LECTURE 23

Fourier Transforms and Other Integral Transforms

1. Integral Transforms

Several times in this course we encountered formulas of the sort

$$g(s) = \int f(t)\, K(s,t)\, dt \tag{1}$$

For example, in the last lecture we utilized the the Laplace transform of a function is

$$L[f](x) := \int_0^{\infty} f(t)\, e^{-st} dt$$

and even earlier in our discussion of Fourier series we utilized Fourier transforms via

$$F(n) = \frac{2}{\pi} \int_0^{\pi} f(t) \cos(nt)\, dt$$

We refer to a formula of the general form (1) as an *integral transform.* The function $g(s)$ is called the (integral) transform of $f(t)$ by the *kernel* $K(s,t)$.

A particularly good way to think about such transforms is that they are analogous to a linear transformaion between two vector spaces: Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation between the space of n-dimensional vectors and the space of m-dimensional vectors. Then the transformation is completely determined by its action on the standard basis (or any basis) of $\mathbb{R}^n$. Suppose

$$T(\mathbf{e}_i) = \sum_{j=1}^{m} A_{ji} \mathbf{f}_j \tag{2}$$

Thinking of the "components" of a function f as being the set of its values $f(t)$ at points t, and thinking of replacing the discrete sum over components in (2) by a continuous integral over the components of f, the analogy should now be clear. It is worth remarking that the similarity between formulas (1) and (2) is mitigated by much more that simply reinterpreting a sum as an integral. The real point is that (1), like (2) is a linear transformation between two vector spaces.

Also, just as in linear algebra where one uses linear transformations to transport a problem about an abstract vector space (e.g. a set of polynomial functions) to a concrete setting like $\mathbb{R}^n$ in order to carry out computations, one can use integral transforms to transport problems, in particular differential equations, to a setting where they're much easier to solve. The general scheme is something like this

$$\begin{tikzcd} \text{original problem} \arrow[r, "\text{integral transform}"] & \text{problem in transform space} \arrow[d, "\text{relatively easy solution}"] \\ \text{solution to original problem} & \text{solution in transform space} \arrow[l, "\text{inverse integral transform}"'] \end{tikzcd}$$

By far the most useful integral transform is the Fourier transform of a function

$$F(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{i\nu t} dt$$

In many applications this is generally thought of as a way of identifying the frequency components of a time-varying function; but the utility of this transform really extends far beyond such applications. Two other common integral transforms are

(Fourier-Bessel) $$g(\alpha) = \int_0^\infty f(t)\, J_n(\alpha t)\, dt$$

(Mellin transform) $$g(\alpha) = \int_0^\infty f(t)\, t^{\alpha-1} dt$$

2. The Fourier Transform

The big advantage of the Laplace transform is that it can be used to reduce a differential equation to an algebraic equation. However, in practice the Laplace transform also comes with a disadvantage, the inverse Laplace transform taking the solution in the transform space back to the original space is often difficult to compute.

Recall that when we were utilizing Fourier series, we were using formulas

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt \end{aligned}$$

to expand functions on the integral $[-L, L]$ in terms of functions. Let's consider what happens with these formulas as $L \to \infty$. We have

$$\begin{aligned} f(x) &= \frac{1}{2L}\int_{-L}^{L} f(t)\, dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dt \\ &\quad + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dt \\ &= \frac{1}{2L}\int_{-L}^{L} f(t)\, dt + \sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}(t-x)\right) dt \end{aligned}$$

Now if $f(t)$ is an integrable function on the real line; i.e.,

$$\left|\int_{-L}^{L} f(t)\, dt\right| < \infty$$

Then

$$\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} f(t)\, dt = 0$$

On the other hand, if we set

$$\begin{aligned} \omega &= \frac{n\pi}{L} \\ \Delta w &= \frac{\pi}{L} \end{aligned}$$

then

$$\sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}(t-x)\right) dt = \sum_{n=1}^{\infty}\frac{\Delta w}{\pi}\int_{-L}^{L} f(t)\cos(w(t-x))\, dt$$

and so when we let $L \to \infty$, $\Delta w \to 0$ and the expression on the right, interpreted as a Riemann sum over Δw, becomes

$$\frac{1}{\pi}\int_0^{\infty} d\omega \int_{-\infty}^{\infty} f(t)\cos(\omega(t-x))\,dx$$

We can thus infer that, in the limit $L \to \infty$, the Fourier expansion formula (1) becmes

$$f(x) = \frac{1}{\pi}\int_0^{\infty} d\omega \int_{-\infty}^{\infty} f(t)\cos(\omega(t-x))\,dx$$

Now since $f(t)\cos(\omega(t-x))$ is a even function of ω we can rewrite (2) as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(t)\cos(\omega(t-x))\,dx$$

and because $f(t)\sin(\omega(t-x))$ is an odd function of ω, we also have

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(t)\sin(\omega(t-x))\,dx$$

If we now use

$$e^{iy} = \cos(y) + i\sin(y)$$

we can write

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(t)\,e^{i(t-x)\omega}dx$$

because the integral over the odd part of $e^{i(t-x)\omega}$ will not contribute to the total integral.

Reversing the order of integration we obtain the formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left(\int_{-\infty}^{\infty} e^{i(t-x)\omega}d\omega\right)dt$$

which suggests that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(t-x)\omega}d\omega = \delta(t-x)$$

On the other hand, suppose we define the Fourier Transform of f to be

$$\mathcal{F}[f](\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\,e^{-ix\omega}dx$$

and the Inverse Fourier Transform of a function $g(\omega)$ to be

$$\mathcal{F}^{-1}[g](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(w)\,e^{i\omega t}d\omega$$

Then

$$\mathcal{F}^{-1}[\mathcal{F}[f]](x) = f(x)$$

For

$$\begin{aligned}
\mathcal{F}^{-1}[\mathcal{F}[f]](x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}dt\right]e^{i\omega x}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left[\int_{-\infty}^{\infty} e^{i\omega(x-t)}d\omega\right]dt \\
&= \int_{-\infty}^{\infty} f(t)\,\delta(x-t)\,dt \\
&= f(x)
\end{aligned}$$

and so $\mathcal{F}$ and $\mathcal{F}^{-1}$ are truly inverses of each other.

3. The Wave and Equation and Fourier Transforms

Consider

$$\begin{aligned}\phi_{tt} - c^2\phi_{xx} &= 0 \\ \phi(x,0) &= f(x) \\ \phi_t(x,0) &= 0\end{aligned}$$

Suppose we take the Fourier transforms of this PDE with respect to x :

$$\Phi(\omega,t) := \mathcal{F}_x[\phi] = \int_{-\infty}^{+\infty} \phi(x,t)\, e^{-i\omega x} dx$$

Then

$$\mathcal{F}_x\left[\frac{\partial\phi}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2\phi}{\partial x^2} e^{-i\omega x} dx$$

Integrating by parts twice and assuming $\lim_{x\to\pm\infty}\phi(x,0) = 0$ (which is reasonable for any finite energy solution) so that the contributions from the boundaries vanish, we have

$$\mathcal{F}_x\left[\frac{\partial\phi}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2\phi}{\partial x^2}\left(\frac{\partial^2}{\partial x^2} e^{-i\omega x}\right) dx = -\omega^2\int_{-\infty}^{\infty}\phi(x,t)\, e^{-i\omega x} dx = -\omega^2\mathcal{F}_x[\phi]$$

and so the Fourier transforms of the differential equation and initial condiations are

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\Phi(\omega,t) + c^2\omega^2\Phi(\omega,t) &= 0 \\ \Phi(\omega,0) &= F(\omega) := \mathcal{F}_x[f](\omega) \\ \Phi_t(\omega,0) &= 0\end{aligned}$$

The differential equation for $\Phi(\omega,t)$ is easily solved

$$\Phi(\omega,t) = A(\omega)\, e^{ic\omega t} + B(\omega)\, e^{-ic\omega t}$$

Imposing the boundary conditions on $\Phi(\omega,t)$ yields

$$\begin{aligned}A(\omega) + B(\omega) &= F(\omega) \\ ic\omega A(\omega) - ic\omega B(\omega) &= 0\end{aligned}$$

And so

$$A(\omega) = B(\omega) = \frac{1}{2}F(\omega)$$

We thus have

$$\Phi(\omega,t) = \frac{1}{2}F(\omega)\, e^{ic\omega t} + \frac{1}{2}F(\omega)\, e^{-ic\omega t}$$

as the Fourier transform of the solution $\phi(x,t)$ of the original PDE/BVP. To recover $\phi(x,t)$ all we need to is apply the inverse Fourier transform

$$\begin{aligned}\phi(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{2}F(\omega)\, e^{ic\omega t} + \frac{1}{2}F(\omega)\, e^{-ic\omega t}\right) e^{i\omega x} d\omega \\ &= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega(x+ct)} d\omega + \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega(x-ct)} d\omega \\ &= \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct)\end{aligned}$$

4. Diffusion Equation with a Instantaneous Source

Consider the PDE/BDE

$$\begin{aligned}\phi_t - a^2\phi_{xx} &= 0 \\ \phi(x,0) &= S\delta(x)\end{aligned}$$

which would govern the diffusion of S particles released at a point $x = 0$ at time $t = 0$ in a 1-dimensional medium. Taking the Fourier transform with respect to x of both the PDE and the initial condition yields

$$\begin{aligned}\frac{\partial}{\partial t}\Phi(\omega,t) + a^2\omega^2\Phi &= 0 \\ \Phi(\omega,0) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Se^{-i\omega x}dx = \frac{S}{\sqrt{2\pi}}\end{aligned}$$

The differential equation for $\Phi(\omega,t)$ has as its general solution

$$\Phi(\omega,t) = A(\omega)\, e^{-a^2\omega^2 t}$$

To obtain the coefficient $A(\omega)$, we just apply the Fourier transform of the boundar condition. This gives us

$$\Phi(\omega,t) = \frac{S}{\sqrt{2\pi}}e^{-a^2\omega^2 t}$$

Now it turms out that

$$\mathcal{F}^{-1}\left(e^{p\omega^2}\right) = \frac{1}{\sqrt{2p}}e^{-\frac{x^2}{4p}}$$

(This one can look up in a Table of Fourier Transforms). Finally, we recover the solution to the original problem by applying the inverse Laplace transform. We thus arrive at

$$\begin{aligned}\phi(x,t) &= \frac{S}{\sqrt{2\pi}}\frac{1}{\sqrt{2a^2t}}\exp\left(-x^2/4a^2t\right) \\ &= \frac{S}{\sqrt{4\pi a^2 t}}e^{-x^2/\left(4a^2t\right)}\end{aligned}$$