Solutions to Problem Set 4

1. (Problem 3.4.3 in text)

(a) Consider an infinite-interval problem, \(-\infty < x < +\infty\), for which

\[
\begin{align*}
\mathbf{u}(x,0) &= \begin{cases} 
  h(x) & \text{for } x > 0 \\
  -h(-x) & \text{for } x < 0 
\end{cases} \\
\mathbf{u}_t(x,0) &= 0
\end{align*}
\]

Show that the solution of

\[
\mathbf{u}_{tt} - c^2 \mathbf{u}_{xx} = 0
\]

satisfying these initial conditions also solves the following semi-infinite problem: find \(\mathbf{u}(x,t)\) satisfying \(\mathbf{u}_{tt} - c^2 \mathbf{u}_{xx} = 0, x \in (0, +\infty)\), with initial conditions \(\mathbf{u}(x,0) = h(x), \mathbf{u}_t(x,0) = 0\), and with fixed end condition \(\mathbf{u}(0,t) = 0\). [Here \(h(x)\) is any given function, with \(h(0) = 0\).] Sketch the solution for the case where \(h(x) = \frac{1}{2} - |x - \frac{3}{2}|\) for \(1 < x < 2\), \(h(x) = 0\) elsewhere.

(b) Use a similar idea to explain how you could use

\[
\mathbf{u}(x,t) = \frac{1}{2} \left[ \mathbf{u}(x + ct,0) + \mathbf{u}(x - ct,0) \right] + \frac{1}{2c} \int_{x - ct}^{x + ct} \mathbf{u}_t(\tau,0) \, d\tau
\]

to solve any finite interval problem in which \(\mathbf{u}(0,t) = \mathbf{u}(l,t) = 0\) for all \(t\), with \(\mathbf{u}(x,0) = h(x)\) and \(\mathbf{u}_t(x,0) = 0\) for \(0 < x < l\). [We take \(h(0) = h(l) = 0\).]

(c) Reconsider parts (a) and (b) for situations in which \(\mathbf{u}_t(x,0)\) is prescribed, with \(\mathbf{u}(x,0) = 0\). Sketch the solution for a simple case.

(a) Equation (D.2) gives the unique solution of the wave equation in the region \(-\infty < x < +\infty, 0 < t < +\infty\), in terms of its Cauchy data along the \(x\)-axis. If we use Eq. (D.1) to extend a Cauchy problem on the positive \(x\)-axis to the entire \(x\)-axis, then by restricting Eq. (D.2) to the region \((0, +\infty) \times (0, +\infty)\) we obtain a solution of the wave equation satisfying the boundary conditions \(\mathbf{u}(x,0) = h(x), \mathbf{u}_t(x,0) = 0\), for all \(x \in (0, +\infty)\). We only have to check that if the boundary condition \(\mathbf{u}(0,t) = 0\) is also satisfied. From (D.2) we have

\[
\begin{align*}
\mathbf{u}(x,0) &= \frac{1}{2} \left[ \mathbf{u}(x,0) + \mathbf{u}(x,0) \right] + \frac{1}{2c} \int_{x}^{x + ct} \mathbf{u}_t(\tau,0) \, d\tau \\
&= \frac{1}{2} h(x) \\
&= 0 \\
\end{align*}
\]

\(\square\)

(b) Our problem now is to define an extension \(H(x)\) of the function \(h(x)\) defined on \((0,l)\) to the entire \(x\)-axis so that Eq. (8.22) can be used to write down the solution of the wave equation in the region \((0,l) \times (0, +\infty)\) satisfying

\[
\begin{align*}
\mathbf{u}(x,0) &= h(x) \\
\mathbf{u}_t(x,0) &= 0 \\
\mathbf{u}(0,t) &= 0 \\
\mathbf{u}(l,t) &= 0
\end{align*}
\]

\[\text{Appendix D}\]

\[\text{Solutions to Problem Set 4}\]

1. (Problem 3.4.3 in text)

(a) Consider an infinite-interval problem, \(-\infty < x < +\infty\), for which

\[
\begin{align*}
\mathbf{u}(x,0) &= \begin{cases} 
  h(x) & \text{for } x > 0 \\
  -h(-x) & \text{for } x < 0 
\end{cases} \\
\mathbf{u}_t(x,0) &= 0
\end{align*}
\]

Show that the solution of

\[
\mathbf{u}_{tt} - c^2 \mathbf{u}_{xx} = 0
\]

satisfying these initial conditions also solves the following semi-infinite problem: find \(\mathbf{u}(x,t)\) satisfying \(\mathbf{u}_{tt} - c^2 \mathbf{u}_{xx} = 0, x \in (0, +\infty)\), with initial conditions \(\mathbf{u}(x,0) = h(x), \mathbf{u}_t(x,0) = 0\), and with fixed end condition \(\mathbf{u}(0,t) = 0\). [Here \(h(x)\) is any given function, with \(h(0) = 0\).] Sketch the solution for the case where \(h(x) = \frac{1}{2} - |x - \frac{3}{2}|\) for \(1 < x < 2\), \(h(x) = 0\) elsewhere.

(b) Use a similar idea to explain how you could use

\[
\mathbf{u}(x,t) = \frac{1}{2} \left[ \mathbf{u}(x + ct,0) + \mathbf{u}(x - ct,0) \right] + \frac{1}{2c} \int_{x - ct}^{x + ct} \mathbf{u}_t(\tau,0) \, d\tau
\]

to solve any finite interval problem in which \(\mathbf{u}(0,t) = \mathbf{u}(l,t) = 0\) for all \(t\), with \(\mathbf{u}(x,0) = h(x)\) and \(\mathbf{u}_t(x,0) = 0\) for \(0 < x < l\). [We take \(h(0) = h(l) = 0\).]

(c) Reconsider parts (a) and (b) for situations in which \(\mathbf{u}_t(x,0)\) is prescribed, with \(\mathbf{u}(x,0) = 0\). Sketch the solution for a simple case.

(a) Equation (D.2) gives the unique solution of the wave equation in the region \(-\infty < x < +\infty, 0 < t < +\infty\), in terms of its Cauchy data along the \(x\)-axis. If we use Eq. (D.1) to extend a Cauchy problem on the positive \(x\)-axis to the entire \(x\)-axis, then by restricting Eq. (D.2) to the region \((0, +\infty) \times (0, +\infty)\) we obtain a solution of the wave equation satisfying the boundary conditions \(\mathbf{u}(x,0) = h(x), \mathbf{u}_t(x,0) = 0\), for all \(x \in (0, +\infty)\). We only have to check that if the boundary condition \(\mathbf{u}(0,t) = 0\) is also satisfied. From (D.2) we have

\[
\begin{align*}
\mathbf{u}(x,0) &= \frac{1}{2} \left[ \mathbf{u}(x,0) + \mathbf{u}(x,0) \right] + \frac{1}{2c} \int_{x}^{x + ct} \mathbf{u}_t(\tau,0) \, d\tau \\
&= \frac{1}{2} h(x) \\
&= 0 \\
\end{align*}
\]

\(\square\)

(b) Our problem now is to define an extension \(H(x)\) of the function \(h(x)\) defined on \((0,l)\) to the entire \(x\)-axis so that Eq. (8.22) can be used to write down the solution of the wave equation in the region \((0,l) \times (0, +\infty)\) satisfying

\[
\begin{align*}
\mathbf{u}(x,0) &= h(x) \\
\mathbf{u}_t(x,0) &= 0 \\
\mathbf{u}(0,t) &= 0 \\
\mathbf{u}(l,t) &= 0
\end{align*}
\]
The validity of the first two boundary conditions will be automatic (since the restriction of our extension must give us exactly what we started with).

Setting \( u(x, 0) = H(x) \), \( u_t(x, 0) = 0 \) we obtain from Eq. (8.22)
\[
u(x, t) = \frac{1}{2} [H(x + ct) + H(x - ct)]
\]
The third boundary condition in (D.4) thus leads to
\[
0 = u(0, t) = \frac{1}{2} [H(ct) + H(-ct)]
\]
This will be satisfied automatically if we extend \( h(x) \) is such a way that the new function \( H(x) \) is odd with respect to reflections about \( x = 0 \).

The last boundary condition thus leads to
\[
0 = u(l, t) = \frac{1}{2} [H(l + ct) + H(l - ct)]
\]
This will be satisfied automatically if we extend \( h(x) \) is such a way that the new function \( H(x) \) is odd with respect to reflections about \( x = l \).

We thus define \( H(x) \) as follows:
\[
g(x) = \begin{cases} 
  h(2l + x) & , -2l < x < -l \\
  -h(-x) & , -l < x < 0 \\
  h(x) & , 0 < x < l \\
  -h(x - l) & , l < x < 2l 
\end{cases}
H(x) = g(x - 4nl) , \quad 4nl - 2l < x < 4nl + 2l , \quad n \in \mathbb{Z} \quad .
\]

(c) If instead we had boundary conditions of the form
\[
\begin{align*}
  u(x, 0) &= 0 \quad , \quad 0 < x < l \\
  u_t(x, 0) &= p(x) \quad , \quad 0 < x < l \\
  u(0, t) &= 0 \\
  u(l, t) &= 0
\end{align*}
\]
we would seek to extend the definition of \( p(x) \) to the entire \( x \)-axis so that the last two boundary conditions are satisfied automatically. We would thus need to define \( P(x) \) such that
\[
\begin{align*}
  0 &= u(0, t) = \frac{1}{2c} \int_{-ct}^{ct} P(\tau) d\tau \\
  0 &= u(l, t) = \frac{1}{2c} \int_{l - ct}^{l + ct} P(\tau) d\tau
\end{align*}
\]
automatically. To accomplish this we can simply extend \( p(x) \) in such a way that it is periodic with period \( 4l \) and antisymmetric with respect to reflections about \( x = 0 \) and \( x = l \).
2. (Problem 3.4.4 in text)

Consider the “whip-cracking” problem:

\[ \phi_{tt} - c^2 \phi_{xx} = 0 \]
\[ \phi(x, 0) = 0 \]
\[ \phi_t(x, 0) = 0 \]
\[ \phi(0, t) = \gamma(t) \]
\[ \phi(0, 0) = 0 \]

in the region \( x > 0, t > 0 \).

We know from the discussion in Lecture 9 that

\[ \phi(x, t) = a(x + ct) + \beta(x - ct) \]

is the general solution to the wave equation

\[ \phi_{tt} - c^2 \phi_{xx} = 0 \]

The boundary conditions in (D.11) imply

\[ a(x) + \beta(x) = 0 \]
\[ ca'(x) - c\beta'(x) = 0 \]
\[ a(ct) - \beta(-ct) = \gamma(t) \]

The equation tells us that \( \beta(x) = -a(x) \). Making this substitution, we get from the second equation that

\[ 2ca'(x) = 0 \]

so

\[ a(x) = K \]

It would seem that this trivializes everything; however, the first two conditions in (D.14) are only imposed only for \( x > 0 \). We are therefore free to adjust the functions \( a(x) \) and \( \beta(x) \) in the region where \( x < 0 \). The third equation, in fact, tells us that

\[ a(cx) + \beta(-cx) = \gamma(x) \]

which we can satisfy by extending the definition of \( \beta(x) \) to \( x < 0 \)

\[ \beta(-x) = -K + \gamma \left( \frac{x}{c} \right) , \quad \forall x > 0 \]

It is not necessary to extend the domain of \( a(x) \) to \( x < 0 \), since in the expression (D.12) for \( \phi(x,t) \), the argument of \( a \) is always positive. Thus, we take

\[ a(\zeta) = \begin{cases} K, & \zeta > 0 \\ K \frac{\zeta}{c}, & \zeta < 0 \end{cases} \]

\[ \beta(\eta) = \begin{cases} \gamma \left( \frac{\eta}{c} \right) - K, & \eta < 0 \\ \gamma \left( \frac{\eta}{c} \right) - K, & \eta > 0 \end{cases} \]

Thus,

\[ \Phi(\zeta, \eta) = a(\zeta) + \beta(\eta) = \begin{cases} \gamma \left( \frac{\eta}{c} \right), & \eta < 0 \\ 0, & \eta > 0 \end{cases} \]

and so the solution of (D.11) is

\[ \phi(x, t) = \begin{cases} \gamma \left( t - \frac{x}{c} \right), & x - ct < 0 \\ 0, & x - ct > 0 \end{cases} \]
(a) Let \( u(x,t) \) satisfy the equation

\[
u_{tt} = c^2 u_{xx} \quad , \quad c = \text{constant} ,
\]

in some region of the \((x,t)\) plane. Show that the quantity \((u_t - cu_x)\) is constant along each straight line defined by \(x - ct = \text{constant}\), and that \((u_t + cu_x)\) is constant along each straight line of the form \(x + ct = \text{constant}\). These straight lines are called characteristics; we will refer to typical members of the two families as \(C_+\) and \(C_-\) curves, respectively; thus \((x - ct = \text{constant})\) is a \(C_+\) curve.

Set

\[
\phi_+(x,t) = u_t(x,t) - cu_x(x,t) .
\]

Along a \(C_+\) curve we have

\[
x = k_1 + ct
\]

and so along such a curve

\[
\phi_+(x,t) = \phi_+(t) = u_t(k_1 + ct,t) - cu_x(k_1 + ct,t) .
\]

Differentiating \(\phi_+\) with respect to \(t\) we obtain

\[
\frac{d\phi_+}{dt} = cu_{tx} + u_{tt} - c^2 u_{xx} - cu_{tx}
\]

\[
= u_{tt} - c^2 u_{xx}
\]

\[
= 0
\]

since \(u\) satisfies the wave equation. Therefore, \(\phi_+\) is constant along any curve of the form \((D.18)\).

Similarly, if we set

\[
\phi_-(x,t) = u_t(x,t) + cu_x(x,t) .
\]

Then along the curve

\[
x = k_2 - ct
\]

we have

\[
\phi_-(x,t) = \phi_-(t) = u_t(k_2 - ct,t) + cu_x(k_2 - ct,t)
\]

and so

\[
\frac{d\phi_-}{dt} = -cu_{tx} + u_{tt} - c^2 u_{xx} + cu_{tx}
\]

\[
= u_{tt} - c^2 u_{xx}
\]

\[
= 0
\]

Thus, \(\phi_-\) is constant along any curve of the form \((D.24)\).

(b) Let \(u(x,0)\) and \(u_t(x,0)\) be prescribed for all values of \(x\) between \(-\infty\) and \(+\infty\), and let \((x_s,t_s)\) be some point in the \((x,t)\) plane, with \(t_s > 0\). Draw the \(C_+\) and \(C_-\) curves through \((x_s,t_s)\) and let \(A\) and \(B\) denote, respectively, their intercepts with the \(x\)-axis. Use the properties of \(C_+\) and \(C_-\) derived in part (a) to determine \(u_t(x_s,t_o)\) in terms of initial data at points \((A,0)\) and \((B,0)\). Using a similar technique to obtain \(u_t(x_s,\tau)\) with \(0 < \tau < t_o\), determine \(u(x_s,t_o)\) by integration with respect to \(\tau\), and compare with Equation (8.22). Observe that this “method of characteristics” again shows that \(u(x_s,t_o)\) depends only on that part of the initial data between points \((A,0)\) and \((B,0)\).

Let

\[
k_{\pm} = x_o \mp ct_o .
\]
and set

\[(D.29) \quad c_{\pm} = \{(x,t) \in \mathbb{R}^2 \mid x \mp ct = k_{\pm}\}\]

From part (a) we know that

\[(D.30) \quad \phi_+ = u_t(x,t) - cu_x(x,t) \quad \phi_- = u_t(x,t) + cu_x(x,t)\]

are, respectively, constant along the lines \(c_+\) and \(c_-\).

At the point \((A,0)\) where the line \(c_+\) intersects the \(x\)-axis we have

\[(D.31) \quad \phi_+ = u_t(A,0) - cu_x(A,0)\]

and so the constant \(\phi_+\) is completely determined by the Cauchy data at the point \((A,0)\).

Similarly, at the point \((B,0)\) where the line \(c_-\) intersects the \(x\)-axis we have

\[(D.32) \quad \phi_- = u_t(B,0) + cu_x(B,0)\]

and so the constant \(\phi_-\) is completely determined by the Cauchy data at the point \((B,0)\).

Using (D.31) and (D.32) we can rewrite equations (D.30) as

\[(D.33) \quad u_t(A,0) - cu_x(A,0) = u_t(x,0,t_0) - cu_x(x,0,t_0) \quad u_t(B,0) + cu_x(B,0) = u_t(x,0,t_0) + cu_x(x,0,t_0)\]

Adding the second equation to the first and then dividing by 2 we obtain

\[(D.34) \quad u(x,0,t) = \frac{1}{2} (u_t(A,0) + u_t(B,0) - cu_x(A,0) + cu_x(B,0))\]

We can be even more explicit than this. For the value of \(A\) is precisely \(k_+ = x_o - ct_o\), and the value of \(B\) is precisely \(k_- = x_o + ct_o\). Thus,

\[(D.35) \quad u_t(x_o,t_o) = \frac{1}{2} (u_t(x_o - ct_o,0) + u_t(x_o + ct_o,0)) + \frac{1}{2} \left(-u_x(x_o - ct_o,0) + u_x(x_o + ct_o,0)\right)\]

This equation is perfectly valid for any choice of \(x_o\) and \(t_o\), and so we can write

\[(D.36) \quad u_t(x_o,t) = \frac{1}{2} (u_t(x_o - ct,0) + u_t(x_o + ct,0)) + \frac{1}{2} \left(-u_x(x_o - ct,0) + u_x(x_o + ct,0)\right)\]

Integrating both sides with respect to \(t\) from 0 to \(t_o\) we obtain

\[(D.37) \quad u(x_o,t_o) - u(x_o,0) = \frac{1}{2} \int_0^{t_o} u_t(x_o - ct,0) \, dt + \frac{1}{2} \int_0^{t_o} u_t(x_o + ct,0) \, dt \]

\[-\frac{1}{2} \int_0^{t_o} u_x(x_o - ct,0) \, dt + \frac{1}{2} \int_0^{t_o} u_x(x_o + ct,0) \, dt\]
If we make a change of variables $\zeta = x_o - ct$ in the first and third integrals and a change of variables $\zeta = x_o + ct$ in the second and fourth integrals, the (D.37) becomes

$$u(x_o, t_o) - u(x_o, 0) = -\frac{1}{2c} \int_{x_o}^{x_o-ct_o} u_t(\zeta, 0) \, d\zeta + \frac{1}{2c} \int_{x_o}^{x_o+ct_o} u_t(\zeta, 0) \, d\zeta$$

(D.38)

$$+ \frac{1}{2} \int_{x_o}^{x_o-ct_o} u_x(\zeta, 0) \, d\zeta + \frac{1}{2} \int_{x_o}^{x_o+ct_o} u_x(\zeta, 0) \, d\zeta$$

(D.39)

$$= \frac{1}{2c} \int_{x_o-ct_o}^{x_o} u_t(\zeta, 0) \, d\zeta$$

(D.40)

$$+ \frac{1}{2} u(\zeta, 0) \int_{x_o}^{x_o-ct_o} + \frac{1}{2} u(\zeta, 0) \int_{x_o}^{x_o+ct_o}$$

(D.41)

$$= \frac{1}{2c} \int_{x_o-ct_o}^{x_o} u_t(\zeta, 0) \, d\zeta$$

(D.42)

$$+ \frac{1}{2} u(x_o - ct_o) + u(x_o + ct_o) + u(x_o, 0)$$

or

$$u(x_o, t_o) = \frac{1}{2} (u(x_o - ct_o) + u(x_o + ct_o)) + \frac{1}{2c} \int_{x_o-ct_o}^{x_o+ct_o} u_t(\zeta, 0) \, d\zeta$$

(D.43)

(D.44)

which is precisely Equation (8.22).