LECTURE 8

Secant Method

The idea underlying the **secant method** is the same as the one underlying Newton's method: to find an approximate zero of a function $f(x)$ we find instead a zero for a linear function $F(x)$ that corresponds to a "best straight line fit" to $f(x)$. In Newton's method, the function representing the best straight line fit is determined by the first order Taylor expansion:

$$F(x) = f(x_o) + f'(x_o)(x - x_0)$$

In the secant method, we instead determine a "best straight line fit" by determining the linear function whose graph corresponds to a line that connects two points on the graph of $f(x)$.

Recall that the line that passes through two points (x_0, y_0) and (x_1, y_1) is prescribed by

$$y = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

Let $(x_0, f(x_0))$ and $(x_1, f(x_1))$ be two nearby points on the graph of $f(x)$. The line passing through these two points is thus

$$y = F(x) \equiv f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

Now the function $F(x)$ has a zero at x_2 if

$$0 = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0)$$

or

$$x_2 = x_0 - f(x_0)\left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right)$$

If we regard, x_1, x_0, and x_2 as successive approximations for an actual zero of $f(x)$ we can interprete this calculation as an algorithm for calculating an $(n+1)^{th}$ order approximation to a zero of $f(x)$: indeed, setting $x_1 = x_{n+1}$, $x_0 = x_n$, and $x_1 = x_{n-1}$ we have

$$x_{n+1} = x_n - f(x_n)\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right)$$

Example 8.1. Write a Maple routine that utilizes the secant method to determine a zero of

$$f(x) = x^3 - 4x + 1$$

starting with

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 1 \end{aligned}$$

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M := 10;
delta := 0.000001;
epsilon := 0.000001;
f := x -> x^3 - 4*x + 1;
x0 := 0.0;
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x1 := 1.0;
for i from 1 to M do
    x2 := x1 - f(x1)*(x1 -x0)/(f(x1) -f(x0));
    if (abs(f(x2)) < epsilon) then break; fi;
    x0 := x1;
    x1 := x2;
    if (abs(x1 -x0) < delta) then  break; fi;
    od:
x2;
f(x2);
```

1. Rate of Convergence of Secant Method

Let r be the actual root of $f(x) = 0$, let x_n be the approximate value for r obtained by carrying out n iterations of the secant method, and let e_n be the corresponding error:

$$e_n = x_n - r.$$

We then have

$$\begin{aligned}
e_{n+1} &= x_{n+1} - r \\
&= x_n - f(x_n)\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) - r \\
&= e_n - f(x_n)\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) \\
&= e_n - f(x_n)\left(\frac{x_n - r - x_{n-1} + r}{f(x_n) - f(x_{n-1})}\right) \\
&= e_n - f(x_n)\left(\frac{e_n - e_{n-1}}{f(x_n) - f(x_{n-1})}\right) \\
&= \frac{e_n\left(f(x_n) - f(x_{n-1})\right) - f(x_n)\left(e_n - e_{n-1}\right)}{f(x_n) - f(x_{n-1})} \\
&= \frac{e_n f(x_{n-1}) - e_{n-1} f(x_n)}{f(x_n) - f(x_{n-1})} \\
&= \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right]\left[\frac{e_n f(x_{n-1}) - e_{n-1} f(x_n)}{x_n - x_{n-1}}\right] \\
&= \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right]\left[\frac{f(x_{n-1})/e_{n-1} - f(x_n)/e_n}{x_n - x_{n-1}}\right] e_n e_{n-1}
\end{aligned}$$

Now by Taylor's Theorem

$$\begin{aligned}
f(x_n) &= f(r) + f'(r)e_n + \frac{1}{2}f''(r)e_n^2 + \mathcal{O}\left(e_n^3\right) \\
&= 0 + f'(r)e_n + \frac{1}{2}f''(r)e_n^2 + \mathcal{O}\left(e_n^3\right)
\end{aligned}$$

So

$$\frac{f(x_n)}{e_n} = f'(r) + \frac{1}{2}f''(r)e_n + \mathcal{O}\left(e_n^2\right)$$

and similarly

$$\frac{f(x_{n-1})}{e_{n-1}} = f'(r) + \frac{1}{2}f''(r)e_{n-1} + \mathcal{O}\left(e_{n-1}^2\right)$$

So we have

$$\begin{aligned}\frac{f(x_n)}{e_n}-\frac{f(x_{n-1})}{e_{n-1}} &= \left(f'(r)+\frac{1}{2}f''(r)e_n\right)-\left(f'(r)-\frac{1}{2}f''(r)e_{n-1}\right)+\mathcal{O}\left(e_{n-1}^2\right)\\ &= \frac{1}{2}f''(r)\left(e_n-e_{n-1}\right)+\mathcal{O}\left(e_{n-1}^2\right)\end{aligned}$$

and

$$e_{n+1}\approx\left[\frac{x_n-x_{n-1}}{f(x_n)-f(x_{n-1})}\right]\left[\frac{\frac{1}{2}f''(r)(e_n-e_{n-1})}{x_n-x_{n-1}}\right]e_ne_{n-1}$$

Now

$$e_n-e_{n-1}=(x_n-r)-(x_{n-1}-r)=x_n-x_{n-1}$$

and for x_n and x_{n-1} sufficiently close to r

$$\frac{x_n-x_{n-1}}{f(x_n)-f(x_{n-1})}\approx f'(r)$$

So

$$e_{n+1}\approx[f'(r)]\left[\frac{1}{2}f''(r)\right]e_ne_{n-1}=Ce_ne_{n-1} \tag{8.1}$$

In order to determine the order of convergence, we now suppose an asymptotic relationship of the form

$$|e_{n+1}|\approx A|e_n|^\alpha$$

This relationship also requires

$$|e_n|\approx A|e_{n-1}|^\alpha \quad\Rightarrow\quad |e_{n-1}|=\left(A^{-1}|e_n|\right)^{1/\alpha}$$

In order to be consistent with (1) we need

$$|e_{n+1}|=A|e_n|^\alpha=C|e_ne_{n-1}|=C|e_n|\left(A^{-1}|e_n|\right)^{1/\alpha}$$

or

$$A|e_n|^\alpha=CA^{-\frac{1}{\alpha}}|e_n|^{1+\frac{1}{\alpha}}$$

or

$$\frac{A^{1-\frac{1}{\alpha}}}{C}=|e_n|^{1-\alpha+\frac{1}{a}}$$

Now the left hand side is a product of constants. Therefore the right hand side must also be constant as $n\to\infty$. For this to happen we need

$$0=1-\alpha+\frac{1}{\alpha}\quad\Rightarrow\quad\alpha^2-\alpha-1=0\quad\Rightarrow\quad\alpha=\frac{1\pm\sqrt{1+4}}{2}=\frac{1}{2}\pm\frac{\sqrt{5}}{2}$$

Taking the positive root (otherwise, the error terms asympotically diverge), we find

$$\alpha\approx1.62<2$$

Thus, the rate of convergence of the secant method is superlinear, but not quadratic.

Homework Problems

1. Show that

$$e_{n+1}\approx\frac{|f''(r)|}{2|f'(r)|}e_ne_{n-1}=Ce_ne_{n-1}$$

2. Use the secant method to find a solution of

$$exp(x^2-2)=3ln(x)$$

starting with $x0=1.5, x1=1.4$.