LECTURE 21

Green’s Identities

Let us recall Stokes’ Theorem in \(n\)-dimensions.

**Theorem 21.1.** Let \( \mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a vector field over \( \mathbb{R}^n \) that is of class \( C^1 \) on some closed, connected, simply connected \( n \)-dimensional region \( D \subseteq \mathbb{R}^n \). Then

\[
\int_D \nabla \cdot \mathbf{F} \, dV = \int_{\partial D} \mathbf{F} \cdot \mathbf{n} \, dS
\]

where \( \partial D \) is the boundary of \( D \) and \( \mathbf{n}(\mathbf{r}) \) is the unit vector that is (outward) normal to the surface \( \partial D \) at the point \( \mathbf{r} \in \partial D \).

As a special case of Stokes’ theorem, we may set

\[
\mathbf{F} = \nabla \phi
\]

with \( \phi \) a \( C^2 \) function on \( D \). We then obtain

\[
\int_D \nabla^2 \phi \, dV = \int_{\partial D} \nabla \phi \cdot \mathbf{n} \, dS .
\]

Recall that the identity (21.2) was essential to the proof that any extrema of a solution \( \phi \) of 2-dimensional Laplace’s equation

\[
\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0
\]

must occur on the boundary of region. The analogous proposition about extrema for solutions of Laplace’s equation in \( n \)-dimensions is also true and again it is relatively easy consequence of (21.2).

Another special case of Stokes’ theorem comes from the choice

\[
\mathbf{F} = \phi \nabla \psi .
\]

For this case, Stokes’ theorem says

\[
\int_D \nabla \cdot (\phi \nabla \psi) \, dV = \int_{\partial D} \phi \nabla \psi \cdot \mathbf{n} \, dS .
\]

Using the identity

\[
\nabla \cdot (\phi \mathbf{F}) = \nabla \phi \cdot \mathbf{F} + \phi \nabla \cdot \mathbf{F}
\]

we find (21.4) is equivalent to

\[
\int_D \nabla \phi \cdot \nabla \psi \, dV + \int_D \phi \nabla^2 \psi \, dV = \int_{\partial D} \phi \nabla \psi \cdot \mathbf{n} \, dS .
\]

Equation (21.6) is known as Green’s first identity.

Reversing the roles of \( \phi \) and \( \psi \) in (21.6) we obtain

\[
\int_D \nabla \psi \cdot \nabla \phi \, dV + \int_D \psi \nabla^2 \phi \, dV = \int_{\partial D} \psi \nabla \phi \cdot \mathbf{n} \, dS .
\]
Finally, subtracting (21.7) from (21.6) we get

\begin{equation}
\int_D (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, dV = \int_{\partial D} (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n} \, dS.
\end{equation}

Equation (21.8) is known as **Green’s second identity**.

Now set

\[ \psi(\mathbf{r}) = \frac{1}{|\mathbf{r} - \mathbf{r}_0| + \epsilon} \]

and insert this expression into (21.8). We then get

\[ \int_D \phi \left( \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}_0| + \epsilon} \right) \, dV = \int_D \frac{1}{|\mathbf{r} - \mathbf{r}_0| + \epsilon} \nabla^2 \phi \, dV \\
+ \int_{\partial D} \left( \frac{1}{|\mathbf{r} - \mathbf{r}_0| + \epsilon} \nabla \phi - \phi \left( \nabla \frac{1}{|\mathbf{r} - \mathbf{r}_0| + \epsilon} \right) \cdot \mathbf{n} \, dS \right). \]

Taking the limit \( \epsilon \to 0 \) and using the identities

\[ \lim_{\epsilon \to 0} \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}_0| + \epsilon} = -4\pi \delta^{(n)}(\mathbf{r} - \mathbf{r}_0) \]
\[ \lim_{\epsilon \to 0} \frac{1}{|\mathbf{r} - \mathbf{r}_0| + \epsilon} = \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \]
\[ \lim_{\epsilon \to 0} \nabla \frac{1}{|\mathbf{r} - \mathbf{r}_0| + \epsilon} = \nabla \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \]

we obtain

\begin{equation}
-4\pi \phi(\mathbf{r}_0) = \int_D \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \nabla^2 \phi \, dV \\
+ \int_{\partial D} \left( \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \nabla \phi - \phi \left( \nabla \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \right) \cdot \mathbf{n} \, dS \right).
\end{equation}

Equation (21.9) is known as **Green’s third identity**.

Notice that if \( \phi \) satisfies Laplace’s equation the first term on the right hand side vanishes and so we have

\begin{equation}
\phi(\mathbf{r}_0) = \frac{1}{4\pi} \int_{\partial D} \left( \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \nabla \phi - \phi \left( \nabla \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \right) \cdot \mathbf{n} \, dS \right)
= \frac{1}{4\pi} \int_{\partial D} \left( \phi \frac{\nabla}{\nabla|\mathbf{r} - \mathbf{r}_0|} - \frac{1}{|\mathbf{r} - \mathbf{r}_0|^2} \frac{\partial \phi}{\partial n} \right) \, dS.
\end{equation}

Here \( \frac{\partial}{\partial n} \) is the directional derivative corresponding to the surface normal vector \( \mathbf{n} \). Thus, if \( \phi \) satisfies Laplace’s equation in \( D \) then its value at any point \( \mathbf{r}_0 \in D \) is completely determined by the values of \( \phi \) and \( \frac{\partial}{\partial n} \) on the boundary of \( D \).
1. Green’s Functions and Solutions of Laplace’s Equation, II

Recall the fundamental solutions of Laplace’s equation in $n$-dimensions

$$\Phi_n(r, \psi, \theta_1, \ldots, \theta_{n-2}) = \begin{cases} \log |r|, & \text{if } n = 2 \\ \frac{c_n}{r^{n-1}}, & \text{if } n > 2 \end{cases}$$

Each of these solutions really only makes sense in the region $\mathbb{R}^n - \{0\}$; for each possesses a singularity at the origin.

We studied the case when $n = 3$, a little more closely and found that we could actually write

$$\nabla^2 \left( \frac{1}{r} \right) = -4\pi \delta^3(r) = \begin{cases} 0, & \text{if } r \neq 0 \\ \infty, & \text{if } r = 0 \end{cases}$$

In fact, using similar arguments one can show that

$$\nabla^2 \Phi(r) = -c_n \delta^n(r)$$

where $c_n$ is the surface area of the unit sphere in $\mathbb{R}^n$. Thus, the fundamental solutions can actually be regarded as solutions of an inhomogeneous Laplace equation where the driving function is concentrated at a single point.

Let us now set $n = 3$ and consider the following PDE/BVP

$$\begin{align*}
\nabla^2 \Phi(r) &= f(r), \quad r \in D \\
\Phi(r)|_{\partial D} &= h(r)|_{\partial D}
\end{align*}$$

where $D$ is some closed, connected, simply connected region in $\mathbb{R}^3$. Let $r_s$ be some fixed point in $D$ and set

$$G(r, r_s) = \frac{-1}{4\pi |r - r_s|} + \phi_s(r, r_s)$$

where $\phi_s(r, r_s)$ is some solution of the homogeneous Laplace equation

$$\nabla^2 \phi_s(r, r_s) = 0.$$ 

Then

$$\nabla^2 G(r, r_s) = \delta^3(r - r_s).$$

Now recall Green’s third identity

$$\int_D \left( \Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi \right) dV = \int_{\partial D} \left( \Phi \nabla \Psi - \Psi \nabla \Phi \right) \cdot \mathbf{n} dS.$$

If we replace $\psi$ in (21.18) by $G(r, r_s)$ we get

$$\begin{align*}
\Phi(r_s) &= \int_D \Phi(r) \delta^3(r - r_s) dV \\
&= \int_D \Phi \nabla^2 G dV \\
&= \int_D \nabla^2 \Phi dV + \int_{\partial D} \left( \Phi \nabla G - G \nabla \Phi \right) \cdot \mathbf{n} dS \\
&= \int_D \nabla f dV + \int_{\partial D} h \frac{\partial G}{\partial n} - G \frac{\partial h}{\partial n} dS \\
&= \int_D \nabla f dV + \int_{\partial D} h \frac{\partial G}{\partial n} dS - \int_{\partial D} G \frac{\partial h}{\partial n} dS.
\end{align*}$$

Up to this point we have only required that the function $\phi_s$ satisfies Laplace’s equation. We will now make our choice of $\phi_s$ more particular; we shall choose $\phi_s(r, r_s)$ to be the unique solution of Laplace’s equation in $D$ satisfying the boundary condition

$$\frac{1}{4\pi |r - r_s|}|_{\partial D} = \phi_s(r, r_s)|_{\partial D}$$

so that

$$G(r, r_s)|_{\partial D} = 0.$$
Then the last integral on the right hand side of (21.19) vanishes and so we have

\[(21.21) \quad \Phi (r, \mathbf{r}_0) = \int_D G(r, \mathbf{r}_0) f(\mathbf{r}) \, dV + \int_{\partial D} h(\mathbf{r}) \frac{\partial G}{\partial n} (r, \mathbf{r}_0) \, dS.\]

Thus, once we find a solution \(\phi_\circ (r, \mathbf{r}_0)\) to the homogeneous Laplace equation satisfying the boundary condition (21.20), we have a closed formula for the solution of the PDE/BVP (21.14) in terms of integrals of \(G(r, \mathbf{r}_0)\) times the driving function \(f(\mathbf{r})\), and of \(\frac{\partial G}{\partial n} (r, \mathbf{r}_0)\) times the function \(h(r)\) describing the boundary conditions on \(\Phi\). Note that the Green’s function \(G(r, \mathbf{r}_0)\) is fixed once we fix \(\phi_\circ\) which in turn depends only on the nature of the boundary of the region \(D\) (through condition (21.20)).

**Example**

Let us find the Green’s function corresponding to the interior of sphere of radius \(R\) centered about the origin. We seek to find a solution of \(\phi_\circ\) of the homogenous Laplace’s equation such that (21.20) is satisfied. This is accomplished by the following trick.

Suppose \(\Phi (r, \psi, \theta)\) is a solution of the homogeneous Laplace equation inside the sphere of radius \(R\) centered at the origin. For \(r > R\), we define a function

\[(21.22) \quad \hat{\Phi} (r, \psi, \theta) = \frac{R}{r} \Phi \left( \frac{R^2}{r}, \psi, \theta \right).\]

I claim that \(\hat{\Phi} (r, \psi, \theta)\) so defined also satisfies Laplace’s equation in the region exterior to the sphere.

To prove this, it suffices to show that

\[(21.23) \quad 0 = r^2 \nabla \hat{\Phi} = \frac{\partial}{\partial r} \left( r^2 \frac{\partial \hat{\Phi}}{\partial r} \right) + \frac{1}{\sin \psi} \frac{\partial}{\partial \psi} \left( \sin \psi \frac{\partial \hat{\Phi}}{\partial \psi} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \hat{\Phi}}{\partial \theta^2} \]

or

\[(21.24) \quad \frac{\partial}{\partial r} \left( r^2 \frac{\partial \hat{\Phi}}{\partial r} \right) = - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \hat{\Phi}}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2 \hat{\Phi}}{\partial \psi^2} \]

Set

\[(21.25) \quad u = \frac{R^2}{r} \]

so that

\[(21.26) \quad \hat{\Phi} (r, \psi, \theta) = \frac{R}{u} \Phi \left( u, \psi, \theta \right)
\quad \frac{\partial}{\partial r} = - \frac{\partial u}{\partial r} \frac{\partial}{\partial u} = - \frac{R^2}{r^2} \frac{\partial}{\partial u} = - \frac{u^2}{R^2} \frac{\partial}{\partial u} \]

and so

\[(21.27) \quad \frac{\partial}{\partial r} \left( r^2 \frac{\partial \hat{\Phi}}{\partial r} \right) = \left( \frac{u^2}{R^2} \frac{\partial}{\partial u} \right) \left( \frac{R}{u} \Phi \right) - \frac{u^2}{R^2} \frac{\partial}{\partial u} \left( \frac{R}{u} \frac{\partial \Phi}{\partial u} \right) \]

\[= \frac{u^2}{R^2} \frac{\partial}{\partial u} \left( u \Phi \right) \]

\[= \frac{u}{R} \left( u \frac{\partial \Phi}{\partial u} + 2 \frac{\partial u \Phi}{\partial u} \right) \]

\[= \frac{u}{R} \left( \frac{\partial}{\partial u} \left( u \frac{\partial \Phi}{\partial u} \right) \right) \]

\[= - \frac{u}{R} \left( \frac{\partial}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \Phi}{\partial \psi^2} \]

Notice that

\[(21.28) \quad \lim_{r \to R} \hat{\Phi} (r, \psi, \theta) = \Phi (r, \psi, \theta) \]

This transform is called *Kelvin inversion*. 
Now let return to the problem of finding a Green’s function for the interior of a sphere of radius. Let
\begin{equation}
\hat{r} = r \left( \frac{R^2}{r}, \psi, \theta \right) = \frac{R^2}{r^2} r.
\end{equation}
In view of the preceding remarks, we know that the functions
\begin{align}
\Phi_1 (r) &= \frac{1}{|r - r_s|} \\
\Phi_2 (r) &= \frac{1}{r^2 |r - r_s|} = \hat{\Phi}_1 (r)
\end{align}
will satisfy, respectively,
\begin{align}
\nabla^2 \Phi_1 (r) &= -4\pi \delta^3 (r - r_s) \\
\nabla^2 \Phi_2 (r) &= -\frac{4\pi \delta^3}{r} \left( \frac{R^2}{r^2} - r_s \right).
\end{align}
However, notice that the support of $\nabla^2 \Phi_2 (r)$ lies completely outside the sphere. Therefore, in the interior of the sphere, $\Phi_3$ is a solution of the homogenous Laplace equation. We also know that on the boundary of the sphere that we have
\begin{equation}
\Phi_1 (r) = \Phi_2 (r) = 0.
\end{equation}
Thus, the function
\begin{equation}
G (r, r_s) = \frac{R^2}{r^2 |r - r_s|} - \frac{1}{4\pi |r - r_s|}
\end{equation}
thus satisfies
\begin{equation}
\nabla^2 G (r, r_s) = \delta^3 (r - r_s)
\end{equation}
for all $r$ inside the sphere and
\begin{equation}
G (r, r_s) = 0
\end{equation}
or all $r$ on the boundary of the sphere. Thus, the function $G (r, r_s)$ defined by (21.33) is the Green’s function for Laplace’s equation within the sphere.

Now consider the following PDE/BVP
\begin{align}
\nabla^2 \Phi (r) &= f (r), \quad r \in B \\
\Phi (R, \psi, \theta) &= 0
\end{align}
where $B$ is a ball of radius $R$ centered about the origin. According to the formula (21.21) and (21.33), the solution of (21.36) is given by
\begin{align}
\Phi (r_s) &= \int_B G (r, r_s) f (r) \, dV + \int_{\partial B} h (\psi, \theta) \frac{\partial G}{\partial n} (r, r_s) \, dS \\
&= \int_B G (r, r_s) f (r) \, dV
\end{align}
To arrive at a more explicit expression, we set
\begin{align}
r_s &= (r \cos (\psi) \sin (\theta), r \sin (\psi) \sin (\theta), r \cos (\theta)) \\
r &= (\rho \cos (\alpha) \sin (\beta), \rho \sin (\alpha) \sin (\beta), \rho \cos (\beta))
\end{align}
Then
\begin{align}
dV &= \rho^2 \sin^2 (\theta) \, d\rho \, d\alpha \, d\beta \\
ds &= \rho^2 \sin^2 (\theta) \, d\rho \, d\beta
\end{align}
and after a little trigonometry one finds
\begin{align}
\frac{1}{4\pi |r - r_s|} &= \frac{1}{4\pi \sqrt{r^2 + \rho^2 - 2r \rho (\cos (\psi - \alpha) \sin (\beta) + \cos (\theta) \cos (\beta))}} \\
\frac{1}{4\pi |\frac{2}{r} r_s - \frac{2}{r} r_s|} &= \frac{R}{4\pi \sqrt{R^4 + r^2 \rho^2 - 2R^2 \rho (\cos (\psi - \alpha) \sin (\beta) + \cos (\theta) \cos (\beta))}}.
\end{align}
Thus,

\[ \Phi(r, \psi, \theta) = \int_0^R \int_0^{2\pi} \int_0^\pi \frac{R f(r, \psi, \theta) r^2 \sin(\theta) d\theta d\psi}{4\pi \sqrt{R^4 + r^2 \rho^2 - 2R^2 r \rho (\cos(\psi - \alpha) \sin(\theta) \sin(\beta) + \cos(\theta) \cos(\beta))}} - \int_0^R \int_0^{2\pi} \int_0^\pi \frac{f(r, \psi, \theta) r^2 \sin(\theta) d\theta d\psi}{4\pi \sqrt{r^2 + \rho^2 - 2r \rho (\cos(\psi - \alpha) \sin(\theta) \sin(\beta) + \cos(\theta) \cos(\beta))}} \]

**Homework:** 9.3.1, 9.3.9