LECTURE 20

Green’s Functions and Solutions of Laplace’s Equation, I

In our discussion of Laplace’s equation in three dimensions

\[
0 = \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}
\]

I pointed out one solution of special importance, the so-called fundamental solution

\[
\Phi(x, y, z) = \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.
\]

Note that due to the singularity at the point \((0,0,0)\), the solution \((20.2)\) is really only a solution for the region \(\mathbb{R}^3 - (0,0,0)\). The nature of this solution when \(r \to 0\) is worth examining a little closer.

In terms of spherical coordinates

\[
\begin{align*}
\rho &= \sqrt{x^2 + y^2 + z^2} \\
\psi &= \tan^{-1} \left( \frac{y}{x} \right) \\
\theta &= \tan^{-1} \left( \frac{\sqrt{x^2 + y^2}}{z} \right)
\end{align*}
\]

we have

\[
\begin{align*}
\Phi(\rho, \psi, \theta) &= \frac{1}{\rho} \\
\nabla &= \rho \frac{\partial}{\partial \rho} + \hat{\theta} \frac{\partial}{\partial \theta} + \hat{\psi} \frac{1}{\sin(\theta)} \frac{\partial}{\partial \psi} \\
\nabla^2 &= \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \right) + \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \psi^2}
\end{align*}
\]

where \(\hat{r}, \hat{\theta}, \hat{\psi}\) are respectively, the unit vectors indicating the directions of tangent vectors to the corresponding coordinate curves.

Applying the gradient \(\nabla\) and the Laplacian \(\nabla^2\) to our solution \((20.2)\) we get

\[
\begin{align*}
\nabla \Phi &= \rho \frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \right) = \frac{\Phi}{\rho^2} \\
\nabla^2 \Phi &= \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \frac{1}{\rho} \right) = 0.
\end{align*}
\]

However, we should note again that these formula are not really valid when \(r = 0\) (since \(\Phi\) is not continuous when \(r = 0\), we certainly cannot evaluate derivatives of \(\Phi\) when \(r = 0\)). To study the situation near \(r = 0\), let \(\epsilon > 0\) be a small positive parameter and define

\[
\Phi_\epsilon = \frac{1}{r + \epsilon}.
\]

Since \(r\) is never negative, \(\Phi_\epsilon\) is perfectly regular throughout \(\mathbb{R}^3\), and obviously

\[
\Phi = \lim_{\epsilon \to 0} \Phi_\epsilon.
\]
Applying the Laplacian to $\Phi$, yields

$$\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{-2r(r+\epsilon)^2 + 2r^2(r+\epsilon)}{(r+\epsilon)^2} \right) = \frac{1}{r^2} \frac{-2r}{r(r+\epsilon)} .$$

(20.8)

Now let us now consider the volume integral of $\nabla^2 \Phi$ over $\mathbb{R}^3$. We have

$$\int_{\mathbb{R}^3} \nabla^2 \Phi \, dV = \lim_{R \to \infty} \int_0^R \int_0^{2\pi} \int_0^\pi \frac{2r^2}{r(r+\epsilon)} r^2 \sin(\theta) \, dr \, d\theta \, d\phi$$

$$= \lim_{R \to \infty} \left[ \int_0^R \frac{8\pi r^2}{r+\epsilon} \, dr \right]_0^R$$

$$= \lim_{R \to \infty} \left[ \frac{8\pi r^2}{r+\epsilon} - \frac{4\pi r^3}{r^2} \right]_R^R$$

$$= -4\pi .$$

(20.9)

Notice the result we obtain is independent of $\epsilon$.

Thus, we have discovered a sequence of functions $f_\epsilon$

$$(20.10) \quad f_\epsilon(r) = \frac{1}{4\pi} \nabla^2 \Phi_\epsilon(r) = \frac{\epsilon}{2\pi r(r+\epsilon)^3}$$

for which

$$(20.11) \quad \lim_{\epsilon \to 0} f_\epsilon(r) = \begin{cases} 0, & \text{if } r \neq (0, 0, 0) \\ \infty, & \text{if } r = (0, 0, 0) \end{cases}$$

and for which

$$(20.12) \quad \int_{\mathbb{R}^3} f_\epsilon(r) \, dV = 1 \quad \forall \epsilon \neq 0$$

But properties (20.11) and (20.12) are exactly the properties that we demand for a sequence of functions to define a three-dimensional delta-function. (See Lecture 7.)

Indeed, let $g\,(r)$ be a differentiable function on $\mathbb{R}^3$ and consider the limit

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^3} g\,(r) f_\epsilon(r) \, dV .$$

According to (20.11) the support of $f_\epsilon(r)$ for small $\epsilon$ is concentrated around the origin. For example, if we set

$$\epsilon = 10^{-6} f_\epsilon(r) < \frac{10^{-6}}{2\pi}, \quad \forall \, r > 1$$

and if we set $\epsilon = 10^{-30}$

$$f_\epsilon(r) < \frac{10^{-6}}{2\pi}, \quad \forall \, r > 10^{-6} .$$

In the limit the support of $f_\epsilon(r)$ the integrand is precisely the origin $O$. Thus,

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^3} g\,(r) f_\epsilon(r) \, dV = \lim_{\epsilon \to 0} \int_{\mathbb{R}^3} g(O) f_\epsilon(r) \, dr = g(O) .$$

And so we set

$$(20.13) \quad \delta^3(r) = \lim_{\epsilon \to 0} \left( -\frac{1}{4\pi} \nabla^2 \left( \frac{1}{r + \epsilon} \right) \right)$$
with the understanding that the limit is to be taken only after integrating. By an abuse of notation one sometimes writes
\begin{equation}
\nabla^2 \left( \frac{1}{r} \right) = -4\pi \delta^3(r)
\end{equation}
or even more generally,
\begin{equation}
\nabla^2 \frac{1}{|r-r_o|} = -4\pi \delta^3 (r-r_o) .
\end{equation}

Okay, so what is the point of all this? Consider the non-homogeneous equation
\begin{equation}
\nabla^2 \Phi = -4\pi g(r)
\end{equation}
with \( g(r) \) decaying faster than \( \frac{1}{r^{d-\tau}} \) as \( r \to \infty \). Multiplying both sides of (20.16) by
\[
\frac{1}{|r-r_o|}
\]
and integrating over \( \mathbb{R}^3 \) we get
\[
\int_{\mathbb{R}^3} \frac{-4\pi}{|r-r_o|} g(r) dV = \int_{\mathbb{R}^3} \frac{1}{|r-r_o|} \nabla^2 \Phi(r) dV
\]
\[
= - \int_{\mathbb{R}^3} \nabla \left( \frac{1}{|r-r_o|} \right) \cdot \nabla \Phi(r) dV
\]
\[
= \int_{\mathbb{R}^3} \nabla^2 \left( \frac{1}{|r-r_o|} \right) \Phi(r) dV
\]
\[
= \int_{\mathbb{R}^3} -4\pi \delta^3 (r-r_o) \Phi(r) dV
\]
\[
= -4\pi \Phi(r_o) .
\]
(In the second and third steps we have used an integration by parts formula coming from Gauss’s theorem). We thus have the following solution to (20.16)
\[
\Phi(r) = \int_{\mathbb{R}^3} \frac{g(r')}{|r'-r|} dV .
\]

Note how the integral kernel \( G(r,r') = \frac{1}{|r-r'|} \) is used to construct the solution \( \Phi(r) \) directly from the “source function” \( g(r) \). More generally, an integral kernel that interpolates between source functions (inhomogeneous terms) and solutions of a nonhomogeneous PDE is referred to as Green’s function for the PDE.