LECTURE 13

Review Session

1. Mathematical Preliminaries

1.1. Taylor's Theorem.

a. **Taylor's Theorem with Integral Remainder**

$$f(x) = \sum_{i=0}^{n} \frac{f^{(n)}(x_o)}{n!} + \frac{1}{n!} \int_{x_o}^{x} f^{(n+1)}(t) \, (x-t)^n \, dt$$

b. **Taylor's Theorem with Lagrange Remainder**

$$\begin{aligned} f(x) &= \sum_{i=0}^{n} \frac{f^{(n)}(x_o)}{n!} + \frac{1}{(n+1)!} f^{(n+1)}(\xi) \, (x-x_o)^{n+1} \\ &= \sum_{i=0}^{n} \frac{f^{(n)}(x_o)}{n!} + \mathcal{O}\left(|x-x_o|^{n+1} \right) \end{aligned}$$

1.2. Rates of Convergence.

a. **Big $\mathcal{O}$ and little $\mathfrak{o}$**

$$b_n = \mathcal{O}(a_n) \quad \Rightarrow \quad \exists \, C, N \quad s.t. \quad |b_n| \leq C \, |a_n| \quad \forall \, n > N$$

$$b_n = \mathfrak{o}\,(a_n) \quad \Rightarrow \quad \lim_{n \to \infty} \frac{|b_n|}{|a_n|} = 0$$

b. **Orders of Convergence**

A sequence $\{a_n\}$ such that $\lim_{n \to \infty} a_n = 0$ is said to converge with a rate of order α if

$$|a_{n+1}| \leq C \, |a_n|^{\alpha}$$

for sufficiently large n.

2. Computer Arithmetic

2.1. Floating Point Numbers and Machine Numbers.

a. **Construction of machine numbers**

$$x = 1^s \times 1.m \times 2^e$$

b. **Determining numbers corresponding to real numbers**

- Determine binary decimal expansion of real number
- Round down to closest machine number q_-
- Determine next machine number q_+
- Determine closest machine number

2.2. Machine Arithmetic.

a. **Roundoff Errors and Loss of Significance** (Be able to give examples)

3. Solution of Nonlinear Equations

3.1. Bisection Method.

a. $c_n = \frac{a_n + b_n}{2}$
b. if $c_n < 0 \quad \Rightarrow \quad a_{n+1} = c_n \quad , \quad b_{n+1} = b_n$
1. if $c_n > 0 \quad \Rightarrow \quad a_{n+1} = a_n \quad , \quad b_{n+1} = c_n$
c. if $c_n = 0 \quad \Rightarrow \quad r = c_n$

3.2. Newton's Method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

3.3. Secant Method.

$$x_{n+1} = x_n - f\left(x_n\right)\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right)$$

4. Solving Systems of Linear Equations

4.1. LU Factorizations.

a. **Solving Lower Triangular Systems $\mathbf{Lx = b}$**
b. **Solving Upper Triangular Systems $\mathbf{Ux = b}$**
c. **Solving Systems of the form $\mathbf{LUx = b}$**

4.2. Gaussian Elimination.

4.3. Gaussian Elimination with Scaled Row Pivoting.