# Math 4263
## Solutions to Homework Set 4

1. Show that a function $f(z) = u(z) + iv(z)$ of a complex variable $z = x + iy$ that satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

also has the property that both its real part $u(z)$ and its imaginary part $v(z)$ satisfy Laplace's equation: i.e.,

$$u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$$

- We have

$$u_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = -u_{xx}$$

and so

$$u_{xx} + u_{yy} = 0 \quad .$$

Similarly,

$$v_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = -v_{yy}$$

and so

$$v_{xx} + v_{yy} = 0 \quad .$$

2. Let $g(x)$ be any piecewise continuous function on $\mathbb{R}$. Show directly from the definition, that the mapping $\phi_g : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ given by

$$\phi_g(f) := \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$

defines a distribution.

- Let $c_1 f_1(x) + c_2 f_2(x)$ be an arbitary linear combination of functions in $C_c^\infty(\mathbb{R})$. We have

$$\begin{aligned}
\phi_g(c_1 f_1 + c_2 f_2) &\equiv \int_{-\infty}^{\infty} (c_1 f_1(x) + c_2 f_2(x))\, g(x)\, dx \\
&= c_1 \int_{-\infty}^{\infty} f_1(x)\, g(x)\, dx + c_2 \int_{-\infty}^{\infty} f_2(x)\, g(x)\, dx \\
&= c_1 \phi_g(f_1) + c_2 \phi_g(f_2)
\end{aligned}$$

and so $\phi_g$ is a linear functional on $C_c^\infty(\mathbb{R})$. Strictly speaking, however, this is not enough to prove that $\phi_g$ is a distribution. For a distribution is a *continuous* linear functional on $C_c^\infty(\mathbb{R})$. To prove the continuity of $\phi_g$, we must check that if $\{f_1, f_2, \ldots\}$ is a sequence of functions in $C_c^\infty(\mathbb{R})$ that vanish outside a common interval and converge uniformly to a function $f \in C_c^\infty(\mathbb{R})$ then

$$\lim_{n\to\infty} \phi_g(f_n) = \phi_g(f)$$

In the case at hand, we thus need

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx \qquad \text{whenever } \{f_1, f_2, \ldots\} \text{ converges uniformly to } f$$

Now uniform convergence means the following: $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to $f$ if for every $\varepsilon > 0$, there exists a natural number $N$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in \mathbb{R}$ and for all $n > N$. Now consider

$$\left| \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx - \int_{-\infty}^{\infty} f(x)\, g(x)\, dx \right| \leq \int_{-\infty}^{\infty} |f_n(x) - f(x)|\, |g(x)|\, dx$$

Since $\{f_n\}$ converges uniformly to $f$, for any $\varepsilon > 0$ we can find an $N$ such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n > N \text{ and all } x \in \mathbb{R}$$

In fact, we are assuming, moreover, that the functions $\{f_n\}$ and $f$ all vanish outside a certain finite interval $I$. Let $L$ be the length of that interval and let $G$ be the maximum value of $g(x)$ in $I$. Then we have for all $n > N$

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx - \int_{-\infty}^{\infty} f(x)\, g(x)\, dx\right| &\leq \int_{-\infty}^{\infty} |f_n(x) - f(x)|\, |g(x)|\, dx \\
&= \int_I |f_n(x) - f(x)|\, |g(x)|\, dx \\
&\leq LG\varepsilon \\
&= (\text{some finite constant}) \times \varepsilon
\end{aligned}$$

Letting $\varepsilon \to 0$, we see we can force

$$\left|\int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx - \int_{-\infty}^{\infty} f(x)\, g(x)\, dx\right| \to 0$$

and so

$$\lim_{n\to\infty} \phi_g(f_n) = \phi_g(f)$$

as required.

3. Let $\psi$ be any distribution. Verify that the functional $\psi'$ defined by

$$\psi'(f) := -\psi\left(\frac{df}{dx}\right)$$

is a distribution.

- The linearity of $\psi'$ is easily verifed:

$$\begin{aligned}
\psi'(c_1 f_1 + c_2 f) &\equiv -\psi\left(\frac{d}{dx}(c_1 f_1 + c_2 f_2)\right) \\
&= -\psi\left(c_1 \frac{df_1}{dx} + c_2 \frac{df_2}{dx}\right) && \text{by properties of differentiation} \\
&= -c_1 \psi\left(\frac{df_1}{dx}\right) - c_2 \psi\left(\frac{df_2}{dx}\right) && \text{by linearity properties of the distribution } \psi \\
&= c_1 \psi'(f_1) + c_2 \psi'(f_2)
\end{aligned}$$

The continuity of $\psi'$ follows from the well-known[1] fact that if $\{f_n\}$ is a sequence of smooth functions that converges uniformly to $f$ on a compact sets then $\left\{\frac{df_n}{dx}\right\}$ converges uniformly to $\frac{df}{dx}$. Thus,

$$\lim_{n\to\infty} f_n = f \quad uniformly \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{df_n}{dx} = \frac{df}{dx} \quad uniformly$$

and so

$$\lim_{n\to\infty} \psi'(f_n) \equiv \lim_{n\to\infty} -\psi\left(\frac{df_n}{dx}\right) = -\psi\left(\frac{df}{dx}\right) \quad \text{since } \psi \text{ is a distribution and } \frac{df_n}{dx} \text{ converges uniformly to } \frac{df}{dx}$$

Thus,

$$\lim_{n\to\infty} f_n = f \quad uniformly \qquad \Longrightarrow \quad \lim_{n\to\infty} \psi'(f_n) = -\psi\left(\frac{df}{dx}\right)$$

and so $\psi'$ is a continuous linear functional on $C_c^{\infty}(\mathbb{R})$, hence a distribution.

[1] This is typically proved somewhere in Math 4023: Introduction to Modern Analysis.

4. Let $u(\mathbf{x}) = u(x,y)$ be a harmonic function on a planar domain $D$. Derive the *representation formula*

$$u(\mathbf{x}_0) = \frac{1}{2\pi}\int_{\partial D} \left[u(\mathbf{x})\left(\boldsymbol{\nabla}\ln\|\mathbf{x}-\mathbf{x}_0\|\right) - \left(\boldsymbol{\nabla}u(\mathbf{x})\right)\ln\|\mathbf{x}-\mathbf{x}_0\|\right]\cdot\mathbf{n}\, dS$$

that expresses $u(\mathbf{x}_0)$ at an interior point $\mathbf{x}_0$ as a certain integral of $u(\mathbf{x})$ and its gradient over the boundary of $D$.

- Consider the function

$$\phi(\mathbf{x}) = \ln\|\mathbf{x}\| = \ln\left(\sqrt{x^2+y^2}\right) = \frac{1}{2}\ln\left|x^2+y^2\right|$$

We have

$$\begin{aligned}
\phi_{xx} &= \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{2x}{x^2+y^2}\right) = \frac{-x^2+y^2}{\left(x^2+y^2\right)^2}\\
\phi_{yy} &= \frac{\partial}{\partial y}\left(\frac{1}{2}\frac{2y}{x^2+y^2}\right) = \frac{x^2-y^2}{\left(x^2+y^2\right)^2}
\end{aligned}$$

Thus,

$$\phi_{xx}+\phi_{yy}=0$$

and so $\phi(\mathbf{x})$ is a solution of the Laplace equation – at least **everywhere it is defined**. This is the 2-dimensional analogue of the fundamental solution $1/\|\mathbf{x}\|$ to the Laplace equation in 3-dimensions.

- Now consider, for any $\varepsilon > 0$,

$$\phi_\varepsilon(\mathbf{x}) := \ln\left(\|\mathbf{x}\|+\varepsilon\right) = \ln|r+\varepsilon| \qquad \text{(in polar coordinates)}$$

This is a well-defined smooth function of $\mathbf{x}$ for all $\mathbf{x}\in\mathbb{R}^2$ and we have

$$\begin{aligned}
\boldsymbol{\nabla}^2\phi_\varepsilon(\mathbf{x}) &= \left(\frac{\partial^2}{\partial r^2}\phi_\varepsilon(r,\theta) + \frac{1}{r}\frac{\partial}{\partial r}\phi_\varepsilon(r,\theta) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi_\varepsilon(r,\theta)\right) = \\
&= \frac{\varepsilon}{r\left(r+\varepsilon\right)^2}
\end{aligned}$$

Now consider

$$\begin{aligned}
\int_{\mathbb{R}^2}\boldsymbol{\nabla}^2\phi_\varepsilon(\mathbf{x})\,d^2\mathbf{x} &= \int_0^\infty\int_0^{2\pi}\frac{\varepsilon}{r\left(r+\varepsilon^2\right)}rdrd\theta\\
&= 2\pi\int_0^\infty\frac{\varepsilon}{\left(r+\varepsilon\right)^2}dr\\
&= \lim_{R\to\infty}\left.\frac{-2\pi\varepsilon}{r+\varepsilon}\right|_0^R\\
&= 2\pi \quad \text{(independent of } \varepsilon\text{)}
\end{aligned}$$

And so

$$\frac{1}{2\pi}\boldsymbol{\nabla}^2\phi_\varepsilon$$

has the properties that

$$\begin{aligned}
\lim_{\varepsilon\longrightarrow 0}\frac{1}{2\pi}\boldsymbol{\nabla}^2\phi_\varepsilon(\mathbf{x}) &= \begin{cases} 0 & \text{if } \mathbf{x}\neq 0\\ \text{undefined} & \text{if } \mathbf{x}=\mathbf{0}\end{cases}\\
\int_{\mathbb{R}^2}\frac{1}{2\pi}\boldsymbol{\nabla}^2\phi_\varepsilon(\mathbf{x}) &= 1 \qquad \text{independent of } \varepsilon
\end{aligned}$$

and so we are justified in setting

$$-\frac{1}{2\pi}\boldsymbol{\nabla}^2\phi_\varepsilon(\mathbf{x}) = \delta^{(2)}(\mathbf{x})$$

and even, upon making a change of variables,

$$-\frac{1}{2\pi}\boldsymbol{\nabla}^2\phi_\varepsilon(\mathbf{x}-\mathbf{x}_o) = \delta^{(2)}(\mathbf{x}-\mathbf{x}_0)$$

- Now let $D \subset \mathbb{R}^2$ be a closed, solid, domain in $\mathbb{R}^2$ and let $\partial D$ be its boundary. Green's second identity says

$$\int_D \left(\phi \nabla^2 \psi - \psi \nabla 2\phi\right) dA = \int_{\partial D} \left(\phi \nabla \psi - \psi \nabla \phi\right) \cdot \mathbf{n} \; dC$$

Now let $\phi(\mathbf{x})$ be any solution of Laplace's equation on $D$ and take $\psi(\mathbf{x}) = \psi_\varepsilon = \frac{1}{2\pi} \ln \|\mathbf{x} - \mathbf{x}_o\|$. We then have

$$\int_D \phi(\mathbf{x}) \left(\frac{1}{2\pi} \nabla \ln\left(\|\mathbf{x} - \mathbf{x}_o\| + \varepsilon\right)\right) dA = \frac{1}{2\pi} \int_{\partial D} \left(\phi \nabla \ln\left(\|\mathbf{x} - \mathbf{x}_o\| + \varepsilon\right) - \ln\left(\|\mathbf{x} - \mathbf{x}_o\| + \varepsilon\right) \nabla \phi\right) \cdot \mathbf{n} \; dC$$

Taking the limit $\varepsilon \to 0$, we get

$$\phi(\mathbf{x}_o) = \int_D \phi(\mathbf{x}) \, \delta^{(2)}(\mathbf{x} - \mathbf{x}_o) \, dA = \frac{1}{2\pi} \int_{\partial D} \left(\phi \nabla \ln\left(\|\mathbf{x} - \mathbf{x}_o\|\right) - \ln\left(\|\mathbf{x} - \mathbf{x}_o\|\right) \nabla \phi\right) \cdot \mathbf{n} \; dC$$

5. A **Green's function** $G_\mathbf{y}(\mathbf{x})$ for the Laplace operator $\nabla^2$ and domain $D$ and a point $\mathbf{y} \in D$, is a function defined for all $\mathbf{x}$ in $D$ such that

(i) $G_\mathbf{y}(\mathbf{x})$ posseses continuous second derivatives and $\nabla^2 G_\mathbf{y}(\mathbf{x}) = 0$; except at the point $\mathbf{x} = \mathbf{y}$.
(ii) $G_\mathbf{y}(\mathbf{x}) = 0$ for all $\mathbf{x}$ on the boundary $\partial D$ of $D$.
(iii) The function

$$G_\mathbf{y}(\mathbf{x}) + \frac{1}{4\pi \|\mathbf{x} - \mathbf{y}\|}$$

is finite at $\mathbf{y}$, has continuous second partial derivatives everywhere and is harmonic at $\mathbf{y}$.

Show that such a function is unique. (You can assume such a function always exists - this is, in fact, true.)

- Consider

$$\varphi_0(\mathbf{x}) = \mathbf{G}_\mathbf{y}(\mathbf{x}) + \frac{1}{4\pi \|\mathbf{x} - \mathbf{y}\|}$$

with $G_\mathbf{y}(\mathbf{x})$ satisfying conditions (i), (ii), (iii) above. Evidently, $\varphi_0(\mathbf{x})$ satifies Laplace's equation at **all** points of $D$ including $\mathbf{x} = \mathbf{y}$, because

$$\nabla^2 \left(G_\mathbf{y}(\mathbf{x}) + \frac{1}{4\pi \|\mathbf{x} - \mathbf{y}\|}\right) = 0 + 0 = 0 \qquad \forall \; \mathbf{x} \neq \mathbf{y}$$

and because of (iii). Moreover, $\varphi_0(\mathbf{x})$ satisfies

(*) $$\left.\varphi(\mathbf{x})\right|_{\partial D} = G_\mathbf{y}(\mathbf{x}) + \left.\frac{1}{4\pi \|\mathbf{x} - \mathbf{y}\|}\right|_{\partial D} = \left.\frac{1}{4\pi \|\mathbf{x} - \mathbf{y}\|}\right|_{\partial D}$$

by (ii). Hence, $\varphi_0(\mathbf{x})$ is unique because there is exactly one solution of Laplace's equation on $D$ satisfying the Dirichlet boundary conditions (*). But then if $\phi_0(\mathbf{x})$ is unique, then

$$\mathbf{G}_\mathbf{y}(\mathbf{x}) = \varphi_0(\mathbf{x}) - \frac{1}{4\pi \|\mathbf{x} - \mathbf{y}\|}$$

is unique.