LECTURE 2

Sequences and Limits

1. Definition of Sequences and Limits

The most common situation In Numerical Analysis, is when one is faced with a problem in which an exact number can never be found; yet by successive approximations a highly accurate determination of that number can be made. For example, it is impossible to determine the full decimal expansion of the natural logarthimic base e, yet if we compute

$$\left(1+\frac{1}{n}\right)^n$$

for larger and larger values of n we get values which get closer and closer to the actual value of the irrational number e.

Below we formalize this idea.

Definition 2.1. *A* ***sequence*** *of real numbers is an ordered set* $\{a_1, a_2, a_3, \ldots\}$*of real numbers which can be viewed as the image of a map* A *from the natural numbers* $\mathbb{N} = \{1, 2, 3, \ldots\}$ *into* $\mathbb{R}$*.*

Remark 2.2. The map $A : \mathbb{N} \to \mathbb{R}$ in the above definition may seem like a bit of a red herring, but the important thing is that it is the basis of the ordering of the elements of the sequence: if $n > m$ then the $a_n \equiv A(n)$ occurs after $a_m \equiv A(m)$ in the sequence. Another thing to note is that because the set $\mathbb{N}$ is infinite, any sequence must also be an infinite set.

Definition 2.3. *Let* $\{a_1, a_2, a_3, \ldots\}$ *be a sequence. We say that this sequence* ***converges*** *to a* ***limit*** L *and write*

$$\lim_{n\to\infty} a_n = L$$

if for every $\varepsilon > 0$ *there is a number* r *such that*

$$|a_n - L| < \varepsilon \qquad \textit{whenever} \qquad n > r$$

Remark 2.4. In other words, if a sequence converges we can get as close as we want to the limit by looking far enough down the sequence. It is important in what follows to understand how the notation in the above definition guarantees this.

Example 2.5. Show that

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = 1$$

- Let ε be any arbitrarily small positive real number. We need to show that by choosing n large enough we can guarantee that $\frac{n+1}{n}$ is within ε of the limit $L = 1$. Now

  $$|a_n - L| = \left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right| = \frac{1}{n}$$

  so if we choose n so that $n > \frac{1}{\varepsilon}$ then $\frac{1}{n} < \varepsilon$

  $$\frac{1}{n} = |a_n - L| < \varepsilon$$

  Hence the sequence converges to 1.

2. Rates of Convergence

A fundamental concern when one is developing a sequence of numerical approximations is *how rapidly is the sequence of approximate values converging to the limit point*? Below we develop some basic terminology to deal with this issue.

DEFINITION 2.6. *Let $\{a_1, a_2, a_3, \ldots\}$ be a sequence with limit L. We say that its rate of convergence is* ***linear*** *if there is a constant $c < 1$ and an integer N such that*

$$|a_{n+1} - L| \leq c\,|a_n - L| \quad , \quad \textit{for all } n > N$$

We say that its rate of convergence is ***at least superlinear*** *if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with limit 0, and an integer N such that*

$$|a_{n+1} - L| \leq \varepsilon_n\,|a_n - L| \quad , \quad \textit{for all } n > N$$

We say that the rate of convergence is ***quadratic*** *if there is a constant C and an integer N such that*

$$|a_{n+1} - L| \leq C\,|a_n - L|^2 \quad , \quad \textit{for all } n > N$$

and, more generally, we saty that the rate of convergence is of ***order*** *α if there are positive constants C, α and an integer N such that*

$$|a_{n+1} - L| \leq C\,|a_n - L|^\alpha \quad , \quad \textit{for all } n > N$$

REMARK 2.7. Naturally, the greater the order of convergence the more rapid the convergence of the corrre-sponding sequence.

EXAMPLE 2.8. In the preceding example we showed

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = 1$$

This sequence however does not converge particularly rapidly. For

$$\frac{|a_{n+1} - L|}{|a_n - L|} = \frac{\left|\frac{n+2}{n+1} - 1\right|}{\left|\frac{n+1}{n} - 1\right|} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1}$$

While this number is less than 1 for any finite n there is no fixed number $c < 1$ that guarantees that

$$\frac{n}{n+1} < c$$

for all n suffciently large, this sequence does not converge linearly.

DEFINITION 2.9. *Let $\{a_n\}$ and $\{b_n\}$ be two different sequences. We write*

$$b_n = \mathcal{O}(a_n)$$

if there are constants C and N such that

$$|b_n| \leq C\,|a_n| \quad \textit{whenever} \quad n > N$$

and say that the sequence $\{b_n\}$ is "big oh" of the sequence $\{a_n\}$.

REMARK 2.10. In other words, a sequence $\{b_n\}$is "big oh" of $\{a_n\}$ if, for sufficiently large n, the elements b_n grow no more rapidly than a fixed constant times the corresponding elements of $\{a_n\}$. Note that these sequences do not necessarily converge; However if $\{a_n\}$ converges to a number L then $\{b_n\}$ must converge to a number $L' \leq CL$.

Below are some examples that should clarify the utility of this notion.

Example 2.11. If

$$a_n = \frac{n}{n+1} \quad \text{and} \quad b_n = 1 \quad \forall n$$

then

$$b_n = \mathcal{O}(a_n) \quad \text{and} \quad a_n = \mathcal{O}(b_n)$$

Example 2.12. If

$$a_n = n^3 \quad \text{and} \quad b_n = 3n^3 + 2n^2 + 1$$

then

$$b_n = \mathcal{O}(a_n)$$

To see this note that

$$b_n = \mathcal{O}(a_n) \quad \Longleftrightarrow \quad \exists\, C \text{ such that } \frac{|b_n|}{|a_n|} \leq C \text{ for sufficiently large } n$$

But

$$\frac{|b_n|}{|a_n|} = \frac{3n^3 + 2n^2 + 1}{n^3} = 3 + \frac{2}{n} + \frac{1}{n^3} \leq 4 \text{ for all } n > 3$$

So taking $C = 4$ will do the trick for us.

Example 2.13. If

$$a_n = n^2 \quad \text{and} \quad b_n = n$$

then

$$b_n = \mathcal{O}(a_n) \quad \text{but} \quad a_n \neq \mathcal{O}(b_n)$$

Definition 2.14. *Let $\{a_n\}$ and $\{b_n\}$ be two different sequences. We write*

$$b_n = \mathfrak{o}(a_n)$$

if

$$\lim_{n\to\infty} \left| \frac{b_n}{a_n} \right| = 0$$

and say that the sequence $\{b_n\}$ is "little oh" of the sequence $\{a_n\}$.

Remark 2.15. Thus, for a sequence $\{b_n\}$ to be little oh of $\{a_n\}$ means that for larger and larger n the terms b_n are insignificant compaired to the corresponding terms of $\{a_n\}$.

Example 2.16. If

$$a_n = \frac{1}{n \ln |n|} \quad \text{and} \quad b_n = \frac{1}{n}$$

then

$$b_n = \mathfrak{o}(a_n)$$

since

$$\lim_{n\to\infty} \left| \frac{b_n}{a_n} \right| = \lim_{n\to\infty} \frac{1}{\ln |n|} = 0$$

Example 2.17. If

$$a_n = n^2 + 1 \quad \text{and} \quad b_n = n + 1$$

then

$$b_n = \mathfrak{o}(a_n)$$

since

$$\lim_{n\to\infty} \left| \frac{b_n}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{n^2+1} = 0$$

3. Upper and Lower Bounds

DEFINITION 2.18. *Let S be a subset of the real numbers. S is **bounded** if there exists real numbers a and b such that*

$$a \leq x \leq b \quad \textit{for all } x \in S$$

*We call a and b, respectively, **lower** and **upper bounds** for the set S.*

AXIOM 1. *Every non-empty set of real numbers that has an upper bound has a least upper bound. Equivalently, every non-empty set of real numbers that has a lower bound has a greatest lower bound.*

4. Problems

1. Prove that $x_n = L + \mathfrak{o}(1)$ iff and only if $\lim_{n\to\infty} x_n = L$.

2. Show that if $b_n = \mathcal{O}(a_n)$ then $b_n / \ln|n| = \mathfrak{o}(a_n)$.

3. Show that if $b_n = \mathfrak{o}(a_n)$ then $b_n = \mathcal{O}(a_n)$, but that the converse is not true.

4. Show that if $b_n = \mathcal{O}(a_n)$ and $c_n = \mathcal{O}(a_n)$, then $b_n + c_n = \mathcal{O}(a_n)$.

5. Show that if $b_n = \mathfrak{o}(a_n)$ and $c_n = \mathfrak{o}(a_n)$, then $b_n + c_n = \mathfrak{o}(a_n)$.

6. Show that for any $r > 0$, $x^r = \mathcal{O}(e^x)$ as $x \to \infty$.

7. Show that for any $r > 0, \ln|x| = \mathcal{O}(x^r)$ as $x \to \infty$.