LECTURE 15

Laplace's Equation on a Disc

Last time we solved the Dirichlet problem for Laplace's equation on a rectangular region. Today we'll look at the corresponding Dirichlet problem for a disc.

Thus, we consider a disc of radius a

(1) $$D = \left\{[x, y] \in \mathbb{R}^2 \mid x^2 + y^2 = a^2\right\}$$

upon which the following Dirichlet problem is posed:

(2a) $$u_{xx} + u_{yy} = 0 \qquad , \qquad \forall\ [x, y] \in D$$

(2b) $$u\left(a\cos\theta, a\sin\theta\right) = h\left(\theta\right) \qquad , \qquad 0 \leq \theta \leq 2\pi$$

We shall solve this problem by first rewriting Laplace's equation in terms of a polar coordinates (which are most natural to the region D) and then separating variables and preceding as in Lecture 14.

Now under the change of variables

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \iff \left\{\begin{array}{l} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(y/x\right) \end{array}\right.$$

we have

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} - \frac{y}{x^2+y^2}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} + \frac{x}{x^2+y^2}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

After a short but tedious calculation one finds

(3) $$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and so in terms of polar coordinates Laplace's equation becomes

(3) $$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad .$$

We'll now apply Separation of Variables to this PDE. Setting $u\left(r,\theta\right) = R\left(r\right)T\left(\theta\right)$ and plugging in we get

$$R''T + \frac{R'}{r}T + \frac{R}{r^2}T'' = 0$$

Multiplying both sides by r^2/RT we get

$$\frac{r^2}{R}R'' + \frac{r}{R}R' = -\frac{T''}{T}$$

Observing that each side depends only a variable that does not appear on the opposite side we conclude that both sides must be equal to a constant. Let's denote this constant by λ^2. We then have

$$\frac{r^2}{R}R'' + \frac{r}{R}R' = \lambda^2 = -\frac{T''}{T}$$

or the following pair of ordinary differential equations

$$T'' = -\lambda^2 T \tag{4a}$$

$$r^2 R'' + r R' = \lambda^2 R \tag{4b}$$

The first equation (4a) should be quite familiar by now. It has as its general solution

$$T(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta) \tag{5}$$

The second equation (4b) is an Euler type equation. Such equations can (almost always) be solved using the ansatz $R(r) = r^m$, regarding m as an adjustable parameter. Inserting r^m in place of R in (4b) we get

$$0 = r^2\left(m(m-1)r^{m-2}\right) + r\left(mr^{m-1}\right) - \lambda^2\left(r^m\right) = \left(m(m-1) + m - n^2\right)r^m = \left(m^2 - \lambda^2\right)r^m$$

Evidently, we must take $m^2 = \lambda^2$ which in turn requires $m = \pm\lambda$. The general solution of (4b) is thus

$$R(r) = ar^{\lambda} + br^{-\lambda} \tag{6}$$

Err, except that something screwy happens when $\lambda = 0$; in that case we only get one linearly independent solution $R(r) = a$ some constant. To get a second linearly independent solution for the $\lambda = 0$ case, we can employ Reduction of Order, or simply solve

$$r^2 R'' + r R' = 0 \tag{7}$$

afresh. The latter is easy enough. Dividing out by r and setting $S = R'$, (7) becomes

$$rS' + S = 0 \quad\Longrightarrow\quad \frac{1}{S}\frac{dS}{dr} = -r \quad\Longrightarrow\quad \frac{dS}{S} = -\frac{dr}{r} \quad\Longrightarrow\quad \int\frac{dS}{S} = -\int\frac{dr}{r} + C$$
$$\Longrightarrow\quad \ln|S| = -\ln|r| + C$$

Setting $C = \ln|c|$ we then have

$$\ln|S| = -\ln|r| + \ln|c| = \ln\left|\frac{c}{r}\right| \quad\Longrightarrow\quad S = \frac{c}{r}$$

Finally, we integrate S to recover R

$$R(r) = \int R'dr + d = \int Sdr + d = \int\frac{c}{r}dr + d = c\ln|r| + d \qquad (\lambda = 0 \text{ case})$$

In conclusion, the general solution of (4b) is

$$R(r) = \begin{cases} cr^{\lambda} + dr^{-\lambda} & \text{if } \lambda \neq 0 \\ c\ln|r| + d & \text{if } \lambda = 0 \end{cases} \tag{8}$$

Putting (5) and (8) together, we obtain

$$u_{\lambda}(r,\theta) = \begin{cases} \left(A_{\lambda}\cos(\lambda\theta) + B_{\lambda}\sin(\lambda\theta)\right)r^{\lambda} + \left(C_{\lambda}\cos(\lambda\theta) + D_{\lambda}\sin(\lambda\theta)\right)r^{-\lambda} & \lambda \neq 0 \\ A_0 + C_0\ln|r| & \lambda = 0 \end{cases}$$

as Separation-of-Variables-type solutions to Laplace's equation in polar coordinates (n.b. when $\lambda = 0$, $\cos(\lambda\theta) = 1$ and $\sin(\lambda\theta) = 0$).

Now we can whittle down this set of possible solutions even further by imposing some hidden boundary conditions (besides (2b)).

One thing we expect of any viable solution is that $u(r,\theta) = u(r,\theta + 2\pi)$, for after all a point with coordinates $[r\cos(\theta), r\sin(\theta)]$ is the same as the point with coordinates $[r\cos(\theta + 2\pi), r\sin(\theta + 2\pi)]$. But

$$\left.\begin{array}{c} \cos(\lambda\theta) = \cos(\lambda(\theta + 2\pi)) \\ \sin(\lambda\theta) = \sin(\lambda(\theta + 2\pi)) \end{array}\right\} \quad\Longleftrightarrow\quad \lambda = n \in \mathbb{Z} := \{0, \pm1, \pm2, \pm3, \dots\}$$

That is, λ must be an integer, we may as well take to be nonnegative since the solutions where $\lambda = n$ and $\lambda = -n$ are actually coincide (up to changing the signs of the arbitary constants B and D).

Secondly, we expect any viable solution to be continuous at $r = 0$. This will require us to throw out the solutions where C and D are non-zero; for both r^{-n} and $\ln|r|$ become unbounded as $r \to 0$. We thus have

$$u_n(r,\theta) = \begin{cases} A_n \cos(n\theta) r^n + B_n \sin(n\theta) r^n & n = 1,2,3,\ldots \\ A_0 & n = 0 \end{cases}$$

or even more succinctly

$$u_n(r,\theta) = A_n \cos(n\theta) r^n + B_n \sin(n\theta) r^n \quad , \quad n = 0,1,2,3,\ldots \tag{9}$$

Now let's take a general linear combination of the solutions (9) and impose the boundary condition (2b). Thus, we set

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n \cos(n\theta) r^n + \sum_{n=1} B_n \sin(n\theta) r^n$$

and impose

$$h(\theta) = u(a,\theta) = \sum_{n=0}^{\infty} A_n \cos(n\theta) a^n + \sum_{n=1} B_n \sin(n\theta) a^n \tag{10}$$

Now from Fourier theory we now that any continuous function on the circle has a unique Fourier expansion whose coefficients can be explicitly determined in terms of certain Fourier integrals. Applying this theory to the case at hand we have

$$h(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + \sum_{n=1}^{\infty} b_n \sin(n\theta) \tag{11a}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos(n\theta)\, d\theta \tag{11b}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin(n\theta)\, d\theta \tag{11c}$$

Comparing (10) with (11a) we conclude that our solution must be

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) r^n + \sum_{n=1} B_n \sin(n\theta) r^n$$

with

$$\begin{aligned} A_0 &= \frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\, d\theta \\ A_n &= \frac{1}{a^n \pi}\int_0^{2\pi} h(\theta)\cos(n\theta)\, d\theta \quad , \quad n = 1,2,3,\ldots \\ B_n &= \frac{1}{a^n \pi}\int_0^{2\pi} h(\theta)\sin(n\theta)\, d\theta \quad , \quad n = 1,2,3,\ldots \end{aligned}$$

1. Poisson Sum Formula

We now have

$$\begin{aligned} u(r,\theta) &= A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) r^n + \sum_{n=1} B_n \sin(n\theta) r^n \\ &= \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\, d\theta \\ &+ \sum_{n=1}^{\infty} \frac{1}{a^r \pi}\int_0^{2\pi} h(\theta') \cos(n\theta') \cos(n\theta)\, d\theta' r^n \\ &+ \sum_{n=1} \frac{1}{a^r \pi}\int_0^{2\pi} h(\theta') \sin(n\theta') \sin(n\theta)\, d\theta r^n \end{aligned}$$

or, interchanging the order of summation and integration (which we can do so long as the series converges in the first place)

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} h(\theta') \left(1 + 2\sum_{n=1}^{\infty} \left(\cos(n\theta)\cos(n\theta') + \sin(n\theta)\sin(n\theta')\right) \left(\frac{r}{a}\right)^n d\theta'\right) \\ &= \frac{1}{2\pi}\int_0^{2\pi} h(\theta') \left(1 + 2\sum_{n=1}^{\infty} \cos(n(\theta - \theta')) \left(\frac{r}{a}\right)^n\right) d\theta \end{aligned} \tag{13}$$

where we have employed the cosine angle sum formula

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \tag{14}$$

Next, inserting

$$\cos(\phi) = \frac{e^{i\phi} + e^{-i\phi}}{2} \tag{15}$$

into the series

$$1 + 2\sum_{n=1}^{\infty} \cos(n\phi)\, t^n \tag{16}$$

we get

$$1 + \sum_{n=1}^{\infty} \left(e^{in\phi} + e^{-in\phi}\right) t^n = 1 + \sum_{n=1}^{\infty} \left(e^{i\phi} t\right)^n + \sum_{n=1}^{\infty} \left(e^{-i\phi} t\right)^n$$

Using the well-known geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \tag{17}$$

we obtain

$$\begin{aligned} 1 + 2\sum_{n=1}^{\infty} \cos(n\phi)\, t^n &= 1 + \left(\frac{1}{1 - e^{i\phi} t} - 1\right) + \left(\frac{1}{1 - e^{-i\phi} t} - 1\right) \\ &= \frac{1 - t^2}{1 - e^{i\phi} t - e^{-i\phi} t + t^2} \\ &= \frac{1 - t^2}{1 - 2\cos(\phi) t + t^2} \end{aligned}$$

or

$$1 + 2\sum_{n=1}^{\infty} \cos(n\phi)\, t^n = \frac{1 - t^2}{1 - 2\cos(\phi) t + t^2} \tag{18}$$

Applying this last formula to

$$\left(1+2\sum_{n=1}^{\infty}\cos\left(n\left(\theta-\theta'\right)\right)\left(\frac{r}{a}\right)^{n}\right)$$

we find

$$\left(1+2\sum_{n=1}^{\infty}\cos\left(n\left(\theta-\theta'\right)\right)\left(\frac{r}{a}\right)^{n}\right)=\frac{1-\left(\frac{r}{a}\right)^{2}}{1-2\cos\left(\theta-\theta'\right)\left(\frac{r}{a}\right)+\left(\frac{r}{a}\right)^{2}}=\frac{a^{2}-r^{2}}{a^{2}-\cos\left(\theta-\theta'\right)ar+r^{2}}$$

Thus,

$$u\left(r,\theta\right)=\frac{1}{2\pi}\int_{0}^{2\pi}h\left(\theta'\right)\frac{a^{2}-r^{2}}{a^{2}-2\cos\left(\theta-\theta'\right)ar+r^{2}}d\theta' \tag{20}$$

Finally, we'll convert this result into something that's understandable in a coordinate free way. Set

$$\mathbf{x}=\left[r\cos\left(\theta\right),r\sin\left(\theta\right)\right]$$
$$\mathbf{y}=\left[a\cos\left(\theta'\right),a\sin\left(\theta'\right)\right]$$

then

$$\|\mathbf{x}\|=r^{2}$$
$$\|\mathbf{y}\|=a^{2}$$

and

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|^{2}&=\left(r\cos\theta-a\cos\theta'\right)^{2}+\left(r\sin\theta-a\sin\theta'\right)^{2}\\&=r^{2}\cos^{2}\theta-2ar\cos\theta\cos\theta'+a^{2}\cos^{2}\theta'\\&+r^{2}\sin^{2}\theta-2ar\sin\theta\sin\theta'+a^{2}\sin^{2}\theta'\\&=r^{2}\left(\cos^{2}\theta+\sin^{2}\theta\right)-2ar\left(\cos\theta\cos\theta'+\sin\theta\sin\theta'\right)+a^{2}\left(\cos^{2}\theta'+\sin^{2}\theta'\right)\\&=r^{2}-2ar\cos\left(\theta-\theta'\right)+a^{2}\end{aligned}$$

and so we can write

$$u\left(\mathbf{x}\right)=\frac{1}{2\pi a}\int_{\mathbf{y}\in\partial D}h\left(\mathbf{y}\right)\frac{a^{2}-\|\mathbf{x}\|^{2}}{\|\mathbf{x}-\mathbf{y}\|^{2}}ds \tag{21}$$

where ∂D is the circle bounding D. The additional factor of $1/a$ arises because

$$ad\theta=ds$$

is the correct expression for the infinitesimal arc-length when we interprete (20) as a path integral about the boundary of D.