LECTURE 9

Reflections off Boundaries

In the last lecture we considered the wave equation with Cauchy boundary conditions

$$\begin{aligned} u_{tt}-c^2u_{xx} &= 0 \quad , \quad -\infty<x<+\infty && \text{(1a)}\\ u(x,0) &= \phi(x) \quad , \quad -\infty<x<+\infty && \text{(1b)}\\ u_t(x,0) &= \psi(x) \quad , \quad -\infty<x<+\infty && \text{(1c)} \end{aligned}$$

defined on the entire real line. This PDE/BVP was to describe an infinitely long wire, which was given an initial configuration (prescribed by the function $\phi(x)$) and certain arrangement of transverse velocities (prescribed by the function $\psi(x)$).

Today we shall discuss PDE/BVP problem corresponding to a semi-infinite wire and try to understand the phenomenon of reflections arises. So we start with the following modification of the PDE/BVP (1)

$$\begin{aligned} u_{tt}-c^2u_{xx} &= 0 \quad , \quad 0<x<+\infty && \text{(2a)}\\ u(x,0) &= \phi(x) \quad , \quad 0<x<+\infty && \text{(2b)}\\ u_t(x,0) &= \psi(x) \quad , \quad 0<x<+\infty && \text{(2c)}\\ u(0,t) &= 0 \quad , \quad \forall\, t && \text{(2d)} \end{aligned}$$

The last condition simply means that we keep the endpoint at $x=0$ fixed for all time.

From Lecture 5 we know that the unique solution of (1a) – (1c) is

(3) $$u(x,t)=\frac{1}{2}\left(\phi(x+ct)+\phi(x-ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\tau)\,d\tau$$

However, having changed our boundary conditions, we can no expect to have exactly the same solution. Also, from experience with a Slinky, for example, we should expect that a wave coming towards the fixed end, hits the boundary, and then gets reflected back. In fact, it bounces back "upside-down". Below we'll explain this phenomenon mathematically.

Recall that a function $f(x)$ on the real line is said to be *even* if $f(-x)=f(x)$ for all x and *odd* if $f(-x)=-f(x)$ for all x.

LEMMA 9.1. *If the functions $\phi(x)$ and $\psi(x)$ are odd functions of x then the solution $u(x,t)$ of (1a) – (1c) is an odd function of x for all t.*

Proof. *Consider what happens when we substitute $x\to -x$ in the solution (3) of (1)*

$$u(-x,t)=\frac{1}{2}\left(\phi(-x+ct)+\phi(-x-ct)\right)+\frac{1}{2c}\int_{-x-ct}^{-x+ct}\psi(\tau)\,d\tau$$

We have

$$\begin{aligned} \phi(-x+ct) &= \phi(-(x-ct))=-\phi(x-ct)\\ \phi(-x-ct) &= \phi(-(x+ct))=-\phi(x+ct) \end{aligned}$$

and under the change of variables $\tau \to \tau' = -\tau$ the integral on the right hand side becomes

$$\begin{aligned} \frac{1}{2c}\int_{-x-ct}^{-x+ct}\psi(\tau)\,d\tau &= \frac{1}{2c}\int_{-(-x-ct)}^{-(-x+ct)}\psi(-\tau')(-\delta\tau') \\ &= \frac{1}{2c}\int_{x+ct}^{x-ct}\psi(\tau')\,d\tau' \\ &= -\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\tau')\,d\tau' \end{aligned}$$

and so

$$\begin{aligned} u(-x,t) &= \frac{1}{2}\left(-\phi(x-ct)+(-\phi(x+ct))\right)-\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\tau')\,d\tau' \\ &= -\frac{1}{2}\left(\phi(-x+ct)+\phi(-x-ct)\right)-\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\tau)\,d\tau \\ &= -u(x,t) \end{aligned}$$

□

Next note that if $u(x,t)$ is an odd function of x for all t then, since $0=-0$,

$$u(0,t)=u(-0,t)=-u(0,t) \quad \Longrightarrow \quad u(0,t)=0 \qquad \forall\ t$$

So here's the basic idea as to how we'll solve (2). Start with the functions $\phi(x)$, $\psi(x)$ defined on the half-line $[0,+\infty)$ and extend them to odd functions on the entire real line by defining

(4) $$\phi_{odd}(x)=\begin{cases}\phi(x) & \text{if } x\geq 0\\ -\phi(-x) & \text{if } x<0\end{cases} \quad , \quad \psi_{odd}(x)=\begin{cases}\psi(x) & \text{if } x\geq 0\\ -\psi(-x) & \text{if } x<0\end{cases}$$

With these functions in hand, we can consider the PDE/BVP

$$u_{tt}-c^2u_{xx} = 0 \quad , \quad -\infty<x<+\infty \tag{5a}$$
$$u(x,0) = \phi_{odd}(x) \quad , \quad -\infty<x<+\infty \tag{5b}$$
$$u_t(x,0) = \psi_{odd}(x) \quad , \quad -\infty<x<+\infty \tag{5c}$$

defined on the whole real line. For this PDE/BVP we have the solution (3)

$$u_{odd}(x,t)=\frac{1}{2}\left(\phi_{odd}(x+ct)+\phi_{odd}(x-ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{odd}(\tau)\,d\tau$$

which in view of the lemma must be an odd function of x. This function will automatically satisfy the PDE and boundary conditions (2a), (2b) and (2c), because all we have to do is restrict the (5a), (5b) and (5c) to the half line $[0,+\infty)$. The last boundary condition

$$u_{odd}(0,t)=0$$

is also automatic since $u_{odd}(x,t)$ is an odd function for all t. Since the solution to the PDE/BVP (2a) – (2d) is unique we can conclude

$$u(x,t)=\left.u_{odd}(x,t)\right|_{x\in[0,+\infty)}$$

is the desired solution.

Okay, now for the reflection phenomenon. Consider the simple case where the initial transverse velocities $\psi(x)$ are all zero, and we have a localized spike in the wire at time $t=0$. Let's suppose further the spike is initially at some distance d from the endpoint $x=0$. Then, as we discussed in the preceding lecture, what happens is that (starting at $t=0$) the hump splits into two smaller spikes of the same shape, but half the magnitude, one travelling to the left towards $x=0$ and the other travelling off to the left towards $+\infty$. We can idealize this situation by considering a intial configuation function $\phi(x)$ such that

$$\phi(x)=\begin{cases}a & \text{if } x=d\\ 0 & \text{if } x\neq d\end{cases} \qquad \Longrightarrow \qquad \phi_{odd}(x)=\begin{cases}\pm a & \text{if } x=\pm d\\ 0 & \text{if } x\neq \pm d\end{cases}$$

Where is the solution $u(x,t)$ non-zero at time t?

$$u(x,t) = u_{odd}(x,t)|_{x\in[0,+\infty)} = \frac{1}{2}\phi_{odd}(x+ct) + \frac{1}{2}\phi_{odd}(x-ct)$$

Consider the first (left-moving) piece of the solution.

$$0 \neq \frac{1}{2}\phi_{odd}(x+ct) \quad \Longrightarrow \quad d = \pm(x+ct) \quad \Longrightarrow \quad x = \pm d - ct$$

We can ignore the solution $x = -d - ct$, because this x will always be negative and so outside our solution domain. As for the other solution $x = d - ct$, it says that at time t this left moving spike is at position $ct - d$. It has amplitude

$$\frac{1}{2}\phi_{odd}((d-ct)+ct) = \frac{1}{2}\phi_{odd}(d) = \frac{1}{2}a \qquad ,$$

so this spike will be upright. Eventually though, once t reaches d/c, this value $d - ct$ no longer be positive and so this value of x will be outside of the domain of our solution.

Let's now look at the second (right-moving) piece of the solution and look at where it has a non-zero contributions.

$$0 \neq \frac{1}{2}\phi_{odd}(x-ct) \quad \Longrightarrow \quad \pm d = x - ct \quad \Longrightarrow \quad x = \pm d + ct$$

The first solution $x = d + ct$ is always positive and so within our solution domain. It corresponds to the original positive $1/2a$ spike that continues to move off to the right. But we also have a solution $x = -d + ct$ that becomes positive once $t > d/c$. This is the reflected wave. Notice that it first appears at time $t = d/c$ and then moves off to the right with velocity c and amplitude

$$\frac{1}{2}\phi_{odd}((-d+ct)-ct) = \frac{1}{2}\phi_{odd}(-d) = -\frac{1}{2}a \qquad ;$$

Thus, the reflected wave is upside-down.