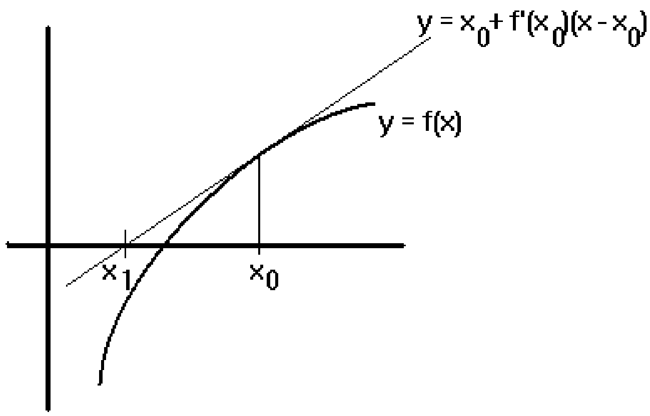
LECTURE 7

Newton's Method

Newton's method is another technique for finding the zeros of an equation of the form

$$f(x) = 0$$

Suppose f is both continuous and differentiable. Then f will have a smooth graph looking (for example) something like



In this picture we have also displayed the line

$$y = x_0 + f'(x_0)(x - x_0)$$

that represents the best straight line fit to the curve $y = f(x)$ near the point x_0. (Note that the right hand side of this equation is just the first order Taylor expansion of f about the point x_0.) Since

$$F_1(x) \equiv f(x_0) + f'(x_0)(x - x_0)$$

is a linear function of x it is trivial to find its zero x_1:

$$0 = f(x_0) + f'(x_0)(x_1 - x_0) \quad \Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now, in all likelyhood, x_1 is not also zero for $f(x)$; but if $F_1(x)$ is a sufficiently good approximation of $f(x)$ near the point x_1 then its zero there should be close to a zero of $f(x)$. Indeed, it should be closer to a zero of $f(x)$ then x_0. We therefore improve our approximation of $f(x)$ near its zero by looking at

$$F_2(x) = f(x_1) + f'(x_1)(x - x_1)$$

This function has a zero at

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

which should be an even better approximate zero of $f(x)$. The algorithm for finding successively better approximations for the zeros of $f(x)$ should now be clear.

1. Choose an initial point x_0, hopefully one that is close to a zero of $f(x)$.
2. Calculate
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
3. Replace the value of x_0 by the value of x_1 and loop back to Step 2.
4. Repeat Steps 2 and 3 until either
   (a) $f(x_0)$ is sufficiently close to zero
   (b) x_0 is sufficiently close to the actual zero
   (c) a fixed number of iterations is completed.

Example 7.1. Write a Maple routine that utilizes Newton's method to find a zero of
$$f(x) = x^5 - 3x + 1$$
starting with an initial point $x_0 = 1$.

```
f := x -> x^5 - 3*x + 1;
f1 := x -> 5*x^4 - 3;
x0 := 1.0;
for i from 1 to 100 while ( abs(f(x0)) > 0.000001) do
    x0 := x0 - f(x0)/f1(x0);
    od;
x0;
f(x0);
```

Notice how quickly this routine achieves the desired accuracy. This is an example of an algorithm with **quadratic convergence**. To see this explicitly, let r denote the actual root of $f(x) = 0$, let x_n denote the approximate value of r obtained by applying n iterations of the algorithm above. Then
$$e_n = x_n - r$$
is the absolute error after n iterations. Since
$$\begin{aligned} e_{n+1} &= x_{n+1} - r \\ &= x_n - \frac{f(x_n)}{f'(x_n)} - r \\ &= e_n - \frac{f(x_n)}{f'(x_n)} \\ &= \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)} \end{aligned}$$
Now by Taylor's Theorem
$$0 = f(r) = f(x_n - e_n) = f(x_n) + f'(x_n)(-e_n) + \frac{1}{2} f''(\xi_n)(-e_n)^2$$
for some ξ_n between r and x_n. So
$$e_n f(x_n) - f(x_n) = \frac{1}{2} f''(\xi_n)(e_n)^2$$
for some ξ_n between r and x_n. Hence,
$$e_{n+1} = \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)} = \frac{1}{2}\frac{f''(\xi_n)}{f'(x_n)}(e_n)^2 \approx \frac{1}{2}\frac{f''(r)}{f'(r)}(e_n)^2 = C|e_n|^2$$

Hence, the error terms converge quadratically to zero. We summarize this argument with the following theorem.

THEOREM 7.2. *Suppose that $f \in C^2(\mathbb{R})$ and let r be a simple zero of f. Then there is a neighborhood of r and a constant C such that if Newton's method is started in that neighborhood, the sucessive points become steadily closer to r and satisfy*

$$|x_{n+1} - r| \leq C\,|x_n - r|^2 \ .$$

(A simple zero of f is a point at which $f(x) = 0$ but $f'(x) \neq 0$.)