Reliability Analysis of a Communications Satellite Subsystem

Math 4583 Project #2

Fall 2014

1. The Setting

Certain electronic components used in a communications satellite subsystem are prone to failure at known rates. The aim of this project to develop a means of computing the failure rate for a multiple component subsystem as a function of time. In the communication satellite under consideration, the critical (failure prone) components are two sets of vacuum tubes, referred to as TWTAs (Travelling Wave Tube Amplifiers). This subsystem consists of 4 high power TWTAs with a mean-time-to-failure of 6.67×10^5 hours and 11 low power TWTAs with a mean-time-to-failure of 1.00×10^6 per hour. In addition the subsystem is equipped with 2 spare high power TWTAs with a mean-time-to-failure of 6.67×10^6 hours, and 2 spare low power TWTAs with a mean-time-to-failure of 1.00×10^7 hours.

2. Working Hypotheses

(1) The failure of any individual TWTA does not influence any of the other TWTAs.
(2) If an operational TWTA fails then one of the spare TWTAs of the same type is immediately activated (so long as one is available).
(3) The probability that an active TWTA fails during a (sufficiently short) time interval $[t, t+h]$ is

$$P_{failure,active} = \lambda h + o(h) \tag{1}$$

independent of the particular time t and where $o(h)$ represents some function of h such that

$$\lim_{h \to 0} \frac{o(h)}{h} = 0 \quad .$$

Here the factor λ is what we'll call the *failure rate coefficient* of the TWTA. (I'll suggest a way of relating λ to the measured mean-time-to-failure of an TWTA below.)
(4) The probability of a inactive TWTA failing during a time interval $[t, t+h]$ is given by

$$P_{failure,inactive} = q\lambda h + o(h) \quad . \tag{2}$$

The factor q is about $1/10$, reflecting the observed fact that inactive TWTAs fail at $1/10$ the rate at which active TWTAs fail.

3. Objectives

Let

$$P_{F,T}^{(i,k)}(t) = \text{ probability that exactly } i \text{ out of an initial } k \text{ TWTAs fail by time } t$$

(i) Show that under the hypotheses above, the probability of a single TWTA remaining operational after time t is

$$P_{F,T}^{(0,1)}(t) = e^{-\lambda_T t} \tag{3}$$

where λ_T is the proportionality constant in formula (1) appropriate to TWTAs of type T. (The one additional assumption needed to derive formula (3) is that $P_{F,T}^{(0,1)}(0) = 1$; which is equivalent to assuming one is starting with a working TWTA.)

(ii) A standard measure of the reliability of a system is the so-called Mean-Time-To-Failure (MTTF) of a system. This is defined as follows.

Let $P_R(t)$ denote the *reliability function* of a component (for a single TWTA, this would be the function $P_{F,T}^{(0,1)}(t)$ determined in (i)). The function

$$P_F(t) = 1 - P_R(t)$$

then gives the probability of the subsystem failing sometime between time 0 and time t. We define *probability density function* $f(t)$ by

$$f(t) = \frac{d}{dt}P_F(t) = \lim_{h\to 0}\frac{P_F(t+h) - P_F(t)}{h}$$

and interprete $f(t)\,dt$ as prescribing the likilyhood of a subsystem failure between times t and $t+dt$. The Mean-Time-To-Failure, MTTF, is defined as the weighted average of t with respect to the density function

$$MTTF = \int_0^\infty t f(t)\,dt \ .$$

For a single TWTA, relate its experimentally determined MTTF to the parameter λ_T in equation (2).

(ii) Derive an expression for $P_{F,T}^{(i,m)}(t)$.

(iii) Consider a subsystem consisting of a set of $m+k$ TWTAs in which m are to be in operation at all times, but one initially has k TWTAs to use as backups when an operation TWTA fails. Assume the TWTAs that operation have a failure rate of λ_a and those that are not in use have a failure rate of λ_o. Find a function $R_{subsystem}(t)$ expressing the probability that this subsystem remains operational until time t.

(iv) Consider Find a function $P_R(t)$ expressing the probablity that the communication satellite described above remains operational up between time 0 and time t.

(v) Compute MTTF for the communication satellite described in the Setting section.

4. The Failure Rate Coefficient and Mean Time To Failure

Let's consider a particular TWTA with a characteristic failure rate coefficient of λ. We will first try to develop a formula for

$$P_{F,T}^{(0,1)}(t) = \text{probability that exactly 0 out of 1 TWTAs of type } T \text{ fails between time 0 and time } t.$$

In this section, we'll abbreviate this function as simply $P_R(t)$. (Here the R stands for *reliability*, since the probabilty of not failing between times 0 and t is the same as the probability of the TWTA staying reliable between times 0 and t.) Let

$$\widetilde{P}_F(t_1, t_2) = \text{ probability of failure between times } t_1 \text{ and } t_2 \quad .$$

Hypothesis 3 stipulates that

$$\widetilde{P}_F(t, t+h) = \lambda h + o(h) \qquad \text{indepedent of } t \quad .$$

Observation 4.1. The probability of a single TWTA **not failing** during the time interval $[t, t+h]$ is $1 - \lambda h + o(h)$:

This follows from a basic tenet of probability theory: Suppose a experiment has only two possible outcomes A and B and that the probability of outcome A is P_A and the probability of outcome B si P_B. Then

$$P_A + P_B = 1 \quad .$$

Put another way,

$$P_{not-A} = 1 - P_A \quad .$$

In the situation at hand, the probility of not failing between times t and $t+h$ should be 1 minus the probability of failing between times t and $t+h$. Thus,

$$\widetilde{P}_R(t, t+h) = 1 - (\lambda h + o(h)) \quad \approx \quad 1 - \lambda h + o(h)$$

Another basic tenet of probability theory is that the combined probability of two independent events occurring is the product of their i8ndividual probabilities:

$$P_{A\ and\ B} = P_A * P_B$$

We'll use this rule in the next step.

Consider the function $P_R(t+h)$ which is supposed to provide the probability of 0 failures occuring between times 0 and $t+h$. Now the time interval $[0, t+h]$ can be regarded as the union of the two independent time intervals $[0, t]$ and $[t, t+h]$. Thus, for a TWTA not to fail between times 0 and $t+h$,

(i) it must not fail between times 0 and t;
(ii) it must not fail between times t and $t+h$.

(i) and (ii) are two independent situations, and so correspond to independent probabilities. Thus, viewing $P_R(t+h)$ as a combined probability, we get

$$\begin{aligned} P_R(t+h) &= P_R(t) * \widetilde{P}_R(t, t+h) \\ &= P_R(t) * (1 - \lambda h + o(h)) \end{aligned}$$

or

$$P_R(t+h) - P_R(t) = (-\lambda h + o(h))\, P_R(t)$$

Dividing by h and then dropping terms of order $o(h)/h$ on the right hand side, we get

$$\frac{P_F^{(0)}(t+h) - P_F^{(0)}(t)}{h} = -\lambda P_F^{(0)}(t)$$

If we now take the limit $h \to 0$ we get a differential equation for $P_R(t)$:

$$\frac{d}{dt} P_R(t) = -\lambda P_R(t)$$

This differential equation is just the differentical equation of an exponential function. It has as its general solution

$$P_R(t) = Ce^{-\lambda t} \qquad , \qquad C \text{ being some constant}$$

If we choose the constant of integration C to be 1 we would have

$$P_R(0) = 1$$

which would be appropriate to the assumption that at time $t = 0$ at least the TWTA was functional. We thus have

$$P_R(t) = e^{-\lambda t}$$

as the reliability function for the TWTA.

Let us now calculate the MTTF (mean-time-to-failure) for this system consisting of a single TWTA. According to the definition of MTTF

$$\begin{aligned} MTTF &= \int_0^\infty t\frac{d}{dt}\left(1 - P_R\ (t)\right)dt = \\ &= \lim_{T\to\infty}\int_0^T t\frac{d}{dt}\left(1 - e^{-\lambda t}\right)dt \\ &= \lim_{T\to\infty}\int_0^T t\left(\lambda e^{-\lambda t}\right) \\ &= \frac{1}{\lambda} \end{aligned}$$

And so λ corresponds to 1 over the measured MTTF.

4.1. **A Simpler Formula for MTTF in terms of $P_R(t)$.** By the way, since it's so close at hand, I may as well derive here a cleaner formula for the Mean-Time-To-Failure in terms of the system reliability probability. Starting with

$$\begin{aligned} MTTF &= \int_0^\infty t\frac{d}{dt}\left(1 - P_R\ (t)\right)dt \\ &= -\int_0^\infty t\frac{dP_R}{dt}dt \end{aligned}$$

we can apply Integration by Parts to get

$$\begin{aligned} MTTF &= \left(-tP_R(t)\right)|_0^\infty + \int_0^\infty P_R\ (t)\,dt \\ &= \int_0^\infty P_R(t)\,dt \end{aligned}$$

because

$$\left(tP_R(t)\right)|_0^\infty = \lim_{t\to\infty} -tP_R(t) - 0 = 0$$

because $P_R(t)$ (normally) goes to zero as $t \to \infty$ like a decaying exponential function; that is to say as $t \to \infty$, $P_R(t) \to 0$ faster than any power of t. Thus, in terms of the reliability probability function

$$MTTF = \int_0^\infty P_R(t)\,dt \quad . \tag{4}$$

5. An Intermediary Problem: the reliability a single subsystem

The analysis above concerns a single TWTA. Let's now extend our analysis to handle a particular (high power or low power) subsystem, with say m active tubes and k backups. Mimicing the analysis above we'll first look at the possibilities and probabilities for small time intervals.

Let's suppose first that at time t exactly i TWTAs have failed; that leaves us with m active TWTAs and $k-i$ spares in reserve. I will next write down the probability that none of these $m+(k-i)$ TWTAs fails during the time interval $[t, t+h]$, and denote this probability by $P_0^{(i)}(t, t+h)$.

$$P_0^{(i)}(t, t+h) = \left(1 - \lambda h + o(h)\right)^m \left(1 - q\lambda h + o(h)\right)^{k-i}$$

This is just the product of the probabilities of the individual tubes not failing. We have m factors of $(1 - \lambda h + o(h))$ because we are expecting m active TWTAs not to fail, and $(k-i)$ factors of $(1 - q\lambda h + o(h))$ because we are expecting $(k-i)$ of the inactive TWTAs not to fail. Keeping only terms of order h or less we get

$$P_0^{(i)}(t, t+h) = 1 - m\lambda h - (k-i)\,q\lambda h \tag{5}$$

Let me introduce a little short-hand here that will prove useful latter on. Setting

$$a_i \equiv m\lambda + (k-i)\, q\lambda \tag{7}$$

we can rewrite (5) as

$$P_0^{(i)}(t, t+h) = 1 - a_i h \tag{8}$$

Next, let's again suppose that exactly i TWTAs have failed between times 0 and t, and write down an expression for the probability of a single tube (active or inactive) has failed during the time interval $[t, t+h]$. This probability we'll denote by $P_1^{(i)}(t, t+h)$ and it will be (ignoring terms of order $o(h)$)

$$P_1^{(i)}(t, t+h) = m(\lambda h)(1 - \lambda h + o(h))^{m-1}(1 - q\lambda h)^{(k-i)} \quad + \quad (1-\lambda h)^m (k-i)(q\lambda h)(1-q\lambda h)^{k-i-1} \tag{9}$$

This formula comes about as follows. We have two basic possibilities for a single failure: either one of the active tubes fails, or one of the inactive tubes fails. Each of the two terms on the right hand side of (9) accounts for one of these basic possibilities.

The first term on the right hand side of (9) is the probability that one of the active TWTA fails. The factor (λh) is the probability of one active TWTA not failing, the factor $(1-\lambda h)^{m-1}$ is the probability that the other $m-1$ active TWTAs do not fail and the factor $(1-q\lambda h)^{k-i}$ is the probability that none of the $(k-i)$ inactive TWTAs fail. Finally, the initial factor m is included because while only 1 active TWTA fails, there are m possibilities for which one of the m TWTAs fails.

The second term on the right hand side of (9) is the probability that one of the inactive TWTAs fails. This will be the combined probability of having none of the m active TWTAs failing and 1 out of $(k-i)$ inactive TWAs failing and $(k-i-1)$ of the inactive TWTAs not failing.

Keeping only terms of order h or less, we can rewrite (9) as

$$P_1^{(i)}(t, t+h) \approx m\lambda h + (k-i)\, q\lambda h = a_i h \tag{10}$$

where I have used again the notation (7).

Now while I could similarly define a more general function $P_j^{(i)}(t+t+h)$ for describing the probability of having j tubes fail; we won't need it. You see the probability of having one failure during the time interval $[t, t+h]$ is already of order h, having more than one failure with be of order h^2 or worse. So, so long as we're thinking h is very, very small, we can ignore such possibilities.

Now let me set

$$P_i(t) = \text{probability that exactly } i \text{ TWTAs fail during the time interval } [0, t].$$

We'll have

$$P_i(t+h) = P_i(t+h) * P_0^{(i)}(t, t+h) + P_{i-1}(t) * P_1^{(i-1)}(t, t+h)$$

This formula arises as follows, the interval $[0, t+h]$ is the union of $[0, t]$ and $[t, t+h]$. The first term corresponds to i TWTAs failing between times 0 and t with no addition TWTAs failing between times t and $t+h$. The second term corresponds to $(i-1)$ TWTAs failing sometime between 0 and t and one more TWTA failing between times t and $t+h$.

We thus have, using (7) and (10),[1]

$$P_i(t+h) = P_i(t)(1 - a_i h) + P_{i-1}(t)(a_{i-1} h)$$

or, after a little algebra

$$\frac{P_i(t+h) - P_i(t)}{h} = -a_i P_i(t) + a_{i-1} P_{i-1}(t)$$

[1] You may need to think for a minute why I used $a_{i-1}h$ in the second term.

Letting $h \to 0$ we get

$$\frac{dP_i}{dt}(t) = -a_i P_i(t) + a_{i-1} P_{i-1}(t) \tag{11}$$

Now, in fact, equation (11) is valid even when $i = 0$ so long as we agree that $P_{-1}(t)$ is zero (which kind of makes sense since that would be the probability of -1 of the TWAs failing). Indeed, when $i = 0$, equation (11) becomes

$$\frac{dP_0}{dt}(t) = -a_0 P_0(t) = -(m\lambda + kq\lambda) P_0(t) \quad \Rightarrow \quad P_0(t) = e^{-(m\lambda + kq\lambda)t}$$

which, as per the preceding section, would be the probabilty of m active and k inactive tubes not failing.

For $i = 1, 2, \ldots$, one needs to know $P_{i-1}(t)$ in order to solve the differential equation for $P_i(t)$. So what we really have is a sequence of differential equations that need to be solved sequentially:

$$\begin{aligned} \frac{dP_0}{dt} &= -a_0 P_0(t) \\ \frac{dP_1}{dt} &= -a_1 P_1(t) + a_0 P_0(t) \\ \frac{dP_2}{dt} &= -a_2 P_2(t) + a_1 P_1(t) \\ &\vdots \end{aligned} \tag{12}$$

These differential equations are accompanied by the following natural initial conditions

$$\begin{aligned} P_0(0) &= 1 \\ P_1(0) &= 0 \\ P_2(0) &= 0 \\ &\vdots \end{aligned} \tag{13}$$

which simply correspond to the hypothesis that at time $t = 0$ all tubes were working.

6. The Reliability Functions

Before solving equations (12) and (13) for our two subsystems, let me point out how we intend to use these results.

Now a subsystem will remain functional so long as we don't have more than two failures. In other words, we can have 0, 1 or 2 failures between times 0 and t, and subsystem will remain functional. Becuase these three possibilities are mutually exclusive, the probability of the subsystem remaining functional at time t will be

$$P_R(t) = P_0(t) + P_1(t) + P_2(t) \tag{14}$$

We'll refer to this probability as the *reliabiity function* for a subsystem. By solving equations (12) and (13) above using the appropriate constants a_0, a_1,and a_2 (which in depend on the values of m, k, q and λ for a subsystem), we can obtain explicit reliability functions for the high power and low power subsystems.

Suppose we have done this for both the high power and low power subsystems and let $P_R^{HP}(t)$, $P_R^{LP}(t)$ be the corresponding reliability functions, Since the two subsystems are independent, the probability that the boths subsystems remain functional until time t will be

$$P_R^{sat}(t) = P_R^{HP}(t)\, P_R^{LP}(t)$$

This function $P_R^{sat}(t)$ will be reliability function the satellite's mean time to failure can now be calculated using the formula (*) in the preceding section:

$$MTTF_{sat} = \int_0^\infty P_R^{sat}(t)\, dt$$