1. Use the Taylor series method to calculate a solution to

$$\frac{dx}{dt} = x^2 t , \quad x(0) = 1$$

that’s accurate to order $t^4$.

- We have

$$x(0) = 1$$
$$x'(0) = x^2 t \bigg|_{t=0} = (1)^2(0) = 0$$
$$x''(0) = (2xx' + x^2) \bigg|_{t=0} = 2(1)(0) + (1)^2 = 1$$
$$x'''(0) = (2(x')^2 + 2xx'' + 4xx') \bigg|_{t=0} = 0 + 0 + 0 + 0 = 0$$
$$x^{(iv)}(0) = (4x'x''t + 2(x')^2 + 2x'x'' + 2xx'' + 2xx'' + 4(x')^2 + 4xx'') \bigg|_{t=0} = 2 + 4 = 6$$

so

$$x(t) = x(0) + x'(0)t + \frac{1}{2}x''(0)t^2 + \frac{1}{6}x'''(0)t^3 + \frac{1}{24}x^{(iv)}(0)t^4 + O(t^5)$$
$$= 1 + \frac{1}{2}t^2 + \frac{1}{4}t^4 + O(t^5)$$

2. Use a five step Euler method to calculate the solution of

$$\frac{dx}{dt} = x^2 t , \quad x(0) = 1$$

on the interval $[0, 1]$.

- To divide the interval $[0, 1]$ into five subintervals, we use a step size of

$$\Delta t = \frac{1-0}{5} = 0.2$$
\[
\begin{align*}
t_0 &= 0 \\
x_0 &= 1 \\
m_0 &= x^2 t \bigg|_{t_0} = x_0 t_0^2 = 0 \\
t_1 &= 0.2 \\
x_1 &= x_0 + m_0 \Delta t = 1 + 0 = 1 \\
m_1 &= x_1^2 t_1 = (1)^2(0.2) = 0.2 \\
t_2 &= 0.4 \\
x_2 &= x_1 + m_1 \Delta t = 1.04 \\
m_2 &= x_2^2 t_2 = .43264 \\
t_3 &= 0.6 \\
x_3 &= x_2 + m_2 \Delta t = 1.126528000 \\
m_3 &= x_3^2 t_3 = .7614392010 \\
t_4 &= 0.8 \\
x_4 &= x_3 + m_3 \Delta t = 1.278815840 \\
m_4 &= x_4^2 t_4 = 1.308295962 \\
t_5 &= 1.0 \\
x_5 &= x_4 + m_4 \Delta t = 1.540475032
\end{align*}
\]

This calculation could also be carried out using Maple:

```maple
n := 5;
t[0] := 0;
x[0] := 1;
f := (t, x) -> t*x^2;
dt := (1.0 - 0.0)/n;
for i from 0 to 4 do
    m[i] := f(t[i], x[i]);
    t[i+1] := t[i] + dt;
    x[i+1] := x[i] + m[i]*dt;
end;
```

3. Use a five step second order Runge-Kutta method to calculate the solution of

\[
\frac{dx}{dt} = x^2 t , \quad x(0) = 1
\]

on the interval \([0, 1]\).

- Again we have \(t_0 = 0, x_0 = 1\), and \(\Delta t = 0.2\). Successive values of \(t_i\) and \(x_i\) are determined by computing

\[
\begin{align*}
t_{i+1} &= t_i + \Delta t \\
F_{1,i} &= \Delta t f(t_i, x_i) \\
F_{2,i} &= \Delta t f(t_i + \Delta t, x_i + F_{1,i}) \\
x_{i+1} &= x_i + \frac{1}{2}(F_{1,i} + F_{2,i})
\end{align*}
\]

The following Maple program automates this calculation.

```
#second order Runge-Kutta
```
4. Use a five step fourth order Runge-Kutta method to calculate the solution of
\[ \frac{dx}{dt} = x^2 t \quad , \quad x(0) = 1 \]
on the interval \([0,1]\).

- Again we have \(t_0 = 0\), \(x_0 = 1\), and \(\Delta t = 0.2\). Successive values of \(t_i\) and \(x_i\) are determined by computing

\[ t_{i+1} = t_i + \Delta t \]
\[ F_{1,i} = \Delta t f(t_i, x_i) \]
\[ F_{2,i} = \Delta t f(t_i + \Delta t/2, x_i + F_{1,i}/2) \]
\[ x_{i+1} = x_i + \frac{1}{2} (F_{1,i} + F_{2,i}) \]

The following Maple program automates this calculation.

```maple
#fourth order Runge-Kutta
n := 5;
t[0] := 0.0;
x[0] := 1.0;
f := (t,x) -> t*x^2;
dt := (1.0 - 0.0)/n;
for i from 0 to 4 do
    t[i+1] := t[i] + dt;
    F1[i] := dt*f(t[i],x[i]);
    F2[i] := dt*f(t[i]+dt/2,x[i]+F1[i]/2);
    F3[i] := dt*f(t[i]+dt/2,x[i]+F2[i]/2);
    F4[i] := dt*f(t[i]+dt,x[i]+F3[i]);
    x[i+1] := x[i] + (F1[i] + 2*F2[i] + 2*F3[i] + F4[i])/6;
od;
```
This program calculates the following table

<table>
<thead>
<tr>
<th>i</th>
<th>t</th>
<th>F1</th>
<th>F2</th>
<th>F3</th>
<th>F4</th>
<th>x</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0000</td>
<td>0.000000</td>
<td>0.020000</td>
<td>0.020400</td>
<td>0.041651</td>
<td>1.00000</td>
</tr>
<tr>
<td>1</td>
<td>0.2000</td>
<td>0.041649</td>
<td>0.066505</td>
<td>0.066521</td>
<td>0.094511</td>
<td>1.02040</td>
</tr>
<tr>
<td>2</td>
<td>0.4000</td>
<td>0.094518</td>
<td>0.128650</td>
<td>0.132540</td>
<td>0.178461</td>
<td>1.08701</td>
</tr>
<tr>
<td>3</td>
<td>0.6000</td>
<td>0.178434</td>
<td>0.260576</td>
<td>0.251171</td>
<td>0.346071</td>
<td>1.21950</td>
</tr>
<tr>
<td>4</td>
<td>0.8000</td>
<td>0.346021</td>
<td>0.486251</td>
<td>0.528631</td>
<td>0.793981</td>
<td>1.47060</td>
</tr>
<tr>
<td>5</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.99980</td>
</tr>
</tbody>
</table>

5. Write a Maple program to solve

\[ \frac{dx}{dt} = e^{xt} - \cos(x - t) \]

Use fourth order Runge-Kutta formulas with \( h = 0.001 \). Stop the computation just before the solution overflows and graph the solution.

\[ x(1) = 3 \]

```
N := 1000; # hopefully a sufficient number of iterations
t[0] := 1.0;
x[0] := 3.0;
f := (t,x) -> exp(t*x)-cos(x-t);
dt := 0.001;
for i from 0 to N do
    t[i+1] := t[i] + dt;
    F1[i] := dt*f(t[i],x[i]);
    F2[i] := dt*f(t[i]+dt/2,x[i]+F1[i]/2);
    F3[i] := dt*f(t[i]+dt/2,x[i]+F2[i]/2);
    F4[i] := dt*f(t[i]+dt,x[i]+F3[i]);
    x[i+1] := x[i] + (F1[i] + 2*F2[i] + 2*F3[i] + F4[i])/6;
    if F2[i] > 10^-6 then break fi; # or else f(t,x) is getting too big to handle
od;
n := i-1;
datapoints := [seq([t[i],x[i]],i=0..n)];
with(plots):
    pointplot(datapoints);
```

This program produces the following graph:
6. Derive the second order Runge-Kutta formula

\[ x(t + h) = x(t) + hF\left(t + \frac{1}{2}h, x + \frac{1}{2}hF(t, x)\right) \]

by performing a Richardson Extrapolation on Euler's method using step sizes \( h \) and \( h/2 \). (Hint: assume the error term is \( Ch^2 \).)

7. Derive the third order Runge-Kutta formulas

\[ x(t + h) = x(t) + \frac{1}{9} (2F_1 + 3F_2 + 4F_3) \]

where

\[ F_1 = hF(t, x) \]
\[ F_2 = hF\left(t + \frac{1}{2}h, x + \frac{1}{2}F_1\right) \]
\[ F_3 = hF\left(t + \frac{3}{4}h, x + \frac{3}{4}F_2\right) \]