Sturm-Liouville Theory: Examples

Recall that a Sturm-Liouville problem is

- a second order linear ordinary differential equation of the form

\[
\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) - q(x) y = -\lambda r(x) y , \quad a \leq x \leq b
\]

with \( \lambda \) some constant parameter and \( p(x) \) and \( r(x) \) non-negative functions on the interval \([a, b] \);

- homogeneous linear boundary conditions of the form

\[
\begin{align*}
\alpha_1 y(a) + \alpha_2 y'(b) &= 0 \\
\beta_1 y(b) + \beta_2 y'(b) &= 0
\end{align*}
\]

A solution of a Sturm-Liouville problem consists of a value of the parameter \( \lambda \) and a solution \( \phi(x) \) of the ODE/BVP (1), (2a), (2b). The number \( \lambda \) is called a (Sturm-Liouville) \textit{eigenvalue} and the solution \( \phi(x) \) is called a (Sturm-Liouville) \textit{eigenfunction}.

Here are the fundamental properties of Sturm-Liouville problems:

- Solutions exist only for a discrete set of eigenvalues: \( \{\lambda_1, \lambda_2, \lambda_3, \ldots\} \)
- The eigenvalues can be ordered such that

\[
\lambda_1 < \lambda_2 < \lambda_3 < \cdots
\]

and with such a ordering

\[
\lim_{n \to \infty} \lambda_n \to +\infty
\]

- For each \( \lambda_n \) the space of solutions of (1), (2a), (2b) is 1-dimensional (that is, the solutions are unique up to a scalar factor).
- The solutions \( \phi_n \) corresponding \( \lambda_n \) can be normalized so that

\[
\int_a^b \phi_n(x) \phi_m(x) r(x) \, dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}
\]

- Any piece-wise continuous function \( f(x) \) on the interval \([a, b] \) can be expanded in terms of the normalized Sturm-Liouville eigenfunctions \( \{\phi_1, \phi_2, \ldots\} \)

\[
f(x) \approx \sum_{n=1}^{\infty} a_n \phi_n(x)
\]

with

\[
a_n = \int_a^b f(x) \phi_n(x) r(x) \, dx
\]

If \( f(x) \) is actually continuous then the expansion (6) is an equality, otherwise

\[
\sum_{n=1}^{\infty} a_n \phi_n(x) = \frac{1}{2} \left( \lim_{\varepsilon \to 0^+} f(x + \varepsilon) + \lim_{\varepsilon \to 0^+} f(x - \varepsilon) \right)
\]

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Example 1:

Setting \( p(x) = 1, q(x) = 0, r(x) = 1, \alpha_1 = 1 = \beta_1, \alpha_2 = 0 = \beta_2, a = 0, b = L \), we have arrive at the following Sturm-Liouville system

\[
\begin{align*}
y''(x) &= -\lambda y(x) & x \in [0, L] \\
y(0) &= 0 \\
y(L) &= 0
\end{align*}
\]

We’ve seen this system before. Its solution is

\[
\lambda = \left( \frac{n\pi}{L} \right)^2, \quad y_n(x) = \sin \left( \frac{n\pi}{L} x \right)
\]

Moreover, we know that

\[
\frac{2}{L} \int_0^L \sin \left( \frac{n\pi}{L} x \right) \sin \left( \frac{m\pi}{L} x \right) \, dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}
\]

So the normalized Sturm-Liouville eigenfuctions for this problem would be

\[
\phi_n(x) = \sqrt{\frac{L}{2}} \sin \left( \frac{n\pi}{L} x \right)
\]

Example 2:

Let’s keep \( p(x) = 1, q(x) = 0, r(x) = 1 \), but now impose less trivial homogeneous boundary conditions

\[
\begin{align*}
y'(0) &= 0 \\
y(1) + y'(1) &= 0
\end{align*}
\]

The differential equation is again

\[
y'' = -\lambda y
\]

and so we know that the Sturm-Liouville eigenfunctions must be of the form

\[
y(x) = A \cos \left( \sqrt{\lambda} x \right) + B \sin \left( \sqrt{\lambda} x \right)
\]

Actually, life will be simpler if we get rid of square-roots by replacing the parameter \( \lambda \) by its square. That is, we consider

\[
\begin{align*}
y'' &= -\lambda^2 y \\
y'(0) &= 0 \\
y(1) + y'(1) &= 0
\end{align*}
\]

as our Sturm-Liouville problem, and look for functions

\[
y(x) = A \cos (\lambda x) + B \sin (\lambda x)
\]

that solve the boundary conditions (8b) and (8c). Plugging (9) into (8b) yields

\[
0 = y'(0) = -\lambda A \sin (0) + \lambda B \cos (0) = \lambda B
\]

We thus need either \( \lambda = 0 \) or \( B = 0 \). In the first case, we’d have

\[
y'' = 0 \quad \text{and} \quad y'(0) = 0 \quad \implies \quad y(x) = c \quad \text{a constant}
\]

and then (8c) would require \( c = 0 \). So we just get a trivial solution when \( \lambda = 0 \). The alternative is to set \( B = 0 \), and so we do.

Now we must solve

\[
0 = y(1) + y'(1) = B \sin (\lambda) - B \lambda \cos (\lambda)
\]

or

\[
\lambda = \frac{\sin (\lambda)}{\cos (\lambda)} = \cot (\lambda)
\]
This happens to be a transcendental equation for $\lambda$, but there are, in fact, an infinite number of roots:

\begin{align*}
\lambda_1 &= .8603335890 \\
\lambda_2 &= 3.425618459 \\
\lambda_3 &= 6.437298179 \\
&\vdots
\end{align*}