1. Solve the following PDE/BVP

\[ 2u_t + 3u_x = 0 \]  \hspace{1cm} (1a)
\[ u(x,0) = \sin(x) \]  \hspace{1cm} (1b)

- As in the example in §3 of Lecture 1, we can view the left hand side of the PDE as saying the directional derivative of \( u(x,t) \) along the direction \([3, 2]\) is equal to zero.

\[ 0 = 3u_x + 2u_x = [3, 2] \cdot \left[ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t} \right] = [3, 2] \cdot \nabla u = D_{[3,2]}u \]

This means that along lines of the form

\[ \gamma(s) = [x_0, t_0] + s [3, 2] = [x_0 + 3s, t_0 + 2s] \]  \hspace{1cm} (2)

the solution must be constant. Consider

\[ \gamma_{x_0}(s) = [x_0 + 3s, 2x] \]

This will be a line in the direction of \([2, 3]\) that passes through the point \((x_0, 0)\) at \(s = 0\) (in other words, I’m specializing (2) to the case when \(t_0 = 0\)). Along this line the solution of (1a) and (1b) is constant. Thus,

\[ u\left(\gamma_{x_0}(t)\right) = C_{x_0} \]

where \(C_{x_0}\) is some constant depending on the choice the parameter \(x_0\) that characterizes which line where looking at. At the point \((x_0, 0)\), the boundary condition (1b) requires

\[ \sin(x_0) = u(x_0, 0) = u\left(\gamma_{x_0}(0)\right) = C_{x_0} \]

Thus, we have

\[ u\left(\gamma_{x_0}(0)\right) = \sin(x_0) \]

This gives us the solution of (1) at any point along the line \(\gamma_{x_0}(s)\). To get the solution at an arbitrary point \((x, t)\) we just need to figure out which line \(\gamma_{x_0}(x, y)\) lives on. So we set

\[ [x, t] = \gamma_{x_0}(s) = [x_0 + 3s, 2s] \implies \begin{cases} x = x_0 + 3s \\ t = 2s \end{cases} \implies \begin{cases} s = \frac{t}{2} \\ x_0 = x - \frac{3}{2}t \end{cases} \]

So

\[ u(x, t) = u\left(\gamma_{x_0}(t)\right) = \sin(x_0) = \sin\left(x - \frac{3}{2}t\right) \]

We conclude

\[ u(x, t) = \sin\left(x - \frac{3}{2}t\right) \]

is our solution.

2. Solve the following PDE/BVP

\[ u_x + e^x u_y = 0 \]  \hspace{1cm} (1a)
\[ u(0, y) = y^2 \]  \hspace{1cm} (1b)
• Quick solution: (following the method described on page 8 of the text)

\[ \frac{dy}{dx} = e^x \]

\[ \int dy = \int e^x \, dx \Rightarrow y = e^x + C \]

\[ \frac{d}{dx} u(x, e^x + C) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} e^x = u_x + u_y e^x = 0 \]

\[ > u(x, e^x + C) = u(0, 1 + C) = u(0, 1 + y - e^x) \]

\[ u(x, y) = f(1 + y - e^x) \]

\[ x = 0 \Rightarrow y^2 = u(0, y) = f(y) \Rightarrow u(x, y) = (1 + y - e^x)^2 \]

•Verbose solution: We again try to understand the (first order linear) PDE as saying something about the behaviour of solutions along a particular family of curves. In this problem, however, the appropriate curve is not a straight line, but the graph of a function of \( x \). Set

\[ \gamma_1(t) = [x(t), y(t)] = [t, e^t + C_1] \]

Then we’ll have

\[ \frac{dx}{dt} = 1 \quad , \quad \frac{dy}{dt} = e^t \]

and so along the curve \( \gamma_1(t), \) we’ll have

\[ \frac{d}{dt} u(\gamma_1(t)) = \frac{d}{dt} u(x(t), y(t)) = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} = u_x + u_y e^t = u_x + e^x u_y = 0 \]

the last equality because \( u(x, y) \) satisfies (1a)). We thus conclude that, along the curves \( \gamma_1(t), \) the solutions of (1a) are constant: thus

\[ u(\gamma_1(t)) = C_{2,C_1} \]

where \( C_{2,C_1} \) is a constant which might depend on which curve \( \gamma_1(t) \) we are looking at. We can figure out the constants \( C_{2,C_1} \) by imposing (1b). We have

\[ C_{2,C_1} = u(\gamma_1(0)) = u(0, e^0 + C_1) = u(0, 1 + C_1) = (1 + C_1)^2 \]

So inserting (5) in (4) we have

\[ u(\gamma_1(t)) = (1 + C_1)^2 \]

giving us the value of the solution of (1) along any curve (3).

- It remains to find the value of the solution at an arbitrary point \( [x, y] \). For this, we thus need only figure out upon which curve \( \gamma_1(t) \) the point \( [x, y] \) sits.

\[ [x, y] = [t, e^t + C_1] \quad \Rightarrow \quad \left\{ \begin{array}{l} x = t \\ y = e^t + C_1 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} t = x \\ C_1 = y - e^x \end{array} \right. \]

Thus,

\[ u(x, y) = u(\gamma_1(t)) \bigg|_{C_1=y-e^x}^{t=x} = (1 + C_1)^2 \bigg|_{C_1=y-e^x}^{t=x} = (1 + y - e^x)^2 \]

We conclude that

\[ u(x, y) = (1 + y - e^x)^2 \]

is the solution of (1).

3. (a) Find the curves \( \gamma: t \rightarrow (x(t), y(t)) \) such that

\[ \frac{dx}{dt} = x \quad , \quad \frac{dy}{dt} = y \]

that cross the line \( y = 1 \) at \( t = 0 \).
Both the ODE for \( x(t) \) and the ODE for \( y(t) \) are the differential equations for the natural exponential function. Their general solutions will be
\[
\begin{align*}
x(t) &= C_1 e^t \\
y(t) &= C_2 e^t
\end{align*}
\]
We want to find solutions that cross the line \( y = 1 \) at \( t = 0 \). Let \( [x_0, 1] \) be an arbitrary point on this line. We set
\[
[x_0, 1] = [C_1 e^0, C_2 e^0] \quad \Rightarrow \quad \begin{cases} C_1 = x_0 \\ C_2 = 1 \end{cases}
\]
Thus, the curves
\[
\gamma_{x_0}(t) = [x_0 e^t, e^t]
\]
will obey
\[
\frac{dx}{dt} = x, \quad \frac{dy}{dt} = y
\]
and cross the line \( y = 1 \).

(b) Solve the following PDE/BVP
\[
\begin{align*}
x \phi_x + y \phi_y &= y \\
\phi(x, 1) &= x + 1
\end{align*}
\]
by first finding solutions of the PDE/BVP along the curves \( \gamma \) determined in Part (a) and then extending these solutions to arbitrary points in

- Suppose \( \phi(x, y) \) is a solution of equations (4). Then if we define
\[
\psi_{x_0}(t) = \phi(\gamma_{x_0}(t))
\]
where \( \gamma_{x_0}(t) \) is one of the solutions (3) of (1). By the chain rule
\[
\frac{d\psi_{x_0}}{dt} = \frac{\partial \phi}{\partial x} \frac{dx}{dt} + \frac{\partial \phi}{\partial y} \frac{dy}{dt} = x \phi_x + y \phi_y
\]
and then by virtue of the PDE (4a) we have
\[
\frac{d\psi_{x_0}}{dt} = x \phi_x + y \phi_y = e^t. \tag{6}
\]
The general solution of (6) will be
\[
\psi_{x_0}(t) = e^t + C_{x_0} \tag{7}
\]
- To determine the constant \( C_0 \) we impose the boundary condition on \( \phi \) along the line \( y = 1 \).
\[
1 + C_{x_0} = \psi_{x_0}(0) = \phi(\gamma_{x_0}(0)) = \phi(x_0, 1) = x_0 + 1
\]
or
\[
C_{x_0} = x_0 \tag{8}
\]
Thus, along one of the curves \( \gamma_{x_0}(t) \) we must have
\[
\phi(\gamma_{x_0}(t)) = \psi_{x_0}(t) = e^t + x_0
\]
Now suppose the point \([x, y]\) lives on the curver \( \gamma_{x_0} \). We have
\[
[x, y] = [x_0 e^t, e^t] \quad \Rightarrow \quad \begin{cases} x = x_0 e^t \\ y = e^t \end{cases} \quad \Rightarrow \quad \begin{cases} t = \ln |y| \\ x_0 := x/y \end{cases}
\]
and so
\[
\phi(x, y) = \phi(\gamma_{x_0}(t)) \big|_{t=\ln|y|} = (e^t + x_0) \big|_{t=\ln|y|} = e^{\ln|y|} + y.x
\]
We conclude that the solution of (3.4) is
\[
\phi(x, y) = y + \frac{x}{y}
\]
4.

(a) Find the curves $\gamma : t \rightarrow [x(t), y(t)]$ such that
\[
\frac{dx}{dt} = y \\
\frac{dy}{dt} = 2y
\]

- The second equation is just a simple first order linear ODEs with constant coefficients whose solutions are exponential functions:
\[
\frac{dy}{dt} = 2y \implies y(t) = y_0 e^{2t}
\]
Plugging this into the right hand side of the first equation yields
\[
\frac{dx}{dt} = y_0 e^{2t}
\]
which can be solved by integrating both sides with respect to $t$:
\[
x(t) = \frac{1}{2} y_0 e^{2t} + C
\]
The solution curves are thus
\[
\gamma_{y_0,C}(t) = \left[\frac{1}{2} y_0 e^{2t} + C, y_0 e^{2t}\right]
\]
(Varying the constants $y_0$ and $C$ we get all the solutions.)

(b) Solve the following PDE/BVP:
\[
y \frac{\partial \phi}{\partial x} + 2y \frac{\partial \phi}{\partial y} = xy
\]
\[
\phi(x, 1) = x + 2
\]

- We first choose the solution curves found in (a) so that they cross the line $\Sigma = \{[s, 1] \mid s \in \mathbb{R}\}$ when $t = 0$. This requires
\[
s = \left(\gamma_{y_0,C}(0)\right)_x = \left(\frac{1}{2} y_0 e^{2t} + C\right)|_{t=0} = \frac{1}{2} y_0 + C
\]
\[
1 = \left(\gamma_{y_0,C}(0)\right)_y = y_0 e^{2t}|_{t=0} = y_0
\]
So we must take
\[
y_0 = 1
\]
and
\[
C = s - \frac{1}{2} y_0 = s - \frac{1}{2}.
\]
The characteristic curve that passes through the point $[s, 1]$ at $t = 0$ is thus
\[
\gamma_s(t) = \left[\frac{1}{2} e^{2t} + s - \frac{1}{2}, e^{2t}\right]
\]
Next we suppose $\phi(x, y)$ is a solution of the PDE/BVP and set
\[
\Phi_{\gamma_s}(t) = \phi(\gamma_s(t)) = \phi(x(t), y(t))
\]
We then have
\[
\frac{\partial \Phi_{\gamma_s}}{dt} = \frac{\partial \Phi}{\partial x} \frac{dx}{dt} + \frac{\partial \Phi}{\partial y} \frac{dy}{dt}
\] by the Chain Rule

\[
y(t) \frac{\partial \Phi}{\partial x} + 2y(t) \frac{\partial \Phi}{\partial y}
\] by the differential equations for \(\gamma_s(t)\)

\[
x(t) y(t)
\] by the PDE

\[
\left( \frac{1}{2} e^{2t} s - \frac{1}{2} \right) (e^{2t})
\] by the solution to the differential equations for \(\gamma_s(t)\)

\[
\frac{1}{2} e^{4t} + \left( s - \frac{1}{2} \right) e^{2t}
\]

Integrating both sides of this last equation yields
\[
\Phi_{\gamma_s}(t) = \frac{1}{8} e^{4t} + \frac{1}{2} \left( s - \frac{1}{2} \right) e^{2t} + c
\]

To fix the constant \(c\) we look at what happens at \(t = 0\). On the one hand, we have
\[
\Phi_{\gamma_s}(0) = \Phi(\gamma_s(0)) = \phi(s, 1) = s + 2
\]

On the other, we have
\[
\Phi_{\gamma_s}(0) = \frac{1}{8} e^{4\cdot0} + \frac{1}{2} \left( s - \frac{1}{2} \right) e^{2\cdot0} + c = \frac{1}{8} + \frac{1}{2} \left( s - \frac{1}{2} \right) + c
\]

Solving
\[
s + 2 = \frac{1}{8} + \frac{1}{2} \left( s - \frac{1}{2} \right) + c \quad \Rightarrow \quad c = s + \frac{17}{8}
\]

So now we know that along the curve \(\gamma_s(t)\) the solution \(\Phi\) of the PDE/BVP is given by
\[
\phi(\gamma_s(t)) = \Phi_{\gamma_s}(t) = \frac{1}{8} e^{4t} + \frac{1}{2} \left( s - \frac{1}{2} \right) e^{2t} + s + \frac{17}{8}
\]

However, we want to know the values of \(\phi\) at points \((x, y)\). We thus set
\[
x = x(t) = \frac{1}{2} e^{2t} + s - \frac{1}{2}
\]
\[
y = e^{2t}
\]

Solving these equations for \(s\) and \(t\) yields
\[
t = \frac{1}{2} \ln(y)
\]
\[
s = x - \frac{y}{2} + \frac{1}{2}
\]

Thus, finally,
\[
\phi(x, y) = \Phi_{\gamma_s}(t)
\]
\[
= \frac{1}{8} e^{4\left( \frac{1}{2} \ln(y) \right)} + \frac{1}{2} \left( x - \frac{y}{2} + \frac{1}{2} - \frac{1}{2} \right) e^{2\left( \frac{1}{2} \ln(y) \right)} + \frac{1}{2} \left( x - \frac{y}{2} + \frac{1}{2} \right) + \frac{17}{8}
\]
\[
= \frac{1}{8} y^2 + \frac{1}{4} \left( 2x - y \right) y + \frac{1}{4} \left( 2x - y + 1 \right) + \frac{17}{8}
\]
\[
= \frac{1}{2} x - \frac{1}{4} y + \frac{1}{2} xy - \frac{1}{8} y^2 + \frac{19}{8}
\]

5. Use the method of characteristics to show that the solution of
\[
u u_x + u_y = 0 \quad , \quad u(x, 0) = f(x)
\]
is given implicitly by
\[
u = f(x - uy)
\]
and verify this result by direct differentiation. In what region is this solution valid?
• The differential equations satisfied by the characteristic curves \((x(t), y(t), u(t))\) are

\[
\begin{align*}
\frac{dx}{dt} &= u(t) \quad \text{(1a)} \\
\frac{dy}{dt} &= 1 \quad \text{(1b)} \\
\frac{du}{dt} &= 0 \quad \text{(1c)}
\end{align*}
\]

This system of ODEs is easily integrated to produce

\[
\begin{align*}
u(t) &= C_1 \quad \text{(2a)} \\
y(t) &= t + C_2 \quad \text{(2b)} \\
x(t) &= C_1 t + C_3 \quad \text{(2c)}
\end{align*}
\]

Let \(t = 0\) be the value of the parameter \(t\) when characteristic curves pass through the plane \(y = 0\) and let \((x_o, 0, u_o)\) be the point where the characteristic through \((x, y, u)\) passes through this plane. We then have

\[
\begin{align*}
C_1 &= u_o \\
C_2 &= 0 \\
C_3 &= x_o
\end{align*}
\]

Our initial conditions \(u(x, 0) = f(x)\), implies that

\[
C_1 = u_o = f(x_o)
\]

We can thus rewrite (2) as

\[
\begin{align*}
x &= f(x_o) t + x_o \quad \text{(3a)} \\
y &= t \quad \text{(3b)} \\
u &= f(x_o) \quad \text{(3c)}
\end{align*}
\]

We can use the last two equations to replace \(f(x_o)\) by \(u\), \(x_o\) by \(f^{-1}(u)\), and \(t\) by \(y\) in the first equation. This leads us to

\[
x = uy + f^{-1}(u) \quad .
\]

or

\[
(4)
u = f(x - uy) \quad .
\]

It is apparent from (4) that

\[
u(x, 0) = f(x + 0) = f(x) \quad .
\]

On the other hand, writing \(w(x, y) = x - uy\) so that the solution (4) can be written

\[
u(x, y) = f(w(x, y))
\]

we have

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \left. \frac{d u}{d w} \right|_{w=x-uy} \frac{\partial w}{\partial x} \bigg|_{(x,y)} = f'(x - uy) \cdot 1 \\
\frac{\partial u}{\partial y} &= \left. \frac{d u}{d w} \right|_{w=x-uy} \frac{\partial u}{\partial y} \bigg|_{(x,y)} = f'(x - uy) \cdot (-u)
\end{align*}
\]

and so

\[
u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = uf'(x - uy) - uf'(x - uy) = 0
\]

It is thus clear that (4) is the solution of (1a) and (1b).

As for the regions where this result would be valid. Well, in general, a couple of problems can occur in several places:

• It may happen that the system of ODEs equations for the characteristic curves don’t satisfy the hypothesis of the existence and uniqueness theorem for systems of ordinary differential equations.
• It may happen that two distinct characteristic curves (satisfying different initial conditions) pass over the same point \((x, y)\). In such a case, we could end up with contradictory results depending on which of the characteristic curves we followed back to the line where the initial data is specified.

The first problem is avoided in the case at hand, as we have no problem in applying the Existence and Uniqueness theorem to the system \((1a) - (1c)\). On the other hand suppose the characteristic curves
\[
\gamma_{x_0}(t) = [f(x_0)t + x_0, t, f(x_0)] \\
\gamma_{x'_0}(t) = [f(x'_0)t + x'_0, t, f(x'_0)]
\]
passed over the same point \((x, y)\) in the \(xy\)-plane:
\[
f(x_0)t + x_0 = x = f(x'_0)t' + x'_0 \\
t = y = t'
\]
then the second problem might occur. Eliminating \(t\) and \(t'\) in favor of \(y\), we see that we have to avoid situations where
\[
f(x_0) \neq f(x'_0) \quad \text{(leading to different values of } u(x, y)\text{)}
\]
while
\[
f(x_0)y + x_0 = f(x'_0)y + x'_0 \quad \text{(with characteristic curves crossing above } (x, y)\text{)}
\]
\[
\square
\]

6. Use the Method of Characteristics to solve
\[
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \phi = e^{x+2y} \\
\phi(x,0) = x
\]
• We rewrite the PDE as
\[
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} = e^{x+2y} - \phi
\]
The ODEs for the characteristics are
\[
\begin{align*}
\frac{dx}{dt} &= 1 \quad \Rightarrow \quad x(t) = t + x_0 \\
\frac{dy}{dt} &= 1 \quad \Rightarrow \quad y(t) = t + y_0 \\
\frac{du}{dt} &= e^{x(t)+2y(t)} - u
\end{align*}
\]
The last ODE, once we plug the solutions of the first two ODEs, is a linear first order ODE for \(u(t)\)
\[
\frac{du}{dt} + u = e^{t+x_0+2(t+y_0)} = e^{x_0+2y_0}e^{3t}
\]
Recall that differential equations of the form
\[
\frac{du}{dt} + p(t)u = g(t)
\]
have as general solutions
\[
u(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\,dx + \frac{C}{\mu(t)}
\]
where
\[
\mu(t) = \exp \left( \int p(t)\,dt \right)
\]
In the case at hand we have
\[
p(t) = 1 \quad \Rightarrow \quad \mu(t) = \exp \left( \int dt \right) = e^t
\]
Since (in the case at hand) 

\[ g(t) = e^{x_0 + 2y_0} e^{3t} \]

we thus have 

\[ u(t) = \frac{1}{e^t} \int e^t (e^{x_0 + 2y_0} e^{3t}) \, dt + \frac{C}{e^t} \]

\[ = e^{-t} \left( e^{x_0 + 2y_0} \left( \frac{1}{4} e^{4t} \right) \right) + Ce^{-t} \]

\[ = \frac{1}{4} e^{x_0 + 2y_0} e^{3t} + Ce^{-t} \]

Thus, a general characteristic for this PDE will be a curve of the form 

\[ \gamma_{x_0,y_0,C}(t) = [x_0 + t, y_0 + t, \frac{1}{4} e^{x_0 + 2y_0} e^{3t} + Ce^{-t}] \]

- We now force such a characteristic \[ \gamma_{x_0,y_0,C}(t) = [x(t), y(t), u(t)] \] to pass through a point \([s, 0, s]\) on the boundary value curve at \(t = 0\).

\[ [s, 0, s] = \gamma_{x_0,y_0,C}(0) = [x_0, y_0, \frac{1}{4} e^{x_0 + 2y_0} + C] \]

This tells us that we need to take 

\[ x_0 = s \]
\[ y_0 = 0 \]
\[ C = s - \frac{1}{4} e^s \]

Now the question is: what are the appropriate values for \(s\) and \(t\) so that such a characteristic curves lies above the point \([x, y]\) in the coordinate plane.

- We now have 

\[ \gamma_{x_0,y_0,C}(t) = \gamma_{s,0,s-\frac{1}{4}e^s}(t) = [s + t, t, \frac{1}{4} e^s e^{3t} + \left( s - \frac{1}{4} e^s \right) e^{-t}] = [x(t), y(t), \phi(x(t), y(t))] \]

Now setting 

\[ x = s + t \]
\[ y = t \]

we can replace \(s\) and \(t\) by 

\[ t = y \]
\[ s = x - y \]

Thus, 

\[ \phi(x, y) = \frac{1}{4} e^s e^{3t} + \left( s - \frac{1}{4} e^s \right) e^{-t} \]

\[ = \frac{1}{4} e^{x-y} e^{3y} + \left( x - y - \frac{1}{4} e^{x-y} \right) e^{-y} \]