LECTURE 24

Error Analysis for Multi-step Methods

1. Review

In this lecture we shall study the errors and stability properties for numerical solutions of initial value problems of the form

\( \frac{dx}{dt} = f(t, x) \)

(24.1)

\( x(t_0) = x_0 \)

(24.2)

Recall that the starting point for multi-step methods such as the Adams-Bashforth and Adams-Moulton methods was actually the integral of equation (24.1)

\( x(t_{n+1}) - x(t_n) = \int_{t_n}^{t_{n+1}} \frac{dx}{dt} \, dt = \int_{t_n}^{t_{n+1}} f(t, x) \, dt \)

By replacing \( f(t, x) \) by its polynomial interpolation at the points \( t_n, t_{n-1}, \ldots, t_{n-k} \) one obtains the fifth order Adams-Bashforth formula

\[
x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} \left( \sum_{i=n-4}^{n} f_i \ell_i(t) \right) dt
\]

\[
= x_n + \frac{h}{720} (1901f_{n} - 2774f_{n-1} + 2616f_{n-2} - 1274f_{n-3} + 251f_{n-4})
\]

where

\( t_n = t_0 + nh \)

\( x_n = x(t_n) \)

\( f_n = f(t_n, x_n) \)

and the numerical coefficients in the second line are just the integrals of the cardinal functions \( \ell_i(t) \).

Similarly, by replacing the function \( f(x, t) \) by its polynomial interpolation at the points \( t_{n+1}, t_n, \ldots, t_{n-3} \) we obtained the Adams-Moulton formula

\[
x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} \left( \sum_{i=n-3}^{n+1} f_i \ell_i(t) \right) dt
\]

\[
= x_n + \frac{h}{720} (251f_{n+1} + 646f_n - 264f_{n-1} + 106f_{n-2} - 19f_{n-3})
\]

2. Linear Multi-step Methods

Of course, there’s nothing to prevent us from calculating even higher order analogs of the Adams-Bashforth and Adams-Moulton formulae. But instead of doing so explicitly, we’ll now assume that we have in our hands a k-step linear multi-step method given by a formula of the form

\( a_k x_n + a_{k-1} x_{n-1} + \cdots + a_1 x_{n-k+1} + a_0 x_{n-k} = h \left[ b_k f_n + b_{k-1} f_{n-1} + \cdots + b_1 f_{n-k+1} + b_0 f_{n-k} \right] \)

(24.3)
relating \( k \) successive values of the \( x(t) \) to \( k \) successive values of the function \( f(t, x) \) (the function that defines the differential equation). The utility of such an equation is, of course, to compute \( x_n \) from the \( k \) preceding values, \( x_{n-k}, x_{n-k+1}, \ldots, x_{n-1} \), and so we shall henceforth assume that the constant \( a_k \neq 0 \). The coefficient \( b_k \) on the right hand side may or not equal zero. If \( b_k = 0 \) then we say the multi-step method corresponding to equation (24.3) is explicit; because, in this case we can solve the equation explicitly for \( x_n \)

\[
x_n = -\frac{1}{a_k} (a_{k-1} x_{n-1} + \cdots + a_1 x_{n-k+1} + a_0 x_{n-k}) + \frac{h}{a_k} [b_k f_n + b_{k-1} f_{n-1} + \cdots + b_1 f_{n-k+1} + b_0 f_{n-k}]
\]

If \( b_k \neq 0 \) we say that the corresponding method is implicit; in this case, the term \( f_n = f(t_n, x_n) \) on the right hand side also depends on \( x_n \) and so we have an implicit algebraic equation for \( x_n \).

**2.1. Convergence of Multi-Step Methods.**

**Definition 24.1.** A multi-step method defined by the a formula of the form (24.3) is said to be convergent in a region \( [t_0, t_1] \) if

\[
\lim_{h \to 0} x_h(t) = x(t) \quad t \in [t_0, t_1]
\]

provided only that

\[
\lim_{h \to 0} x_h(t + jh) = x_0 \quad 0 \leq j < k
\]

Here \( x_h(t) \) is the numerical solution computed using a step size of \( h \) and \( x(t) \) is the exact solution.

This definition is natural enough. The following definitions are not so natural, but nevertheless extremely useful.

**Definition 24.2.** Consider a multi-step method corresponding to a relation of the form (24.3). Set

\[
P(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0 \\
Q(z) = b_k z^k + b_{k-1} z^{k-1} + \cdots + b_1 z + b_0
\]

We shall say that the method (24.3) is stable if the roots of the polynomial \( P(z) \) lie in disk \( |z| \leq 1 \), and if each root such that \( |z| = 1 \) is simple. The method (24.3) is said to be consistent if \( P(1) = 0 \) and \( P'(1) = Q(1) \).

**Theorem 24.3.** Consider a multi-step method corresponding to a relation of the form (24.3). Then this method is convergent if and only if it is both stable and consistent.

A partial proof (that of the necessity of stability and consistency for convergence) is given in the text.

**2.2. The Order of Multi-step Method.** The order of a multi-step method is an integer that corresponds to the number of terms in the Taylor expansion of the solution that a multi-step method simulates. Let us represent the multi-step method (24.3) as a linear functional

\[
I[x] = \sum_{j=0}^{k} [a_j x(jh) - h b_j f(jh)] \\
= \sum_{j=0}^{k} [a_j x(jh) - h b_j x'(jh)]
\]
Here we let \( k = n \) to simplify our notation and assume that the first value of in equation (24.3) begins at \( t = 0 \), rather than \( t = (n - k)h \). Now

\[
x(jh) = \sum_{i=0}^{\infty} \frac{(jh)^i}{i!} x^{(i)}(0)
\]

\[
x'(jh) = \sum_{i=0}^{\infty} \frac{(jh)^i}{i!} x^{(i+1)}(0)
\]

and so we can write

\[
L[x] = \sum_{j=0}^{k} \left[ a_j \sum_{i=0}^{\infty} \frac{(jh)^i}{i!} x^{(i)}(0) - hb_j \sum_{i=0}^{\infty} \frac{(jh)^i}{i!} x^{(i+1)}(0) \right]
\]

Collecting terms proportional to \( x(0), x'(0), \ldots \) (or equivalently by their degree in \( h \)) we have

\[
L[x] = d_0 x(0) + d_1 hx'(0) + d_2 h^2 x''(0) + \cdots
\]

where

\[
d_0 = \sum_{i=0}^{k} a_i
\]

(24.6)

\[
d_1 = \sum_{i=0}^{k} (ia_i - b_i)
\]

(24.7)

\[
d_2 = \sum_{i=0}^{k} \left( \frac{1}{2} i^2 a_i - ib_i \right)
\]

(24.8)

\[
\vdots
\]

(24.9)

\[
d_j = \sum_{i=0}^{k} \left( \frac{j^j}{j!} a_i - \frac{j^j-1}{(j-1)!} b_j \right)
\]

Theorem 24.4. The following three properties of the multi-step method (24.3) are equivalent:

1. \( d_0 = d_1 = \cdots = d_m = 0 \).
2. \( L[P] = 0 \) for each polynomial \( P \) of degree \( \leq m \).
3. \( L[x] = \mathcal{O}(h^{m+1}) \) for all \( x \in C^{m+1} \).

Proof.

\begin{itemize}
  \item (1) \( \Rightarrow \) (2):
    If (1) is true then
    \[
    L[x] = 0 + 0 + \cdots + 0 + d_{m+1} H^{m+1} x^{(m+1)}(0) + d_{m+2} H^{m+2} x^{(m+2)}(0) + \cdots
    \]
    But if \( P \) is a polynomial of degree \( \leq m \) then \( P^{(m+1)}(x) = 0 \). Therefore, for such a polynomial
    \[
    L[P] = 0 + 0 + \cdots + 0 + d_{m+1} H^{m+1} P^{(m+1)}(0) + d_{m+2} H^{m+2} P^{(m+2)}(0) + \cdots
    \]
    \[
    = 0
    \]
  \item (2) \( \Rightarrow \) (3):
    If \( x \in C^{m+1} \), then by Taylor’s Theorem we can write
    \[
x(t) = P(t) + R(t)
    \]
    where \( P(t) \) is a polynomial of degree \( m \) and
    \[
    R(t) = \frac{x^{(m+1)}(\xi)}{(m+1)!} h^{m+1}
    \]
\end{itemize}
Notice that
\[
\frac{d^j R}{dt^j}\bigg|_{t=0} = 0
\]

Hence,
\[
L[x] = L[P] + L[R] = 0 + d_{m+1} h^{m+1} x^{(m+1)}(0) + d_{m+2} h^{m+2} x^{(m+2)}(0) + \cdots = O(h^{m+1})
\]

- (3) \Rightarrow (1)

If (3) is true, then we must have \(d_0 = d_1 = \cdots = d_m = 0\). Hence, (3) implies (1).

**Definition 24.5.** The order of a multi-step method is the unique natural number \(m\) such that
\[
0 = d_0 = d_1 = \cdots = d_m \neq d_{m+1}
\]

**Example 24.6.** What is the order of the following multistep method
\[
x_n - x_{n-2} = \frac{h}{3} (f_n + 4f_{n-1} + f_{n-2})
\]

- This is a three step method \((k = 2)\), with
\[
a_2 = 1 , \quad a_1 = 0 , \quad a_0 = -1
\]
\[
b_2 = \frac{1}{3} , \quad b_1 = \frac{4}{3} , \quad b_0 = \frac{1}{3}
\]

We have
\[
d_0 = \sum_{k=0}^{2} a_i = 1 + 0 - 1 = 0
\]
\[
d_1 = \sum_{k=0}^{2} (ia_i - b_i) = \left( 0 \cdot \frac{1}{3} \right) + \left( 0 - \frac{4}{3} \right) + \left( 2 \cdot \frac{1}{3} \right) = 0
\]
\[
d_2 = \sum_{k=0}^{2} \left( \frac{1}{2} i^2 a_i - ib_i \right) = (0 - 0) + \left( 0 - \frac{4}{3} \right) + \left( \frac{4}{2} \cdot \frac{2}{3} \right) = 0
\]
\[
d_3 = \sum_{k=0}^{2} \left( \frac{1}{6} i^3 a_i - i^2 b_i \right) = (0 - 0) + \left( 0 - \frac{4}{6} \right) + \left( \frac{8}{6} \cdot \frac{4}{6} \right) = 0
\]
\[
d_4 = \sum_{k=0}^{2} \left( \frac{1}{24} i^4 a_i - \frac{1}{6} i^3 b_i \right) = (0 - 0) + \left( 0 - \frac{4}{18} \right) + \left( \frac{(16)(1)}{(24)} \cdot \frac{(8)(1)}{(6)(3)} \right) = 0
\]
\[
d_5 = \sum_{k=0}^{2} \left( \frac{1}{120} i^5 a_i - \frac{1}{24} i^4 b_i \right) = (0 - 0) + \left( 0 - \frac{(16)(4)}{24} \right) + \left( \frac{(32)(1)}{120} \cdot \frac{(16)(1)}{(24)(3)} \right) = -\frac{118}{45}
\]

And so the order is \(m = 4\).

### 2.3. Local Truncation Error

**2.4. Local Truncation Error.** By a local truncation error we mean the error induced at a particular stage of an iterative numerical procedure. In the case at hand, this means the error induced by using a difference relation of the form \((24.3)\) instead to obtain an estimate \(x_n\) of \(x(t_n)\) instead of evaluating the exact solution at \(t_n\).
THEOREM 24.7. Consider a multi-step method corresponding to a relation of the form (24.3). Then if \( x(t) \in C^{k+2}(\mathbb{R}) \) and if \( \frac{\partial f}{\partial x} \) is continuous, we have

\[
x(t_n) - x_n = \frac{d_k + 1}{a_k} h^{m+1} x^{(m+1)}(t_{n-k}) + \mathcal{O}(h^{m+2})
\]

where the coefficients are defined by equations (24.8) - (24.9).

Proof. It suffices to prove the statement for \( n = k \), since \( x_n \) can be expressed as a solution with initial conditions imposed at \( t_{n-k} \). Using the linear function \( L \) of the preceding section we have for the exact solution \( x(t) \)

\[
L[x] = \sum_{i=0}^{k} [a_i x(t_i) - h b_i x'(t_i)] = \sum_{i=0}^{k} [a_i x(t_i) - h b_i f(t_i, x(t_i))]
\]

On the other hand, the numerical solution \( \{x_i\} \) should satisfy

\[
0 = \sum_{i=0}^{k} [a_i x_i - h b_i f(t_i, x_i)]
\]

Subtracting (24.11) from (24.10) yields

\[
L[x] = \sum_{i=0}^{k} [a_i x(t_i) - x_i - h b_i (f(t_i, x(t_i)) - f(t_i, x_i))]
\]

Since we are interested here only in the error induced by the current iteration of the multi-step method we shall assume that the previously determined \( x_i \) are all exactly correct: \( x_i = x(t_i) \). In this case, all but the final terms of the summand vanish and we have

\[
L[x] = a_k (x(t_k) - x_k) - h b_k (f(t_k, x(t_k)) - f(t_k, x_k))
\]

Writing

\[
f(t_k, x_k) = f(t_k, x(t_k)) + \frac{\partial f}{\partial x}(\xi)(x(t_k) - x_k), \quad \text{for some } \xi \in (x(t_k), x_k]
\]

we have

\[
L[x] = a_k (x(t_k) - x_k) - h b_k \frac{\partial f}{\partial x}(\xi)(x(t_k) - x_k)
\]

\[
= [a_k - h b_k F] (x(t_k) - x_k)
\]

We thus have

\[
x(t_k) - x_k = \frac{L[x]}{a_k - h b_k F} = \frac{d_{m+1} h^{m+1} x^{(m+1)}(t_{n-k}) + \cdots}{a_k - h b_k F} \approx \frac{d_{m+1} h^{m+1} x^{(m+1)}(t_{n-k})}{a_k} + \mathcal{O}(h^{m+2})
\]

3. Global Truncation Error

We shall now establish a bound on total error induced by the local truncation errors that occur at each stage of a multi-step numerical method. As before we let \( x_k \) represent the value of the numerical solution after \( n \) iterations of the multi-step method and let \( x(t_n) \) denote the corresponding value of the exact solution (for the same time \( t_n \)). Now the first thing one should realize is that the global truncation error is not simply the sum of the local truncation errors that occur at each stage of the iterative method. For the accuracy of a successive stage depends crucially on the accuracy of the stage that preceded it.

To get a handle on how the error of a preceding stage effects the error of a latter stage we consider a family of initial value problems:

\[
\frac{dx}{dt} = f(t, x)
\]

\[
x(0) = s, \quad s \in \mathbb{R}
\]
We shall assume that $f_x \equiv \frac{\partial f}{\partial x}$ is continuous and satisfies

$$f_x(t, x) \leq \lambda \quad \text{for all } t \in [0, T] \text{ and all } x \in \mathbb{R}$$

The exact solution of (24.12) and (24.13) is, of course, a function of $t$, but in order to make its dependence on the initial condition (24.13) explicit, we shall denote it by $x(t, s)$. Thus, we write

$$\frac{\partial}{\partial t} x(t, s) = f \left( t, x(t, s) \right)$$
$$x(0, s) = s$$

If we differentiate these equations with respect to $s$ we obtain

$$\frac{\partial}{\partial t} \frac{\partial x}{\partial s} = \frac{\partial}{\partial s} \frac{\partial x}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s}$$
$$\frac{\partial x}{\partial s}(0, s) = 1$$

Writing $u = \frac{\partial x}{\partial s}$, and noting that the second equation tells us that $u(0, s)$ does not depend on $s$, we have

$$u' = f_x u \quad \text{(24.14)}$$
$$u(0) = 1 \quad \text{(24.15)}$$

**Theorem 24.8.** If $f_x < \lambda$ then the solution of (24.14) and (24.15) satisfies the inequality

$$|u(t)| \leq e^{\lambda t}, \quad t \geq 0$$

**Proof.** Write

$$\alpha(t) = \lambda - f_x(t)$$

Since $f_x(t) < \lambda$, $\alpha(t)$ is a positive function. And so we have

$$\frac{u'}{u} = f_x = \lambda - \alpha(t)$$

or

$$\frac{d}{dt}(\ln |u|) \leq \lambda - \alpha(t)$$

Integrating both sides between 0 and $t$ yields

$$\ln |u(t)| - \ln |u(0)| = \int_0^t \frac{d}{dt}(\ln |u|) \, dt = \int_0^t (\lambda - \alpha(t)) \, dt = \lambda t - \int_0^t \alpha(t) \, dt$$

or

$$\ln |u(t)| < \lambda t \quad \text{(24.16)}$$

where on the left hand side we have used the initial condition $u(0) = 1$ and on the right hand side we have used the fact that

$$\lambda t - \int_0^t \alpha(t) \, dt < \lambda t$$

since $\alpha(t)$ is a positive function. Noting that the exponential function is monotonically increasing, we can maintain the inequality if we exponentiate both sides of (24.16). We thus have

$$u(t) < e^{\lambda t}$$

**Theorem 24.9.** If the initial value problem corresponding to (24.12) and (24.13) is solved with initial values $s$ and $s + \delta$, then the solution curves at time $t$ differ by at most $\|f\| e^{\lambda t}$. 
Proof. By the Mean Value Theorem we have

\[ |x(t, s) - x(t, s + \delta)| = \left| \frac{\partial x}{\partial s} x(t + \theta \delta) \right| \delta, \quad \text{for some } \theta \in [0, 1] \]

\[ \equiv u(t + \theta \delta) \delta \]

\[ < \delta \epsilon^M \]

**Theorem 24.10.** If the local truncation errors at \( t_1, t_2, \ldots, t_n = t_0 + nh \) do not exceed \( \delta \) in magnitude, then the global truncation error \( \varepsilon \) does not exceed

\[ \frac{\epsilon^M_n - 1}{\epsilon^M - 1} \delta \]

**Proof.**

At the first iteration the total error is just the local truncation error so

\[ \varepsilon_1 = \delta \]

At the second iteration the total error is the sum of the local truncation error and the error arising from the fact that initial condition might be off by \( \varepsilon_1 \). Thus

\[ \varepsilon_2 = \delta + \epsilon^{\lambda h} \delta \]

Similarly at level 3 we would have

\[ \varepsilon_3 = \delta + \epsilon^{\lambda h} \varepsilon_2 = \delta + (\delta + \delta \epsilon^{\lambda h}) \epsilon^{\lambda h} = \delta + \delta \epsilon^{\lambda h} + \delta \epsilon^{2\lambda h} \]

and so on and so on until one has

\[ \varepsilon_n = \delta + \delta \epsilon^{\lambda h} + \delta \epsilon^{2\lambda h} + \cdots + \delta \epsilon^{(n-1)\lambda h} \]

\[ = \delta \sum_{i=0}^{n-1} \epsilon^{i\lambda h} \]

\[ = \delta \frac{\epsilon^{n\lambda h} - 1}{\epsilon^{\lambda h} - 1} \]

\[ = \delta \frac{\epsilon^{M_n} - 1}{\epsilon^{\lambda h} - 1} \]

where in passing from the second to the third equation we have used the fact that

\[ x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1) \]

**Theorem 24.11.** If the local truncation error in a numerical solution of an initial value problem is of order \( \mathcal{O}(h^m) \) then the global truncation error is of order \( \mathcal{O}(h^{m-1}) \).

**Proof.** According to the preceding theorem,

\[ \varepsilon_n = \delta \frac{\epsilon^{M_n} - 1}{\epsilon^{\lambda h} - 1} \]

Now for, assuming we keep the final time \( t_n \) fixed, we have for \( \lambda h << 1 \)

\[ \frac{\epsilon^{M_n} - 1}{\epsilon^{\lambda h} - 1} = \frac{\epsilon^{M_n} - 1}{1 + \lambda h + \frac{1}{2}(\lambda h)^2 + \cdots} - 1 \]

\[ = \frac{\lambda h + \frac{1}{2}(\lambda h)^2 + \cdots}{\lambda h} \]

\[ = \frac{1}{\lambda h} \left( \frac{\epsilon^{M_n} - 1}{1 + \frac{1}{2}(\lambda h) + \cdots} \right) \approx \mathcal{O}(h^{-1}) \]
and so

$$
\varepsilon_n \approx O(h^m) \cdot O(h^{-1}) = O(h^{m-1})
$$