LECTURE 23

Review Session

0.1. Chapter 6: Polynomial Interpolation.

0.1.1. *Newton form of interpolation polynomial.*

$$P(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

0.1.2. *Lagrange form of interpolation polynomial.*

$$\begin{aligned} P(x) &= \sum_{i=0}^{n} f(x_i)\ell_i(x) \\ l_i(x) &= \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)} \end{aligned}$$

0.1.3. *Chebyshev Polynomials.* Roots of Chebyshev polynomials

$$\tilde{x}_i = \cos\left(\frac{2i+1}{2n+2}\pi\right) \quad , \quad i = 0, 1, \ldots, n$$

0.1.4. *Errors in Polynomial Interpolation.*

$$|f(x) - P(x)| = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i)$$

0.1.5. *Divided Differences.*

$$\begin{aligned} f[x_i] &= f(x_i) \\ f[x_i, x_{i+1}, \ldots, x_{i+j}] &= \frac{f[x_{i+1}, \ldots, x_{i+j}] - f\left[x_i, \ldots, x_{i+j-1}\right]}{x_{i+j} - x_i} \\ A_i &= f\left[x_0, x_1, \ldots, x_i\right] \end{aligned}$$

Here the A_i is the coefficient of the $(x - x_0)(x - x_1)\cdots(x - x_{i-1})$ in the Newton form of the polynomial interpolation of $f(x)$

0.2. Chapter 7: Numerical Differentiation and Integration.

0.2.1. *Numerical Differentiation.*

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \mathcal{O}(h)$$

0.2.2. *Richardson Extrapolation.*

$$\begin{aligned} f'(x) &= \phi_0(h) = \frac{f(x+h)-f(x-h)}{2h} + \mathcal{O}\left(h^2\right) \\ &= \phi_i(h) = \frac{4}{3}\phi_0(h/2) - \frac{1}{3}\phi_0(h) \\ &\vdots \\ &= \phi_i(h) = \frac{1}{4^i-1}\left(4^i\phi_{i-1}(h/2) - \phi_{i-1}(h)\right) + \mathcal{O}(h^{2i}) \end{aligned}$$

0.2.3. *Numerical Integration by Interpolation.*

$$\begin{aligned} \int_a^b f(x) &= \sum_{i=0}^{n} f(x_i)A_i \\ A_i &= \int_a^b \ell_i(x)dx \end{aligned}$$

0.2.4. *Simpson's Rule.*

$$\int_a^{a+2h} f(x)dx = \frac{h}{3}\left[f(a) + 4f(a+h) + f(a+2h)\right]$$

$$h = \frac{\Delta x}{2} \Rightarrow \quad \int_a^b f(x)dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)dx \approx \sum_{i=0}^{n-1}\frac{\Delta x}{6}\left[f\left(x_i\right) + 4f(x_i+\Delta x) + f(x_i+\Delta x)\right]$$

0.3. Chapter 8: Numerical Solution of Initial Value Problems.

0.3.1. *Taylor Series Method.*

$$x(t) = x\left(t_0\right) + x'\left(t_0\right)\left(t-t_0\right) + \frac{1}{2}x''\left(t_0\right)\left(t-t_o\right)^3 + \cdots$$

0.3.2. *Euler Method.*

$$x_{i+1} = x_i + f\left(t_i, x_i\right)\Delta t$$

0.3.3. *Runge-Kutta Methods.*

Second Order.

$$\begin{aligned} F_{1,i} &= \Delta t f\left(t_i, x_i\right) \\ F_{2,i} &= \Delta t f\left(t_i+\Delta t, x_i+F_{1,i}\right) \\ x_{i+1} &= x_i + \frac{1}{2}\left(F_{1,i}+F_{2,i}\right) \end{aligned}$$

Fourth Order.

$$\begin{aligned} F_{1,i} &= \Delta t f\left(t_i, x_i\right) \\ F_{2,i} &= \Delta t f\left(t_i+\frac{\Delta t}{2}, x_i+\frac{1}{2}F_{1,i}\right) \\ F_{3,i} &= \Delta t f\left(t_i+\frac{\Delta t}{2}, x_i+\frac{1}{2}F_{2,i}\right) \\ F_{4,i} &= \Delta t f\left(t_i+\Delta t, x_i+F_{3,i}\right) \\ x_{i+1} &= x_i + \frac{1}{6}\left(F_{1,i}+2F_{2,i}+2F_{3,i}+F_{4,i}\right) \end{aligned}$$

0.3.4. *Multi-step Methods.*