for $\mathfrak{sl}(n)$.
- Concatenate $\mathbf{p}_G$ with **two** copies of $\mathbf{p}_A$ and then add as many 1's as necessary to get a partition in $\mathcal{P}_G$.
  N.B. the parity/multiplicity criteria of $\mathcal{P}_G$ will automatically be satisfied.
- The partition $\mathbf{p}_{[\Gamma,\gamma]}$ you end up with will be the partition in $\mathcal{P}_G$ corresponding to $\mathcal{O}_{[\Gamma,\gamma]}$

Example: a CBC diagram for $\mathfrak{sp}(16)$

$$\bullet - \bullet - \circ - \bullet - \circ - \bullet - * \Leftarrow \bullet \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \quad \begin{array}{ll} \bullet - * \Leftarrow \bullet & G\text{-tail} \\ \bullet - \bullet \quad \bullet & A\text{-head} \end{array}$$

$$\begin{array}{rcl} \bullet - \bullet \quad \bullet & \Longrightarrow & [3,3,2,2] \\ \bullet - * \Leftarrow \bullet & \Longrightarrow & [4,2] \quad (\text{from tables for } C_3) \end{array}$$

$$\mathbf{p} = [4,3,3,2,2,2] \in \mathcal{P}_C(16)$$

Example: going the other way

Question: What is the largest Levi subalgebra containing a representative of the nilpotent orbit of $\mathfrak{sp}(16)$ corresponding to the partition $[4, 3, 3, 2, 2, 2]$?

- We first try to view as $[4, 3, 3, 2, 2, 2]$ as a concatenation of two partitions; one consisting of parts with even multiplicities and the other consisting of distinct even parts.

$$[4, 3, 3, 2, 2, 2] \rightsquigarrow [3, 3, 2, 2] \mid [4, 2]$$

- The "subpart" $[3, 3, 2, 2]$ corresponds to a A-head of the form $\bullet - \bullet - \circ - \bullet$.
- Consulting a table of distinguished orbits, one finds that the "subpart" $[4, 2]$ is the partition attached to the distinguished orbit of $\mathfrak{sp}(6)$ with CBC diagram $\bullet - * \Leftarrow \bullet$. This is the G-tail of the CBC of $\mathcal{O}_{[4,3,3,2,2,2]}$.

- Attaching the A-head to the G-tail, with an empty node in between, we obtain the CBC diagram of $\mathcal{O}_{[4,3,3,2,2,2]}$.

$$\bullet - \bullet - \circ - \bullet - \circ - \bullet - * \Leftarrow \bullet$$

- Evidently, the maximal Levi subalgebra containing an $x \in \mathcal{O}_{[4,3,3,2,2,2]}$ will be of type $\mathfrak{gl}(3) + \mathfrak{gl}(2) + \mathfrak{sp}(3)$.
- Note that we get not only the isomorphism class of the desired Levi but also the simple roots generating its semisimple part.
- In fact, we see in **exactly** which distinguished orbit of the Levi subalgebra the element x resides.