Math 4263
Solutions to Homework Set 5

1. Determine if the following ODEs are of the Sturm-Liouville type

\[ \frac{d}{dx} \left[ p(x) \frac{dy}{dx} \right] - q(x) y + \lambda r(x) y = 0, \quad p(x) > 0, \quad r(x) > 0 \]

and if so identify the functions \( p(x), q(x) \) and \( r(x) \).

(a) \( y'' + k^2 y = 0 \)

\[ \bullet \quad \text{If we take } p(x) = 1, \quad q(x) = 0, \quad r(x) = 1 \quad \text{and } \lambda = k^2, \quad \text{we have} \]

\[ 0 = \frac{d}{dx} \left[ p(x) \frac{dy}{dx} \right] - q(x) y + \lambda r(x) y = y'' + 0 + k^2 y = 0 = y'' + k^2 y \]

so this ODE is of Sturm-Liouville type.

(b) \( x^2 y'' + xy' + (x^2 - \nu^2) y = 0 \)

\[ \bullet \quad \text{Dividing (b) by } x \text{ yields} \]

\[ xy'' + y' + (x - \nu^2 x^{-1}) y = 0 \]

We have

\[ \frac{d}{dx} \left( x \frac{dy}{dx} \right) = \frac{d^2 y}{dx^2} + \frac{dy}{dx} \]

and so

\[ xy'' + y' + (x - \nu^2 x^{-1}) y = \frac{d}{dx} \left( x \frac{dy}{dx} \right) + (x - \nu^2 x^{-1}) y \]

The left hand side is in Sturm-Liouville form with

\[ p(x) = x, \quad q(x) = \nu x^{-1}, \quad r(x) = x, \quad \lambda = 1. \]

Note that the positivity requirements on \( p(x) \) and \( r(x) \) are satisfied for all \( x \in (0, +\infty) \).

(c) \( (1 - x^2) y'' - 2xy' + \nu (\nu + 1) y = 0 \)

\[ \bullet \quad \text{Divide the differential equation by } (1 - x^2) \text{ to get} \]

\[ y'' - \frac{2x}{1 - x^2} y + \frac{1}{1 - x^2} \nu (\nu + 1) y = 0 \]

which is in Sturm-Liouville form with \( p(x) = 1, \quad q(x) = \frac{2x}{1 - x^2}, \quad r(x) = \frac{1}{1 - x^2} \quad \text{and } \lambda = \nu (\nu + 1). \)

2. Develop a formal eigenfunction expansion for the solution of the following problem

\[ y'' + \mu y = 0, \quad y'(0) = 0, \quad y(1) + y'(1) = 0 \]

\[ \bullet \quad \text{Consider the following Sturm-Liouville problem} \]

\[ y'' + \lambda^2 y = 0 \]

\[ y'(0) = 0 \]

\[ y(1) + y'(1) = 0 \]

The general solution to (2.1) is of course a linear combination of sine and cosine functions

\[ y'' + \lambda^2 y = 0 \quad \iff \quad y = c_1 \cos(\lambda x) + c_2 \sin(\lambda x) \]

In order to satisfy the boundary conditions \( y'(0) = 0 \) we must have

\[ 0 = -c_1 \lambda \sin(0) + c_2 \lambda \cos(0) \quad \iff \quad c_2 = 0 \quad \text{or} \quad \lambda = 0. \]
Of course, when $\lambda = 0$, we anyway have $y(x) = c_1 \cos(0) + c_2 \sin(0) = c_1$. So we can simply set $c_2 = 0$ without loss of generality. In order to satisfy the second boundary condition, we must have

$$0 = c_1 \cos(\lambda) - c_1 \lambda \sin(\lambda)$$

or

$$\lambda = \cot \lambda$$

This is a transcendental equation for $\mu$, which evidently has an infinite number of solutions. Below the graph of $y = x$ is superimposed on the graph of $y = \cot(x)$. The points where the two graphs intersect are the solutions of $x = \cot(x)$.

Let $\{\lambda_0, \lambda_1, \lambda_2, \ldots\}$ be the set of roots of $x = \cot(x)$ ordered in such a way that

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

Then

$$\phi_n = \cos(\lambda_n x) \quad , \quad n = 0, 1, 2, \ldots$$

will constitute an infinite family of functions satisfying

$$y'' - \lambda_n^2 y = 0$$
$$y'(0) = 0$$
$$y(1) - y'(1) = 0$$

Since this ODE is of Sturm-Liouville type, as are the boundary conditions, the functions $\phi_n(x)$ will constitute a set of mutually orthogonal Sturm-Liouville eigenfunctions. As such, the solution to the original nonhomogeneous problem (2) has an expansion in terms of the $\phi_n(x)$:

$$y(x) = \sum_{n=0}^{\infty} a_n \phi_n(x) = \sum_{n=0}^{\infty} a_n \cos(\lambda_n x)$$
Plugging this expression into the differential equation yields
\[ f(x) = y'' - \mu y = \sum_{n=0}^{\infty} a_n \left( \frac{d^2}{dx^2} \cos(\lambda_n x) + \mu \cos(\lambda_n x) \right) \]
\[ = \sum_{n=0}^{\infty} a_n \left( -\lambda_n^2 + \mu \right) \cos(\lambda_n x) \]

Multiplying both sides of this last equation by \( \cos(\lambda_m x) \) and integrating over the interval \([0,1] \)

\[ \int_0^1 f(x) \cos(\lambda_m x) \, dx = \sum_{n=0}^{\infty} a_n \left( -\lambda_n^2 + \mu \right) \int_0^1 \cos(\lambda_n x) \cos(\lambda_m x) \, dx \]

Now because functions \( \cos(\lambda_n x) \) are Sturm-Liouville eigenfunctions, we know that
\[ \int_0^1 \cos(\lambda_n x) \cos(\lambda_m x) \, dx = \begin{cases} \int_0^1 \cos^2(\lambda_m x) \, dx & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \]
and so
\[ \int_0^1 f(x) \cos(\lambda_m x) \, dx = \sum_{n=0}^{\infty} a_n \left( -\lambda_n^2 + \mu \right) \int_0^1 \cos^2(\lambda_m x) \, dx \]
\[ = a_m \left( -\lambda_m^2 + \mu \right) \int_0^1 \cos^2(\lambda_m x) \, dx \]

or
\[ a_m = \frac{1}{\int_0^1 \cos^2(\lambda_m x) \, dx} \frac{1}{\lambda_m^2} \int_0^1 f(x) \cos(\lambda_m x) \, dx \quad \forall \, m \text{ such that } \lambda_m \neq \mu \]

This formula yields the choice of coefficients \( a_m \) such
\[ y(x) = \sum_{n=0}^{\infty} a_n \cos(\lambda_n x) \]

is a solution to the original nonhomogeneous boundary conditions. (The situation where \( \mu \) is equal
to one of the \( \lambda_m \) is discussed in Lecture 14. I won’t get into that here.)

3. Solve the following PDE/BVP:
\[ \phi_t - k^2 \phi_{xx} = 0 \quad , \quad 0 \leq x \leq 1 \quad , \quad t \geq 0 \]
\[ \phi(x, 0) = f(x) \quad , \quad 0 \leq x \leq 1 \]
\[ \phi(0, t) = 0 \quad , \quad t \geq 0 \]
\[ \phi(1, t) + \phi_x(1, t) = 0 \quad , \quad t \geq 0 \]

- Applying Separation of Variable to the Heat Equation, we quickly discover a large family of solutions:
  viz, functions of the form
\[ \phi_{\lambda}(x, t) = a_\lambda e^{-k^2 \lambda^2 t} \cos(\lambda x) + b_\lambda e^{-k^2 \lambda^2 t} \sin(\lambda x) \]

We aim to find a linear combination of these solutions that satisfies our stated boundary conditions. Luckily, we have so many solutions we can readily discard most of them.

Consider a general linear combination of the solutions \( \phi_{\lambda}(x, t) \):
\[ \phi(x, t) = \sum_\lambda \left( a_\lambda e^{-k^2 \lambda^2 t} \cos(\lambda x) + b_\lambda e^{-k^2 \lambda^2 t} \sin(\lambda x) \right) \]

Imposing the second BC we find the coefficients \( a_\lambda \) must be chosen so that
\[ 0 = \phi(0, t) = \sum_\lambda \left( a_\lambda e^{-k^2 \lambda^2 t} \cos(0) + b_\lambda e^{-k^2 \lambda^2 t} \sin(0) \right) = \sum_\lambda a_\lambda e^{-k^2 \lambda^2 t} \]
since the exponential functions $e^{-k^2\lambda^2 t}$ are linearly independent functions of $t$, we must take each of the coefficients $a_\lambda = 0$.

So now we have
\[
\phi (x, t) = \sum_\lambda b_\lambda e^{-k^2\lambda^2 t} \sin (\lambda x)
\]

Imposing the third boundary condition, we get
\[
0 = \phi (1, t) + \phi_x (1, t) = \sum_\lambda b_\lambda e^{-k^2\lambda^2 t} (\sin (\lambda) + \lambda \cos (\lambda))
\]

Since the right hand side has to vanish for all $t$ and because the functions $e^{-k^2\lambda^2 t}$ are linearly independent we must have
\[
b_\lambda (\sin (\lambda) + \lambda \cos (\lambda)) = 0
\]

This time we don’t want to set all the coefficients $b_\lambda$ equal to zero as that would trivialize our solution (forcing $\phi (x, t) = 0$ for all $x$ and $t$). So instead we use only $\lambda$’s such that
\[
\sin (\lambda) + \lambda \cos (\lambda) = 0 \iff \lambda \text{ is a solution of } \lambda = -\tan (\lambda)
\]

Let us order the positive solutions of $\lambda = -\tan (\lambda)$ as $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ and set
\[
\phi (x, t) = \sum_{n=1}^\infty b_n e^{-k^2\lambda_n^2 t} \sin (\lambda_n x)
\]

We have one last boundary condition to employ, applying it we get

\[
(*)
\]

How does this help us identify the coefficients $b_n$? Well, the way we set things us, the functions $\sin (\lambda_n x)$ satisfy
\[
y'' + \lambda_n^2 y = 0
\]
\[
y (0) = 0
\]
\[
y (1) + y' (1) = 0
\]

That is to say, they all satisfy the differential equation and boundary conditions of a Sturm-Louisville problem. So they’re Sturm-Louisville eigenfunctions and so we can interpret the $(*)$ as the expansion of $f (x)$ in terms of S-L eigenfunctions. The coefficients are therefore determined by integrating $f (x)$ against the normalized S-L eigenfunctions
\[
b_n = \int_0^1 f (x) \frac{\sin (\lambda_n x)}{\left[ \int_0^1 \sin^2 (\lambda_n s) \, ds \right]^{1/2}} dx
\]

Since
\[
\int_0^1 \sin^2 (\lambda x) \, dx = \int_0^1 \left( \frac{1}{2} - \frac{1}{2} \cos 2x\lambda \right) \, dx = \frac{1}{2} - \frac{1}{2} \left( \frac{1}{2\lambda} \right) \sin (2\lambda) = \frac{1}{2} - \frac{\sin (2\lambda)}{4\lambda}
\]

we can be a bit more explicit and write the solution to our PDE/BVP as
\[
\phi (x, t) = \sum_{n=1}^\infty b_n e^{-k^2\lambda_n^2 t} \sin (\lambda_n x)
\]

where $\lambda_1, \lambda_2, \ldots$ are the positive of $\lambda = -\tan (\lambda)$ and the coefficients $b_n$ are given by
\[
b_n = \frac{1}{\left[ \frac{1}{2} - \frac{\sin (2\lambda_n)}{4\lambda} \right]^{1/2}} \int_0^1 f (x) \sin (\lambda_n x) \, dx