Math 4263
SOLUTIONS TO FIRST EXAM
July 2, 2013

1. Find the characteristic curves for the following first order PDE

$$xu_x + uu_y = 1$$

and then use the method of characteristics to find the solution of this PDE in the region $x > 1$ satisfying the following boundary conditions

$$u(1, y) = y \quad .$$

- The characteristic curves will be solution of the following system of ODEs:

$$\begin{aligned} \frac{dx}{dt} &= x \\ \frac{dy}{dt} &= u \\ \frac{du}{dt} &= 1 \end{aligned}$$

This system is easily solved:

$$\begin{aligned} \frac{dx}{dt} &= x \quad \Rightarrow \quad x(t) = x_o e^t \\ \frac{du}{dt} &= 1 \quad \Rightarrow \quad u(t) = t + u_o \\ \frac{dy}{dt} &= u \quad \Rightarrow \quad y(t) = \int (t + u_o)\, dt + y_o = \frac{1}{2} t^2 + u_o t + y_o \end{aligned}$$

Next we find the characteristic curves that pass through a point on the curve Σ where the boundary conditions are defined, at $t = 0$:

$$u(1, y) = y \quad \Rightarrow \quad \Sigma = \{[x, y, u] = [1, s, s] \text{ for some } s \in \mathbb{R}\}$$

$$\begin{aligned} 1 &= x(0) = x_o e^o = x_o \quad \Rightarrow \quad x_o = 1 \\ s &= y(0) = \frac{1}{2}(0)^2 + u_o(0) + y_o \quad \Rightarrow \quad y_o = s \\ s &= u(0) = 0 + u_o \quad \Rightarrow \quad u_o = s \end{aligned}$$

Thus, the characteristic curves we want are the curves

$$\boldsymbol{\gamma}_s(t) = \left[e^t, \frac{1}{2} t^2 + s(t+1), t + s\right]$$

We now set

$$[x, y, u(x, y)] = \boldsymbol{\gamma}_s(t) = \left[e^t, \frac{1}{2} t^2 + s(t+1), t + s\right]$$

From this we see

$$\begin{aligned} t &= \ln|x| \\ s &= \frac{y - \frac{1}{2} t^2}{t + 1} = \frac{y - \frac{1}{2} \ln|x|^2}{1 + \ln|x|} \end{aligned}$$

and so

$$u(x, y) = t + s = \ln|x| + \frac{y - \frac{1}{2} \ln|x|^2}{1 + \ln|x|}$$

is the solution.

2. State the Maximum Principle for solutions of the homogeneous heat equation and then use it to prove the uniqueness of any solution of

$$
\begin{aligned}
u_t - k^2 u_{xx} &= f(x,t) \qquad 0 \leq x \leq L \quad , \quad t > 0 \qquad (*) \\
u(0,t) &= \phi(t) \qquad t > 0 \\
u(L,t) &= \psi(t) \qquad t > 0 \\
u(x,0) &= g(t) \quad 0 \leq x \leq L
\end{aligned}
$$

- The Maximal Principle says that the maximal value of any solution of the homogeneous heat equation $u_t - k^2 u_{xx} = 0$ on a rectangular domain

$$R = [X_1, X_2] \times [T_1, T_2]$$

occurs on one of the three sides

$$\Gamma_{left} = \{[X_1, t] \mid t \in [T_1, T_2]\} \quad , \quad \Gamma_{bottom} = \{[x, T] \mid x \in [X_1, X_2]\} \quad , \quad \Gamma_{right} = \{[X_2, t] \mid t \in [T_1, T_2]\}$$

Step 1. I claim the only one solution of

$$(**) \qquad u_t - k^2 u_{xx} = 0 \quad , \quad u(0,t) = 0 \quad , \quad u(x,0) = 0 \quad , \quad u(L,t) = 0$$

is $u(x,t) = 0$ for all x, t. Indeed, the Maximal Principle and the boundary conditions on the solution require the maximal value of $u(x,t)$ is 0. But if $u(x,t)$ is a solution of (**) so is $-u(x,t)$, and it too must have a maximal value of 0 (applying again the Maximal Principle and the boundary conditions). Thus,

$$u(x,t) \leq 0 \text{ for all } x, t \qquad \text{and} \qquad -u(x,t) \leq 0 \text{ for all } x, t$$

which in turn implies

$$u(x,t) = 0 \quad \text{for all } x, t$$

Step 2. Suppose $u_1(x,t)$ and $u_2(x,t)$ are two solutions of (*). Then it is easy to see that $\Delta u(x,t) \equiv u_1(x,t) - u_2(x,t)$ is a solution of (**). Hence by Step 1,

$$0 = \Delta u(x,t) = u_1(x,t) - u_2(x,t) \qquad \text{for all } x, t$$

or, equivalently,

$$u_1(x,t) = u_2(x,t) \qquad \text{for all } x, t$$

Thus, $u_1(x,t)$ and $u_2(x,t)$ are identical.

3. The temperature ϕ inside a thin wire of length L is governed by the formula

$$\phi_t - a^2\phi_{xx} = 0$$

with boundary conditions

$$\begin{aligned} \phi(x,0) &= 3\sin\left(\frac{4\pi}{L}x\right) \quad , \quad 0 < x < L \\ \phi(0,t) &= 0 \\ \phi(L,t) &= 0 \end{aligned}$$

Use Separation of Variables and the theory of Fourier series to determine the solution to this PDE/BVP.

- We look for solutions of the form $\phi(x,t) = X(x)T(t)$. Plugging $X(x)T(t)$ into the heat equation we obtain

$$X(x)T'(t) = a^2X''(x)T(t) \quad \Rightarrow \quad \frac{1}{a^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

Consistency of this last equation requires both the left and right hand sides must be independent of x and t. Thus, denoting this constant by $-\lambda^2$, we have

$$\frac{1}{a^2}\frac{T'(t)}{T(t)} = -\lambda^2 = \frac{X''(x)}{X(x)}$$

or

$$\begin{aligned} T' + a^2\lambda^2 T &= 0 \quad \Rightarrow \quad T(t) = T_o e^{-a^2\lambda^2 t} \\ X'' + \lambda^2 X &= 0 \quad \Rightarrow \quad X(t) = A\cos(\lambda x) + B\sin(\lambda x) \end{aligned}$$

Thus, for each $\lambda \in \mathbb{C}$, we will have two independent solutions of the heat equation

$$\begin{aligned} \phi_{\lambda,1}(x,t) &= e^{-a^2\lambda^2 t}\cos(\lambda x) \\ \phi_{\lambda,2}(x,t) &= e^{-a^2\lambda^2 t}\sin(\lambda x) \end{aligned}$$

We now toss out the solutions $\phi_{\lambda,1}$ since they do not satisfy the boundary condition $\phi(0,t) = 0$. Imposing next the second boundary condition $\phi(L,t) = 0$ on $\phi_{\lambda,2}$ we requires

$$0 = e^{-a^2\lambda^2 t}\sin(\lambda L) \qquad \text{for all } t$$

To satisfy this last constraint we must take $\lambda = \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$. We are now left with solutions

$$\phi_n(x,t) = e^{-\left(\frac{n\pi}{L}k\right)^2 t}\sin\left(\frac{n\pi}{L}x\right) \qquad , \qquad n = 1, 2, 3, \ldots$$

We still have lots of solutions though, Before imposing the last boundary conditions we form a general linear combination of the $\phi_n(x,t)$ (which will still satisfy the heat equation and the last two boundary conditions)

$$\phi(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{L}k\right)^2 t}\sin\left(\frac{n\pi}{L}x\right)$$

Imposing the initial condition, we have

$$3\sin\left(\frac{4\pi}{L}x\right) = \phi(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

Multiplying both sides by $\frac{2}{L}\sin\left(\frac{m\pi x}{L}\right)$ and integrating from 0 to L yields on the right

$$\sum_{n=1}^{\infty} B_n\left(\frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi x}{L}\right)dx\right) = \sum_{n=1}^{\infty} B_n\delta_{n,m} = B_m$$

and, on the left,

$$3\left(\frac{2}{L}\int_0^L \sin\left(\frac{4\pi}{L}x\right)\sin\left(\frac{m\pi x}{L}\right)dx\right) = \begin{cases} 3 & \text{if } m = 4 \\ 0 & \text{if } m \neq 4 \end{cases}$$

So

$$B_m = \begin{cases} 3 & \text{if } m = 4 \\ 0 & \text{if } m \neq 4 \end{cases}$$

We conclude

$$\phi\left(x,t\right) = 3e^{-\left(\frac{4\pi}{L}k\right)^2 t} \sin\left(\frac{4\pi}{L}x\right)$$

4. Find the eigenvalues and eigenfunctions of the following Sturm-Liouville problem

$$\begin{aligned} y'' + k^2 y &= 0 \\ y'(0) &= 0 \\ y(1) + y'(1) &= 0 \end{aligned}$$

and then show how an arbitrary function on the interval $[0,1]$ can be expanded in terms of these eigenfunctions. Write down an explicit formula for the coefficients in this expansion.

- The differential equation is a second order linear homogeneous ODE with constant coefficients. Its general solution is

(*)
$$y(x) = A\cos(kx) + B\sin(kx)$$

In order to satisfy the first boundary condition we require

$$0 = y'(0) = -kA\sin(0) + kB\cos(0) = kB \quad \Rightarrow \quad B = 0$$

(note that the second solution, $kB = 0 \quad \Rightarrow \quad k = 0$, has exactly the same effect on (*)). Substituting $B = 0$ into (*) and imposing the second boundar condition we find

$$0 = A\cos(k) - kA\sin(k)$$

We cannot set $A = 0$ without trivializing the solution (*), and so instead we must choose k so that

$$0 = \cos(k) - k\sin(k) \quad \Rightarrow \quad k = \cot(k)$$

That is to say, we must choose k to be one of the roots of the transcendal equation

(**)
$$k = \cot(k) \quad .$$

Since $\cos(-kx) = \cos(kx)$, it suffices to consider only the positive roots of (**). Let us order these as

$$0 < k_1 < k_2 < k_3 < \cdots$$

Then functions $\cos(k_n x)$ are the (unnormalized) solutions to the Sturm-Liouville problem.

- Note that for this Sturm-Liouville problem $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$. So the Sturm-Liouville inner product is $(f,g) = \int_0^1 f(x)\, g(x)\,(1)\, dx$. If we set

$$\phi_n(x) = \frac{\cos(k_n(x))}{\left[\int_0^1 \cos^2(k_n x)\right]^{\frac{1}{2}}}$$

then the functions $\phi_n(x)$ will be the normalized solutions to the Sturm-Liouville problem. These functions will satisfy

$$\int_0^1 \phi_n(x)\,\phi_m(x)\,dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

- Sturm-Liouville theory now tells us that any continuous function $f(x)$ on the interval $[0,1]$ can be expanded in terms of the ϕ_n

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) = \sum_{n=1}^{\infty} a_n \cos(k_n x)$$

with the coefficients determined by the formula

$$a_n = \int_0^1 f(x)\,\phi_n(x)\,dx$$