LECTURE 17

Green’s Identities and Green’s Functions

Let us recall The Divergence Theorem in \( n \)-dimensions.

**Theorem 17.1.** Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a vector field over \( \mathbb{R}^n \) that is of class \( C^1 \) on some closed, connected, simply connected \( n \)-dimensional region \( D \subset \mathbb{R}^n \). Then

\[
\int_D \nabla \cdot F \, dV = \int_{\partial D} F \cdot n \, dS
\]

where \( \partial D \) is the boundary of \( D \) and \( n(r) \) is the unit vector that is (outward) normal to the surface \( \partial D \) at the point \( r \in \partial D \).

As a special case of Stokes’ theorem, we may set

(1) \[ F = \nabla \phi \]

with \( \phi \) a \( C^2 \) function on \( D \). We then obtain

(2) \[
\int_D \nabla^2 \phi \, dV = \int_{\partial D} \nabla \phi \cdot dS
\]

Another special case of Stokes’ theorem comes from the choice

(3) \[ F = \phi \nabla \psi \]

For this case, Stokes’ theorem says

(4) \[
\int_D \nabla \cdot (\phi \nabla \psi) \, dV = \int_{\partial D} \phi \nabla \psi \cdot n \, dS
\]

Using the identity

(5) \[ \nabla \cdot (\phi F) = \nabla \phi \cdot F + \phi \nabla \cdot F \]

we find (4) is equivalent to

(6) \[
\int_D \nabla \phi \cdot \nabla \psi \, dV + \int_D \phi \nabla^2 \psi \, dV = \int_{\partial D} \phi \nabla \psi \cdot n \, dS
\]

Equation (6) is known as **Green’s first identity**.

Reversing the roles of \( \phi \) and \( \psi \) in (6) we obtain

(7) \[
\int_D \nabla \psi \cdot \nabla \phi \, dV + \int_D \psi \nabla^2 \phi \, dV = \int_{\partial D} \psi \nabla \phi \cdot n \, dS
\]

Finally, subtracting (7) from (6) we get

(8) \[
\int_D (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, dV = \int_{\partial D} (\phi \nabla \psi - \psi \nabla \phi) \cdot n \, dS
\]

Equation (8) is known as **Green’s second identity**.

Now set

\[
\psi(r) = \frac{1}{|r - r_o| + \epsilon}
\]

78
and insert this expression into (8). We then get
\[
\int_D \phi \left( \nabla^2 \frac{1}{|r - r_o| + \varepsilon} \right) dV = \int_D \frac{1}{|r - r_o| + \varepsilon} \nabla^2 \phi dV + \int_{\partial D} \left( \frac{1}{|r - r_o| + \varepsilon} \nabla \phi - \phi \left( \nabla \frac{1}{|r - r_o| + \varepsilon} \right) \cdot n dS \right).
\]

Taking the limit \( \varepsilon \to 0 \) and using the identities
\[
\lim_{\varepsilon \to 0} \nabla^2 \frac{1}{|r - r_o| + \varepsilon} = -4\pi \delta^n (r - r_o),
\]
\[
\lim_{\varepsilon \to 0} \frac{1}{|r - r_o| + \varepsilon} = \frac{1}{|r - r_o|},
\]
\[
\lim_{\varepsilon \to 0} \nabla \frac{1}{|r - r_o| + \varepsilon} = \nabla \frac{1}{|r - r_o|}
\]
we obtain
\[
-4\pi \phi (r_o) = \int_D \frac{1}{|r - r_o|} \nabla^2 \phi dV + \int_{\partial D} \left( \frac{1}{|r - r_o|} \nabla \phi - \phi \left( \nabla \frac{1}{|r - r_o|} \right) \cdot n dS \right).
\]

Equation (9) is known as Green's third identity.

Notice that if \( \phi \) satisfies Laplace's equation the first term on the right hand side vanishes and so we have
\[
\phi (r_o) = -\frac{1}{4\pi} \int_{\partial D} \left( \frac{1}{|r - r_o|} \nabla \phi - \phi \left( \nabla \frac{1}{|r - r_o|} \right) \cdot n dS \right)
\]
\[
= \frac{1}{4\pi} \int_{\partial D} \phi \frac{\partial}{\partial n} \frac{1}{|r - r_o|} - \frac{1}{|r - r_o|} \frac{\partial \phi}{\partial n} dS.
\]

Here \( \frac{\partial}{\partial n} \) is the directional derivative corresponding to the surface normal vector \( n \). Thus, if \( \phi \) satisfies Laplace's equation in \( D \) then its value at any point \( r_o \in D \) is completely determined by the values of \( \phi \) and \( \frac{\partial \phi}{\partial n} \) on the boundary of \( D \).
1. Green’s Functions and Solutions of Laplace’s Equation, II

Recall the fundamental solutions of Laplace’s equation in $n$-dimensions

$$\Phi_n(r, \psi, \theta_1, \ldots, \theta_{n-2}) = \begin{cases} \log |r|, & \text{if } n = 2 \\ \frac{1}{r^{n-2}}, & \text{if } n > 2 \end{cases}$$

Each of these solutions really only makes sense in the region $\mathbb{R}^n - \mathbf{O}$; for each possesses a singularity at the origin.

We studied the case when $n = 3$, a little more closely and found that we could actually write

$$\nabla^2 \left( \frac{1}{r^2} \right) = -4\pi \delta^3(r) = \begin{cases} 0, & \text{if } r \neq 0 \\ \infty, & \text{if } r = 0 \end{cases}$$

In fact, using similar arguments one can show that

$$\nabla^2 \Phi(r) = -c_n \delta^n(r)$$

where $c_n$ is the surface area of the unit sphere in $\mathbb{R}^n$. Thus, the fundamental solutions can actually be regarded as solutions of an inhomogeneous Laplace equation where the driving function is concentrated at a single point.

Let us now set $n = 3$ and consider the following PDE/BVP

$$\begin{align*}
\nabla^2 \Phi(r) &= f(r), \quad r \in D \\
\Phi(r)|_{\partial D} &= h(r)|_{\partial D}
\end{align*}$$

where $D$ is some closed, connected, simply connected region in $\mathbb{R}^3$. Let $r_o$ be some fixed point in $D$ and set

$$G(r, r_o) = \frac{-1}{4\pi |r - r_o|} + \phi_o(r, r_o)$$

where $\phi_o(r, r_o)$ is some solution of the homogeneous Laplace equation

$$\nabla^2 \phi_o(r, r_o) = 0$$

Then

$$\nabla^2 G(r, r_o) = \delta^3(r - r_o)$$

Now recall Green’s third identity

$$\int_D (\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi) \, dV = \int_{\partial D} (\Phi \nabla \Psi - \Psi \nabla \Phi) \cdot n \, dS$$

If we replace $\psi$ in (18) by $G(r, r_o)$ we get

$$\begin{align*}
\Phi(r_o) &= \int_D \Phi(r) \delta^3(r - r_o) \, dV \\
&= \int_D \Phi \nabla^2 G \, dV \\
&= \int_D G \nabla^2 \Phi \, dV + \int_{\partial D} (\Phi \nabla G - G \nabla \Phi) \cdot n \, dS \\
&= \int_D G f \, dV + \int_{\partial D} \left( h \frac{\partial G}{\partial n} - G \frac{\partial h}{\partial n} \right) \, dS \\
&= \int_D G f \, dV + \int_{\partial D} h \frac{\partial G}{\partial n} \, dS - \int_{\partial D} G \frac{\partial h}{\partial n} \, dS
\end{align*}$$

Up to this point we have only required that the function $\phi_o$ satisfies Laplace’s equation. We will now make our choice of $\phi_o$ more particular; we shall choose $\phi_o(r, r_o)$ to be the unique solution of Laplace’s equation in $D$ satisfying the boundary condition

$$\frac{1}{4\pi |r - r_o|}\bigg|_{\partial D} = \phi_o(r, r_o)|_{\partial D}$$

so that

$$G(r, r_o)|_{\partial D} = 0$$
Then the last integral on the right hand side of (19) vanishes and so we have

\[ (21) \quad \Phi (r, r_o) = \int_D G(r, r_o) f(r) dV + \int_{\partial D} h(r) \frac{\partial G}{\partial n} (r, r_o) dS \ . \]

Thus, once we find a solution \( \phi_o (r, r_o) \) to the homogeneous Laplace equation satisfying the boundary condition (21), we have a closed formula for the solution of the PDE/BVP (14) in terms of integrals of \( G(r, r_o) \) times the driving function \( f(r) \), and of \( \frac{\partial G}{\partial n} (r, r_o) \) times the function \( h(r) \) describing the boundary conditions on \( \Phi \). Note that the Green’s function \( G(r, r_o) \) is fixed once we fix \( \phi_o \) which in turn depends only on the nature of the boundary of the region \( D \) (through condition (20)).

Example

Let us find the Green’s function corresponding to the interior of sphere of radius \( R \) centered about the origin. We seek to find a solution of \( \phi_o (r, r_o) \) of the homogenous Laplace’s equation such that (20) is satisfied. This is accomplished by the following trick.

Suppose \( \Phi (r, \psi, \theta) \) is a solution of the homogeneous Laplace equation inside the sphere of radius \( R \) centered at the origin. For \( r > R \), we define a function

\[ (22) \quad \tilde{\Phi} (r, \psi, \theta) = \frac{R}{r} \Phi \left( \frac{R^2}{r}, \psi, \theta \right) . \]

I claim that \( \tilde{\Phi} (r, \psi, \theta) \) so defined also satisfies Laplace’s equation in the region exterior to the sphere.

To prove this, it suffices to show that

\[ (23) \quad 0 = r^2 \nabla \tilde{\Phi} = \frac{\partial}{\partial r} \left( r^2 \frac{\partial \tilde{\Phi}}{\partial r} \right) + \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial \tilde{\Phi}}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2 \tilde{\Phi}}{\partial \psi^2} \]

or

\[ (24) \quad \frac{\partial}{\partial r} \left( r^2 \frac{\partial \tilde{\Phi}}{\partial r} \right) = - \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial \tilde{\Phi}}{\partial \theta} \right) - \frac{1}{\sin^2(\theta)} \frac{\partial^2 \tilde{\Phi}}{\partial \psi^2} . \]

Set

\[ (25) \quad u = \frac{R^2}{r} . \]

so that

\[ (26) \quad \tilde{\Phi} (r, \psi, \theta) = \frac{R^2}{r} \Phi (u, \psi, \theta) \]

and so

\[ (27) \quad \frac{\partial}{\partial r} \left( r^2 \frac{\partial \tilde{\Phi}}{\partial r} \right) = - \frac{u}{R^2} \frac{\partial}{\partial u} \left( \frac{R^4}{u^2} \right) \left( \frac{\partial}{\partial u} \Phi \right) \]

Notice that

\[ (28) \quad \lim_{r \to R} \tilde{\Phi} (r, \psi, \theta) = \Phi (r, \psi, \theta) \]

This transform is called Kelvin inversion.
1. **GREEN’S FUNCTIONS AND SOLUTIONS OF LAPLACE’S EQUATION, II**

Now let return to the problem of finding a Green’s function for the interior of a sphere of radius. Let

\[ \tilde{r} = r \left( \frac{R^2}{r}, \psi, \theta \right) = \frac{R^2}{r^2} r \]

In view of the preceding remarks, we know that the functions

\[ \Phi_1 (r) = \frac{1}{r - r_o} \]
\[ \Phi_2 (r) = \frac{R}{r} \left( \frac{1}{r - r_o} \right) = \tilde{\Phi}_1 (r) \]

will satisfy, respectively,

\[ \nabla^2 \Phi_1 (r) = -4 \pi \delta^3 (r - r_o) \]
\[ \nabla^2 \Phi_2 (r) = -4 \pi R \delta^3 \left( \frac{R^2 r}{r^2} - r_o \right) \]

However, notice that the support of \( \nabla^2 \Phi_2 (r) \) lies completely outside the sphere. Therefore, in the interior of the sphere, \( \Phi_2 \) is a solution of the homogenous Laplace equation. We also know that on the boundary of the sphere that we have

\[ \Phi_1 (r) = \Phi_2 (r) \]

Thus, the function

\[ G (r, r_o) = -\frac{1}{4 \pi} \frac{1}{|r - r_o|} \]

thus satisfies

\[ \nabla^2 r G (r, r_o) = \delta^3 (r - r_o) \]

for all \( r \) inside the sphere and

\[ G (r, r_o) = 0 \]

or all \( r \) on the boundary of the sphere. Thus, the function \( G (r, r_o) \) defined by (33) is the Green’s function for Laplace’s equation within the sphere.

Now consider the following PDE/BVP

\[ \nabla^2 \Phi (r) = f (r) \quad , \quad r \in B \]
\[ \Phi (R, \psi, \theta) = 0 \]

where \( B \) is a ball of radius \( R \) centered about the origin.

According to the formula (21) and (33), the solution of (36) is given by

\[ \Phi (r_o) = \int_B G (r, r_o) f (r) dV + \int_{\partial B} h (\psi, \theta) \frac{\partial G (r, r_o)}{\partial n} (r, r_o) dS \]
\[ = \int_B G (r, r_o) f (r) dV \]

To arrive at a more explicit expression, we set

\[ r_o = (r \cos(\psi) \sin(\theta), r \sin(\psi) \sin(\theta), r \cos(\theta)) \]
\[ r = (\rho \cos(\alpha) \sin(\beta), \rho \sin(\alpha) \sin(\beta), \rho \cos(\beta)) \]

Then

\[ dV = \rho^2 \sin^2(\theta) d\rho d\alpha d\beta \]
\[ dS = \rho^2 \sin^2(\theta) d\alpha d\beta \]
and after a little trigonometry one finds

$$\frac{1}{4\pi |r - r_o|} = \frac{1}{4\pi \sqrt{r^2 + \rho^2 - 2r\rho \cos(\psi - \alpha) \sin(\theta) \sin(\beta) + \cos(\theta) \cos(\beta)}}$$

$$\frac{1}{4\pi \left| \frac{R}{r} r_o - \frac{R}{\rho} r_o \right|} = \frac{R}{4\pi \sqrt{R^4 + r^2 + \rho^2 - 2R^2 r\rho \cos(\psi - \alpha) \sin(\theta) \sin(\beta) + \cos(\theta) \cos(\beta)}}.$$

Thus,

$$\Phi(r, \psi, \theta) = \int_0^R \int_0^{2\pi} \int_0^\pi \frac{R f(r, \psi, \theta) r^2 \sin(\theta) dr d\theta d\psi}{4\pi \sqrt{R^4 + r^2 + \rho^2 - 2R^2 r\rho \cos(\psi - \alpha) \sin(\theta) \sin(\beta) + \cos(\theta) \cos(\beta)}} - \int_0^R \int_0^{2\pi} \int_0^\pi \frac{f(r, \psi, \theta) r^2 \sin(\theta) dr d\theta d\psi}{4\pi \sqrt{r^2 + \rho^2 - 2r\rho \cos(\psi - \alpha) \sin(\theta) \sin(\beta) + \cos(\theta) \cos(\beta)}}.$$