Suppose we’re given the following PDE/BVP (partial differential equation/boundary value problem);
\[
\begin{align*}
\phi_{tt} - c^2 \phi_{xx} &= f(x, t) \\
\phi(0, t) &= 0 \\
\phi(L, t) &= 0 \\
\phi(x, 0) &= h(x) \\
\phi_t(x, 0) &= p(x)
\end{align*}
\]  
(9.1)

corresponding to a string of length \( L \), fixed at both ends, driven by a varying force \( f(x, t) \), with a given initial shape \( h(x) \) and a given initial transverse velocity \( p(x) \).

We intend to solve this problem by means of an expansion of the form
\[
\phi(x, t) = \sum_{n=1}^{\infty} a_n(t) \beta_n(x),
\]  
(9.2)

where \( \beta_n(x) \) some suitably chosen complete set of functions for the interval \((0, L)\). The criteria by which we suitably choose the functions \( \beta_n(x) \) is the same as in Chapter 2; we choose the \( \beta_n(x) \) to be the eigenfunctions of the Sturm-Liouville problem coming from separation of variables (for the homogeneous problem) and the boundary conditions at time \( t = 0 \).

First, let’s separate variables. Let
\[
\phi(x, t) = F(x)G(t),
\]  
(9.3)

Inserting this expression into
\[
\phi_{tt} - c^2 \phi_{xx} = 0
\]
we get
\[
F''(t)G(x) - c^2 F(t)G''(x) = 0
\]
or
\[
\frac{F''(t)}{F(t)} = c^2 \frac{G''(x)}{G(x)}.
\]

Thus, the function \( G(x) \) should satisfy
\[
\Lambda = c^2 \frac{G''(x)}{G(x)},
\]
or
\[
G''(x) - \frac{\Lambda}{c^2} G(x) = 0.
\]  
(9.4)

In a Sturm-Liouville problem
\[
\frac{d}{dx} \left( p(x) \phi(y) + (q(x) + \lambda(x) r(x)) \right)
\]

\]
the functions \( p(x) \) and \( r(x) \) and the eigenvalues are required to be positive. Therefore, we require
\[
\frac{\lambda}{c^2} = \lambda^2
\]
and rewrite (9.4) as
\[
y'' + \lambda^2 y = 0 \quad .
\]
The eigenvalues and eigenfunctions corresponding to the boundary conditions
\[
y(0) = 0 \quad y(L) = 0
\]
(the analogs of the boundary conditions at \( t = 0 \) in (9.1)), are
\[
\lambda_n = \frac{n\pi}{L}
\]
and
\[
\beta_n(x) = \sin \left( \frac{n\pi x}{L} \right) \quad .
\]
We thus set
\[
\phi(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin \left( \frac{n\pi x}{L} \right) \quad .
\]
Let
\[
\begin{align*}
 f_n(t) &= \frac{1}{L} \int_0^L f(x, t) \sin \left( \frac{n\pi x}{L} \right) \, dx \\
 h_n &= \frac{1}{L} \int_0^L h(x) \sin \left( \frac{n\pi x}{L} \right) \, dx \\
 p_n &= \frac{1}{L} \int_0^L p(x, t) \sin \left( \frac{n\pi x}{L} \right) \, dx
\end{align*}
\]
so that
\[
\begin{align*}
 f(x, t) &= \sum_{n=1}^{\infty} f_n(t) \sin \left( \frac{n\pi x}{L} \right) \\
 h(x) &= \sum_{n=1}^{\infty} h_n \sin \left( \frac{n\pi x}{L} \right) \\
 p(x) &= \sum_{n=1}^{\infty} p_n \sin \left( \frac{n\pi x}{L} \right)
\end{align*}
\]
Plugging the expressions (9.11) into (9.1), and matching the coefficients of \( \sin \left( \frac{n\pi x}{L} \right) \) on all sides, we get
\[
\begin{align*}
 a_n''(t) + \frac{c^2 n^2 \pi^2}{L^2} a_n(t) &= f_n(t) \\
 a_n(0) &= h_n \\
 a_n'(0) &= p_n
\end{align*}
\]
The homogeneous equation corresponding to the ODE in (9.12) is
\[
y'' + \frac{c^2 n^2 \pi^2}{L^2} y = 0 \quad .
\]
The functions
\[
\begin{align*}
 y_1(t) &= \cos \left( \frac{n\pi ct}{L} \right) \\
 y_2(t) &= \sin \left( \frac{n\pi ct}{L} \right)
\end{align*}
\]
are two linearly independent solutions to (9.13) and so the general solution to the ODE in (9.12) is
\[
\begin{align*}
 a_n(t) &= y_1(t) \left[ A_n + \int_0^t \frac{y_1(\xi)}{\partial y_1(\xi) / \partial \xi} \, d\xi \right] + y_2(t) \left[ B_n + \int_0^t \frac{y_2(\xi)}{\partial y_2(\xi) / \partial \xi} \, d\xi \right] \\
 &= \cos \left( \frac{n\pi ct}{L} \right) \left[ A_n + \int_0^t \sin \left( \frac{n\pi \xi}{L} \right) f_x(\xi) \, d\xi \right] \\
 &\quad + \sin \left( \frac{n\pi ct}{L} \right) \left[ B_n - \int_0^t \frac{\cos \left( \frac{n\pi \xi}{L} \right) f_x(\xi)}{\partial f_x(\xi) / \partial \xi} \, d\xi \right]
\end{align*}
\]
In order to satisfy the initial conditions in (9.12) we must have

(9.15) \[ A_n = h_n \]

(9.16) \[ B_n = \frac{L}{n\pi c} p_n \]

Thus,

\[
\alpha_n(t) = \cos \left( \frac{n\pi t}{L} \right) \left[ h_n + \int_0^t \sin \left( \frac{n\pi \zeta}{L} \right) f_n(\zeta) \, d\zeta \right] + \sin \left( \frac{n\pi t}{L} \right) \left[ \frac{L}{\pi n} p_0 - \int_0^t \cos \left( \frac{n\pi \zeta}{L} \right) f_n(\zeta) \, d\zeta \right]
\]

The solution to

\[
\phi_{tt} - c^2 \phi_{xx} = f(x, t) \\
\phi(0, t) = 0 \\
\phi(L, t) = 0 \\
\phi(x, 0) = h(x) \\
\phi_t(x, 0) = p(x)
\]

is thus given by

(9.19) \[ \phi(x, t) = \sum_{n=1}^{\infty} \alpha_n(t) \beta_n(x) \]

where

(9.20) \[ \beta_n(x) = \sin \left( \frac{n\pi x}{L} \right) \]

and the coefficients \(\alpha_n(t)\) are determined by

(9.21) \[ \alpha_n(t) = \cos \left( \frac{n\pi t}{L} \right) \left[ h_n + \int_0^t \sin \left( \frac{n\pi \zeta}{L} \right) f_n(\zeta) \, d\zeta \right] + \sin \left( \frac{n\pi t}{L} \right) \left[ \frac{L}{\pi n} p_0 - \int_0^t \cos \left( \frac{n\pi \zeta}{L} \right) f_n(\zeta) \, d\zeta \right] \]

and

(9.22) \[ f_n(t) = \frac{2}{L} \int_0^L f(x, t) \sin \left( \frac{n\pi x}{L} \right) \, dx \]

(9.23) \[ h_n = \frac{2}{L} \int_0^L h(x) \sin \left( \frac{n\pi x}{L} \right) \, dx \]

(9.24) \[ p_n = \frac{2}{L} \int_0^L p(x) \sin \left( \frac{n\pi x}{L} \right) \, dx \]