LECTURE 13

Conformal Mapping Techniques

DEFINITION 13.1. *Let D be a domain in the complex plane. A mapping $f : D \to \mathbb{C}$ is said to be* ***conformal*** *at a point $z_o \in D$ if f is analytic at every point z_o and $f'(z_o) \neq 0$.*

THEOREM 13.2. *Suppose that a transformation*

$$w = f(z) = u(x, y) + iv(x, y)$$

is conformal on a smooth arc C. If along $f(C)$, a function $h(u, v)$ satisfies either of the conditions

$$h(u, v) = h_o \tag{13.1}$$

$$\frac{dh}{dn} = 0 \quad , \tag{13.2}$$

where h_o is a real constant and $\frac{df}{dn}$ denotes the derivative h along the direction normal to $f(C)$, then the function

$$H(x, y) = h\left(u(x, y), v(x, y)\right)$$

satisfies the corresponding condition

$$H(x, y) = h_o \tag{13.3}$$

$$\frac{dH}{dN} = 0 \tag{13.4}$$

where $\frac{dh}{dN}$ denotes derivatives normal to C.

THEOREM 13.3. *Suppose that the image of an analytic function*

$$f(z) = u(x, y) + iv(x, y)$$

defined on a domain $D \subset \mathbb{C}$ is another domain $f(D) \subset \mathbb{C}$. If $h(u, v)$ is a harmonic function defined on $f(D)$, then the function

$$H(x, y) = h\left(u(x, y), v(x, y)\right)$$

is harmonic in D.

Application: Find the electrostatic potential V in the space enclosed by the half circle $x^2 + y^2 = 1$, $y \geq 0$ and the line $y = 0$ when $V = 0$ on the circular boundary and $V = 1$ on the line segment [-1,1].

Consider the transformation

$$w = f(z) = i\frac{1 - z}{1 + z} \tag{13.5}$$

maps the upper half of the unit circle C onto the first quadrant of the w plane and the interval [-1,1] onto the positive v axis.

We can determine the image of the region described above by figuring out how the boundaries are mapped. The circular part of the boundary can be parameterized by

$$z_1(\theta) = e^{i\theta} \quad , \quad 0 \leq \theta \leq \pi \quad ,$$

and so the image of this boundary by f is the curve

$$f \circ z_1(\theta) = i\frac{1-e^{i\theta}}{1+e^{i\theta}} = \tan(\theta/2) \qquad , \qquad 0 \leq \theta \leq \pi$$

which coincides with the positive real u-axis. The line segment [-1,1] can be parameterized by

$$z_2(t) = t \qquad , \qquad t \in [-1,1]$$

and the image of [-1,1] by f is

$$\left\{ i\frac{1-t}{1+t} \mid t \in [-1,1] \right\} \quad .$$

It is clear that this corresponds to a line running along the positive imaginary axis. To see how the interior of the semi-circular region is mapped, we choose an arbitrary point, say $z = \frac{1}{2} + i\frac{1}{2}$, and compute its image.

$$f\left(\frac{1}{2} + i\frac{1}{2}\right) = i\frac{1-\frac{1}{2}-\frac{1}{2}i}{1+\frac{1}{2}+\frac{1}{2}i} = \frac{\frac{1}{2}i+\frac{1}{2}}{\frac{3}{2}+\frac{1}{2}i} = \frac{1+i}{3+i} = \frac{(1+i)(3-i)}{10} = \frac{4+2i}{10} \quad .$$

This is evidently a point lying in the first quadrant. By continuity arguments we can conclude that all points of the original semi-circular region must be mapped into the first quadrant.

The next step is to find a solution of Laplace's equation that satisfies the boundaries conditions

$$V(u,0) = 1 \qquad , \qquad V(0,v) = 0 \quad . \tag{13.6}$$

Now the imaginary part of the analytic function

$$\frac{2}{\pi}Log(w) = \frac{2}{\pi}(\ln(\rho) + i\phi)$$

is a harmonic function that satisfies the boundary conditions (13.2). In terms of the coordinates u and v, this function is

$$V(u,v) = Im\left[\frac{2}{\pi}Log(u+iv)\right] = \frac{2}{\pi}\arctan\left(\frac{u}{v}\right) \quad .$$

Now all we have to do now is pull back this function to the z plane. From (13.5) we have

$$u + iv = \frac{1-x-iy}{1+x+iy} \tag{13.7}$$

$$= \frac{(1-x-iy)(1+x-iy)}{(1+x+iy)(1+x-iy)} \tag{13.8}$$

$$= \frac{1-x^2-y^2}{1+2x+x^2+y^2} + i\frac{2y}{1+2x+x^2+y^2} \tag{13.9}$$

And so

$$u = \frac{1-x^2-y^2}{1+2x+x^2+y^2} \tag{13.10}$$

$$v = \frac{2y}{1+2x+x^2+y^2} \quad . \tag{13.11}$$

Thus,

$$V(x,y) = \frac{2}{\pi}\arctan\left(\frac{1-x^2-y^2}{2y}\right) \quad . \tag{13.12}$$

□

Homework: 4.8.4, 4.8.5