Math 4263
Solutions to Homework Set 2

1. Use the Maximum Principle for the Heat Equation to demonstrate that there is a unique solution to

$$
\begin{aligned}
u_t - k^2 u_{xx} &= f(x,t) \quad , \quad 0 \le x \le L \quad , \quad t > 0 && \text{(1a)} \\
u(0,t) &= g(t) \quad , \quad t > 0 && \text{(1b)} \\
u(L,t) &= h(t) \quad , \quad t > 0 && \text{(1c)} \\
u(x,0) &= \phi(x) \quad , \quad 0 \le x \le L && \text{(1d)}
\end{aligned}
$$

- According to the Maximum Principle, any solution $v(t,x)$ of the homogeneous Heat Equation

$$u_t - k^2 u_{xx} = 0 \quad , \quad 0 \le x \le L \quad , \quad t > 0$$

attains it maximal value in the rectangle

$$R = \{(x,t) \mid 0 \le x \le L \quad , \quad 0 \le t \le T\}$$

on one of the three sides

$$
\begin{aligned}
\ell_1 &= \{(0,t) \mid 0 \le t \le T\} \\
\ell_2 &= \{(x,0) \mid 0 \le x \le L\} \\
\ell_3 &= \{(L,t) \mid 0 \le t \le T\}
\end{aligned}
$$

Now suppose u_1 and u_2 are two solutions of (1a) - (1b). Then

$$v(x,t) = u_1(x,t) - u_2(x,t)$$

will satisfy

$$
\begin{aligned}
v_t - k^2 v_{xx} &= \frac{\partial}{\partial t}(u_1 - u_2) - k^2 \frac{\partial^2}{\partial x^2}(u_1 - u_2) \\
&= \frac{\partial}{\partial t} u_1 - k^2 \frac{\partial^2}{\partial x^2} u_1 - \frac{\partial}{\partial t} u_2 + k^2 \frac{\partial^2}{\partial x^2} u_2 \\
&= f(x,t) - f(x,t) \\
&= 0
\end{aligned}
$$

And so the Maximum Principle implies that $v(x,t)$ must attain its maximal value on one of the boundary lines ℓ_1, ℓ_2, ℓ_3. But by virtue of equations (1b), (1c) and (1d), we have

$$
\begin{aligned}
v(0,t) &= u_1(0,t) - u_2(0,t) = g(t) - g(t) = 0 \quad , \\
v(L,t) &= u_1(L,t) - u_2(L,t) = h(t) - h(t) = 0 \quad , \\
v(x,0) &= u_1(x,0) - u_2(x,0) = \phi(x) - \phi(x) = 0 \quad .
\end{aligned}
$$

Thus, the maximal value of $v(x,t) = u_1(x,t) - u_2(x,t)$ throughout the rectangle R is 0. Applying the same argument to $v'(x,t) = u_2(x,t) - u_1(x,t) = -u(x,t)$, we can conclude the maximal value of $|u_1(x,t) - u_2(x,t)|$ throughout R is 0. This means $u_1(x,t)$ hast to equal $u_2(x,t)$ throughout R. Hence, any solution of (1a) – (1d) is unique.

2. Prove the following identities

$$
\begin{aligned}
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx &= \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \ne n \end{cases} && \text{(2a)} \\
\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx &= 0 && \text{(2b)} \\
\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx &= \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \ne n \end{cases} && \text{(2c)}
\end{aligned}
$$

- By the addition and subtraction formulas for cosine functions

$$\cos(A+B) = \cos(A)\cos B - \sin(A)\sin(B)$$
$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

we have

$$\begin{aligned}\sin(A)\sin(B) &= \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)\\ \cos(A)\cos B &= \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)\end{aligned}$$

Thus, if $m \neq n$

$$\begin{aligned}\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi}\cos((m-n)x)\,dx - \frac{1}{2}\int_{-\pi}^{\pi}\cos((m+n)x)\,dx\\ &= \frac{1}{2}\frac{1}{m-n}\sin((m-n)x)\Big|_{-\pi}^{\pi} - \frac{1}{2}\frac{1}{m+n}\sin((m+n)x)\Big|_{-\pi}^{\pi}\\ &= \frac{1}{2}\frac{1}{m-n}(\sin(m-n\pi) - \sin((n-m)\pi))\\ &\quad -\frac{1}{2}\frac{1}{m+n}(\sin((m+n)\pi) - \sin(-(m+n)\pi))\\ &= 0\end{aligned}$$

because $\sin(x)$ vanishes whenever x is an integer multiple of π. On the other hand, if $m = n$, then

$$\begin{aligned}\int_{-\pi}^{\pi}\sin(nx)\sin(nx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi}\cos((n-n)x)\,dx - \frac{1}{2}\int_{-\pi}^{\pi}\cos((n+n)x)\,dx\\ &= \frac{1}{2}\int_{-\pi}^{\pi}\cos(0)\,dx - \frac{1}{2}\int_{-\pi}^{\pi}\cos(2nx)\,dx\\ &= \frac{1}{2}\int_{-\pi}^{\pi}1dx - \frac{1}{2}\frac{1}{2n}\sin(2nx)\Big|_{=\pi}^{\pi}\\ &= \frac{1}{2}(\pi-(-\pi)) + 0\\ &= \pi\end{aligned}$$

Formula (2a) now follows.

- Similarly, if $m \neq n$

$$\begin{aligned}\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi}\cos((n+m)x)\,dx + \frac{1}{2}\int_{-\pi}^{\pi}\cos((n-m)x)\,dx\\ &= \frac{1}{2}\frac{1}{(n+m)}\sin((n+m)x)\Big|_{-\pi}^{\pi} + \frac{1}{2}\frac{1}{(n-m)}\sin((n-m)x)\Big|_{-\pi}^{\pi}\\ &= \frac{1}{2}\frac{1}{(n+m)}(\sin((n+m)\pi) - \sin(-(n+m)\pi))\\ &+ \frac{1}{2}\frac{1}{(n-m)}(\sin(\pi(n-m)\pi) - \sin((m-n)\pi))\\ &= 0+0+0+0\\ &= 0\end{aligned}$$

and if $m = n$

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(nx)\cos(nx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((n+n)x)\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos((n-n)x)\,dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \cos(2nx)\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(0)\,dx \\ &= \frac{1}{2}\left(\frac{1}{2n}\right)\sin(nx)\Big|_{-\pi}^{\pi} + \frac{1}{2}x\Big|_{-\pi}^{\pi} \\ &= 0 + 0 + \frac{1}{2}(\pi - (-\pi)) \\ &= \pi\end{aligned}$$

Formula (2c) now follows

- To prove Formula (2b) we use the sine addition formula

$$\begin{aligned}\sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B) \\ \sin(A-B) &= \sin(A)\cos(B) - \cos(A)\sin(B)\end{aligned}$$

to get

$$\sin(A)\cos(B) = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$$

Now suppose $m \neq n$, then

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} \sin((m+n)x)\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin((m-n)x)\,dx \\ &= \frac{1}{2}\frac{-1}{m+n}\left(\cos((m+n)x)\right)\Big|_{-\pi}^{\pi} + \frac{1}{2}\frac{-1}{m-n}\left(\cos((m-n)x)\right)\Big|_{-\pi}^{\pi} \\ &= -\frac{1}{2}\frac{1}{m+n}\left(\cos((m+n)\pi) - \cos(-(m+n)\pi)\right) \\ &\quad -\frac{1}{2}\frac{1}{m-n}\left(\cos((m-n)\pi) - \cos(-(m-n)\pi)\right) \\ &= 0\end{aligned}$$

because $\cos(x) = \cos(-x)$ for all x.

When $m = n$, we have

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(nx)\cos(nx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} \sin((2n)x)\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(0)\,dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \sin((2n)x)\,dx \\ &= -\frac{1}{2}\frac{1}{2n}\left(\cos(2n\pi) - \cos(-2n\pi)\right) \\ &= 0\end{aligned}$$

and Formual (2b) now follows.

3. Consider the following Heat Equation boundary value problem:

$$\begin{aligned} u_t - k^2 u_{xx} &= 0 \quad , \quad 0 \leq x \leq L \quad , \quad t > 0 && \text{(3a)} \\ u(0,t) &= 0 \quad , \quad t > 0 && \text{(3b)} \\ u(L,t) &= 0 \quad , \quad t > 0 && \text{(3c)} \\ u(x,0) &= \phi(x) \quad , \quad 0 \leq x \leq L && \text{(3d)}\end{aligned}$$

(a) Apply the method of Separation of Variables to find a family of solutions of (3a) the form $u(x,t) = X(x)T(t)$.

(b) Impose the boundary conditions (3b) and (3c) to find a more specialized family of solutions $u_n(x,t) = X_n(x)\,T_n(t)$ satisfying (1a)–(1c).

(c) Set

$$u(x,t) = \sum_n a_n u_n(x,t)$$

where the $u_n(x,t)$ are the solutions found in (b), impose (3d), and then use properties of Fourier expansions to determine the coefficients a_n.

- We set $u(x,t) = X(x)\,T(t)$ and plug into the PDE (3a):

  $$u_t - k^2 u_{xx} = 0 \quad \Longrightarrow \quad X(x)\,T'(t) - k^2 X''(x)\,T(t) = 0$$

  Dividing both sides fo the latter equation by $X(x)\,T(t)$ we get

  $$\frac{T'(t)}{T(t)} - k^2 \frac{X''(x)}{X(x)} = 0$$

  or

  $$\frac{1}{k^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

  The circumstance that the left hand side depends only on t while the right hand side depends only on x implies both sides must equal a constant which we shall write as $-\lambda^2$ (which we can do without loss of generality - at this point $-\lambda^2$ might be real or complex). The equations

  $$\begin{aligned} \frac{1}{k^2}\frac{T'(t)}{T(t)} &= -\lambda^2 \quad \Longrightarrow \quad T'(t) = -(k\lambda)^2\,T(t) \\ \frac{X''(x)}{X(x)} &= -\lambda^2 \quad \Longrightarrow \quad X''(x) = -\lambda^2 X(x) \end{aligned}$$

  have as their general solutions

  $$\begin{aligned} X(x) &= A\cos(\lambda x) + B\sin(\lambda x) \\ T(t) &= Ce^{-k^2\lambda^2 t} \end{aligned}$$

  Putting $X(x)$ and $T(t)$ back together we arrive at

  $$u_{\lambda,A,B}(x,t) = Ae^{-k^2\lambda^2 t}\cos(\lambda x) + Be^{-k^2\lambda^2 t}\sin(\lambda x)$$

  This completes part (a). The (infinite) family of solutions of (3a) obtained by letting the parameters λ, A and B vary over the complex numbers.
- We now impose the boundary conditions (3b) and (3c) on the functions $u_{\lambda,A,B}$. (3b) requires

  $$0 = u_{\lambda,A,B}(0,t) = Ae^{-k^2\lambda^2 t}\cos(0) + Be^{-k^2\lambda^2 t}\sin(0) = Ae^{-k^2\lambda^2 t}$$

  Since this must be true for all $t > 0$, we are forced to take $A = 0$. Setting $A = 0$ and imposing (3c) leads to

  $$0 = u_{\lambda,0,B}(L,t) = Be^{-k^2\lambda^2 t}\sin(\lambda L)$$

  Now we can't set $B = 0$ without trivializing our solution completely, and the factor $e^{-k^2\lambda^2 t}$ is never equal to zero for any finite x. We thus need

  $$\begin{aligned} 0 &= \sin(\lambda L) \quad \Longrightarrow \quad \lambda L = n\pi \qquad \text{for some integer } n \\ &\Longrightarrow \quad \lambda = \frac{n\pi}{L} \end{aligned}$$

  We thus arrive at the following family of solutions to (3a), (3b) and (3c).

  $$u_n(x,t) = b_n e^{-\left(\frac{nk\pi}{L}\right)^2 t}\sin\left(\frac{n\pi}{L}x\right)$$

- Finally, we form a linear combination of the solutions $u_n(x,t)$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{nk\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

and impose the last boundary condition

$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} b_n e^0 \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

To determine the coefficients b_n, we multiply both sides of this last equation by $\frac{2}{L}\sin\left(\frac{m\pi}{L}x\right)$ and integrate over the interval $[0,L]$

$$\begin{aligned} \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{m\pi}{L}x\right)dx &= \frac{2.}{L}\int_0^L \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx \\ &= \sum_{n=1}^{\infty} b_n \left(\frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx\right) \end{aligned}$$

Now

$$\frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \delta_{m,n} \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

and so only one term on the right hand side (the one where $m = n$) will contribute to the total sum. Thus,

$$\frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{m\pi}{L}x\right)dx = \sum_{n=1}^{\infty} b_m \delta_{m,n} = b_m$$

Hence, the solution to the original problem is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{nk\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

with the coefficients b_n determined by

$$b_n = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

4. Find the solution of the following PDE/BVP:

$$\begin{aligned} u_t - u_{xx} &= 0 \quad , \quad 0 \leq x \leq 1 \quad , \quad t > 0 & \text{(4a)} \\ u(0,t) &= 0 \quad , \quad t > 0 & \text{(4b)} \\ u(1,t) &= 0 \quad , \quad t > 0 & \text{(4c)} \\ u(x,0) &= 1 - x^2 \quad , \quad 0 \leq x \leq 1 & \text{(4d)} \end{aligned}$$

- This problem is similar to Problem 3 except that we have given an explicit function of x as a Cauchy boundary condition at $t = 0$.

  We thus set $L = 1$, $\phi(x) = 1 - x^2$ and pick up with the formula at the end of part (c) of Problem 3.

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(nk\pi)^2 t} \sin(n\pi x)$$

with

$$\begin{aligned}
b_n &= 2\int_0^1 \left(1-x^2\right)\sin\left(n\pi x\right)dx \\
&= 2\int_0^1 \sin\left(n\pi x\right)dx - 2\int_0^1 x^2\sin\left(n\pi x\right)dx \\
&= \frac{2}{n^3\pi^3}\left(\left(n^2\pi^2x^2 - n^2\pi^2 - 2\right)\cos\left(n\pi x\right) - 2n\pi x\cos\left(n\pi x\right)\right)\Big|_0^1 \\
&= \frac{2}{n^3\pi^3}\left(n^2\pi^2 + 2 - 2\cos\left(n\pi\right)\right)
\end{aligned}$$

Now when n is even $\cos\left(n\pi\right) = 1$ and so for even n,

$$b_n = \frac{2n^2\pi^2}{n^3\pi^3} = \frac{2}{n\pi}$$

When n is odd $\cos\left(n\pi\right) = -1$ and we have

$$b_n = \frac{2n^2\pi^2 + 4}{n^3\pi^3}$$

And so

$$u\left(x,t\right) = \sum_{n,even}\left(\frac{2}{n\pi}\right)e^{-(nk\pi)^2t}\sin\left(n\pi x\right) + \sum_{n,odd}\left(\frac{2n^2\pi^2+4}{n^3\pi^3}\right)e^{-(nk\pi)^2t}\sin\left(n\pi x\right)$$