Secant Method

The idea underlying the secant method is the same as the one underlying Newton’s method: to find an approximate zero of a function \( f(x) \) we find instead a zero for a linear function \( F(x) \) that corresponds to a “best straight line fit” to \( f(x) \). In Newton’s method, the function representing the best straight line fit is determined by the first order Taylor expansion:

\[
F(x) = f(x_0) + f'(x_0) (x - x_0)
\]

In the secant method, we instead determine a “best straight line fit” by determining the linear function whose graph corresponds to a line that connects two points on the graph of \( f(x) \).

Recall that the line that passes through two points \((x_0, y_0)\) and \((x_1, y_1)\) is prescribed by

\[
y = y_0 + \frac{y_1 - y_0}{x_1 - x_0} (x - x_0)
\]

Let \((x_0, f(x_0))\) and \((x_1, f(x_1))\) be two nearby points on the graph of \( f(x) \). The line passing through these two points is thus

\[
y = F(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0)
\]

Now the function \( F(x) \) has a zero at \( x_2 \) if

\[
0 = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x_2 - x_0)
\]

or

\[
x_2 = x_0 - f(x_0) \left( \frac{x_1 - x_0}{f(x_1) - f(x_0)} \right)
\]

If we regard, \( x_1, x_0, \) and \( x_2 \) as successive approximations for an actual zero of \( f(x) \) we can interprete this calculation as an algorithm for calculating an \((n+1)\)th order approximation to a zero of \( f(x) \): indeed, setting \( x_1 = x_{n+1}, x_0 = x_n, \) and \( x_1 = x_{n-1} \) we have

\[
x_{n+1} = x_n - f(x_n) \left( \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right)
\]

**Example 8.1.** Write a Maple routine that utilizes the secant method to determine a zero of

\[ f(x) = x^3 - 4x + 1 \]

starting with

\[
\begin{align*}
x_1 &= 0 \\
x_2 &= 1
\end{align*}
\]

\[
M := 10;  \\
delta := 0.000001;  \\
epsilon := 0.000001;  \\
f := x -> x^3 - 4*x + 1;  \\
x0 := 0.0;
\]
\[ x_1 := 1.0; \]
for i from 1 to M do
\[ x_2 := x_1 - f(x_1) \times (x_1 - x_0)/(f(x_1) - f(x_0)); \]
if (abs(f(x_2)) < epsilon) then break; fi;
\[ x_0 := x_1; \]
\[ x_1 := x_2; \]
if (abs(x_1 - x_0) < delta) then break; fi;
od; 
\[ x_2; \] 
\[ f(x_2); \]

1. Rate of Convergence of Secant Method

Let \( r \) be the actual root of \( f(x) = 0 \), let \( x_n \) be the approximate value for \( r \) obtained by carrying out \( n \) iterations of the secant method, and let \( e_n \) be the corresponding error:

\[ e_n = x_n - r. \]

We then have

\[
\begin{align*}
\epsilon_{n+1} &= x_{n+1} - r \\
&= x_n - f(x_n) \left( \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right) - r \\
&= \epsilon_n - f(x_n) \left( \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right) \\
&= \epsilon_n - f(x_n) \left( \frac{x_n - r - x_{n-1} + r}{f(x_n) - f(x_{n-1})} \right) \\
&= \epsilon_n - f(x_n) \left( \frac{\epsilon_n - e_{n-1}}{f(x_n) - f(x_{n-1})} \right) \\
&= \frac{\epsilon_n (f(x_n) - f(x_{n-1})) - f(x_n) (\epsilon_n - e_{n-1})}{f(x_n) - f(x_{n-1})} \\
&= \frac{\epsilon_n f(x_{n-1}) - \epsilon_{n-1} f(x_n)}{f(x_n) - f(x_{n-1})} \\
&= \left[ \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right] \left[ \frac{\epsilon_n f(x_{n-1}) - \epsilon_{n-1} f(x_n)}{x_n - x_{n-1}} \right] \\
&= \left[ \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right] \left[ \frac{f(x_{n-1})/\epsilon_{n-1} - f(x_n)/\epsilon_n}{x_n - x_{n-1}} \right] \epsilon_n \epsilon_{n-1}
\end{align*}
\]

Now by Taylor’s Theorem

\[
\begin{align*}
f(x_n) &= f(r) + f'(r) \epsilon_n + \frac{1}{2} f''(r) \epsilon_n^2 + O(\epsilon_n^3) \\
&= 0 + f'(r) \epsilon_n + \frac{1}{2} f''(r) \epsilon_n^2 + O(\epsilon_n^3)
\end{align*}
\]

So

\[
\frac{f(x_n)}{\epsilon_n} = f'(r) + \frac{1}{2} f''(r) \epsilon_n + O(\epsilon_n^2)
\]

and similarly

\[
\frac{f(x_{n-1})}{\epsilon_{n-1}} = f'(r) + \frac{1}{2} f''(r) \epsilon_{n-1} + O(\epsilon_{n-1}^2)
\]
So we have
\[ \frac{f(x_n)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}} = \left( f'(r) + \frac{1}{2} f''(r) e_n \right) - \left( f'(r) - \frac{1}{2} f''(r) e_{n-1} \right) + O \left( e_{n-1}^2 \right) \]
\[ = \frac{1}{2} f''(r) (e_n - e_{n-1}) + O \left( e_{n-1}^2 \right) \]
and
\[ e_{n+1} \approx \left[ \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right] \left[ \frac{1}{2} f''(r) (e_n - e_{n-1}) \right] e_n e_{n-1} \]

Now
\[ e_n - e_{n-1} = (x_n - r) - (x_{n-1} - r) = x_n - x_{n-1} \]
and for \( x_n \) and \( x_{n-1} \) sufficiently close to \( r \)
\[ \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \approx f'(r) \]
So
\[ (8.1) \quad e_{n+1} \approx \left[ f'(r) \right] \left[ \frac{1}{2} f''(r) \right] e_n e_{n-1} = C e_n e_{n-1} \]

In order to determine the order of convergence, we now suppose an asymptotic relationship of the form
\[ |e_{n+1}| \approx A |e_n|^a \]
This relationship also requires
\[ |e_n| \approx A |e_{n-1}|^a \quad \Rightarrow \quad |e_{n-1}| = (A^{-1} |e_n|)^{1/a} \]
In order to be consistent with (1) we need
\[ |e_{n+1}| = A |e_n|^a = C |e_n e_{n-1}| = C |e_n| (A^{-1} |e_n|)^{1/a} \]
or
\[ A |e_n|^a = CA^{-1} |e_n|^{1+\frac{1}{a}} \]
or
\[ \frac{A^{1-\frac{1}{a}}}{C} = |e_n|^{-a + \frac{1}{a}} \]

Now the left hand side is a product of constants. Therefore the right hand side must also be constant as \( n \to \infty \). For this to happen we need
\[ 0 = 1 - a + \frac{1}{a} \quad \Rightarrow \quad a^2 - a - 1 = 0 \quad \Rightarrow \quad a = \frac{1 \pm \sqrt{1 + 4}}{2} = \frac{1 \pm \sqrt{5}}{2} \]
Taking the positive root (otherwise, the error terms asymptotically diverge), we find
\[ a \approx 1.62 < 2 \]
Thus, the rate of convergence of the secant method is superlinear, but not quadratic.

Homework Problems

1. Show that
\[ e_{n+1} \approx \left[ \frac{f''(r)}{2 f'(r)} \right] e_n e_{n-1} = C e_n e_{n-1} \]

2. Use the secant method to find a solution of
\[ exp(x^2 - 2) = 3ln(x) \]
starting with \( x_0 = 1.5, x_1 = 1.4 \).