LECTURE 8

The Wave Equation with a Source

We'll now introduce a *source term* to the right hand side of our (formerly homogeneous) wave equation.

$$u_{tt} - c^2 u_{xx} = f(x,t) \tag{1}$$

We shall also impose the usual Cauchy boundary conditions:

$$u(x,0) = \phi(x) \tag{2a}$$

$$u_t(x,0) = \psi(x) \tag{2b}$$

Because the differential operator

$$\mathcal{L} = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}$$

is a linear differential operator we can expect the solution be of the form

$$u(x,t) = u_p(x,t) + u_o(x,t)$$

where $u_p(x,t)$ is a *particular solution* of (1) satisfying homogenous boundary conditions

$$\begin{aligned} u_p(x,0) &= 0 \\ u_{p,t}(x,0) &= 0 \end{aligned}$$

and $u_o(x,t)$ is a solution of the corresponding homogeneous PDE/BVP.

$$u_{tt} - c^2 u_{xx} = 0 \tag{3a}$$

$$u(x,0) = \phi(x) \tag{3b}$$

$$u_t(x,0) = \psi(x) \tag{3c}$$

(See Lecture 1.)

THEOREM 8.1. *The general solution of (1), (2a) and (2b) is given by*

$$u(x,t) = u_p(x,t) + u_o(x,t) \tag{4a}$$

where

$$u_o(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\tau)\, d\tau \tag{4b}$$

is the homogeneous part of the solution and

$$u_p(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\, dy ds \tag{4c}$$

is the particular part of the solution.

Proof. From the remarks just prior to the theorem we really only need to find the solution $u_p(x,t)$ of

$$u_{tt} - c^2 u_{xx} = f(x,t) \tag{5a}$$

$$u(x,0) = 0 \tag{5b}$$

$$u_t(x,0) = 0 \tag{5c}$$

Just as we did in Lecture 5 for the homogeneous case (where $f(x,t)$), let us introduce a change of coordinates

$$\begin{aligned} \xi &= x + ct & \longleftrightarrow & \quad x = \frac{1}{2}(\xi + \eta) \\ \eta &= x - ct & \longleftrightarrow & \quad t = \frac{1}{2c}(\xi - \eta) \end{aligned} \tag{6}$$

Recall that under this change of coordinates the wave operator becomes

$$L = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \quad \longrightarrow \quad -4c^2 \frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} \tag{7}$$

Thus, if we set

$$\begin{aligned} U(\xi,\eta) &= u(x(\xi,\eta), t(\xi,\eta)) \\ F(\xi,\eta) &= f(x(\xi,\eta), t(\xi,\eta)) \end{aligned}$$

we will have

$$\begin{aligned} -4c^2 \frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} U(\xi,\eta) &= -4c^2 \frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} u(x(\xi,\eta), t(\xi,\eta)) \\ &= -4c^2 \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \eta}\right) \\ &= -4c^2 \frac{\partial}{\partial \xi}\left(\frac{1}{2}u_x(x(\xi,\eta), t(\xi,\eta)) - \frac{1}{2c}u_t(x(\xi,\eta), t(\xi,\eta))\right) \\ &= -4c^2 \left(\frac{1}{2}\frac{\partial u_x}{\partial x}\frac{\partial x}{\partial \xi} + \frac{1}{2}\frac{\partial u_x}{\partial t}\frac{\partial t}{\partial \xi} - \frac{1}{2c}\frac{\partial u_t}{\partial x}\frac{\partial x}{\partial \xi} - \frac{1}{2c}\frac{\partial u_t}{\partial t}\frac{\partial t}{\partial \xi}\right) \\ &= -4c^2 \left(\frac{1}{4}u_{xx} + \frac{1}{4c}u_{xt} - \frac{1}{4c}u_{tx} - \frac{1}{4c^2}u_{tt}\right)\Bigg|_{\substack{x=x(\xi,\eta)\\ t=t(\xi,\eta)}} \\ &= \left(-c^2 u_{xx} + u_{tt}\right)\Big|_{\substack{x=x(\xi,\eta)\\ t=t(\xi,\eta)}} \\ &= f(x(\xi,\eta), t(\xi,\eta)) \\ &= F(\xi,\eta) \end{aligned}$$

In other words, $U(\xi,\eta)$ will satisfy

$$-4c^2 U_{\xi\eta} = F(\xi,\eta) \tag{8}$$

(Hopefully, this was already obvious from (5a) and (7). But I thought it worthwhile to do the explicit computation to show how the change of variables works.)

So let us now find the general solution of (8). Recall that the general solution of

$$\Phi_x = f(x,y) \tag{9}$$

is given by

$$\Phi(x,y) = \int f(x,y)\,\partial x + c(y) \tag{10}$$

Here $\int f(x,y)\,\partial x$ means finding the anti-partial derivative of $f(x,y)$ with respect to x; which we can obtain by integrating $f(x,y)$ with respect to x treating y as a constant (a kind of "partial integration" which is why we use the symbol ∂x instead of dx). The term $c(y)$ corresponds some arbitrary function of y - it contains the ambiguity that's left in the solution (10) of (9).

Returning to (8), let us set $W = U_\xi$. Then W must satisfy

$$W_\eta = -\frac{1}{4c^2}F(\xi,\eta)$$

which in view of (9) and (10) requires

$$W(\xi,\eta) = -\frac{1}{4c^2}\int F(\xi,\eta)\,\partial\eta + c_1(\xi)$$

We now solve

$$U_\xi(\xi,\eta) = W(\xi,\eta) = -\frac{1}{4c^2}\int F(\xi,\eta)\,\partial\eta + c_1(\xi)$$

and find

$$U(\xi,\eta) = -\frac{1}{4c^2}\int\int F(\xi,\eta)\,\partial\eta\partial\xi + \int c_1(\xi)\,\partial\xi + c_2(\eta)$$

Notice that since $c_1(\xi)$ is an arbitrary function of ξ, so is $\int c_1(\xi)\,\partial\xi$. Therefore, we may as well write the general solution of (8) as

$$U(\xi,\eta) = -\frac{1}{4c^2}\int F(\xi,\eta)\,\partial\eta\partial\xi + C_1(\xi) + C_2(\eta) \tag{11}$$

with $C_1(\xi)$ and $C_2(\eta)$ arbitrary functions of their arguments.

The next step would be to convert the boundary conditions (8b) and (8c) on $u_p(x,t)$ to conditions on $U(\xi,\eta)$ and then determine the arbitrary functions $C_1(\xi)$ and $C_2(\eta)$. Note that both (8b) and (8c) are defined on the line $t=0$ which correspond to the line $\xi=\eta$ in the (ξ,η) coordinates. Thus,

$$\begin{aligned} u_p(x,0) &= 0 \implies U(\xi,\xi)=0 \\ u_{p,t}(x,0) &= 0 \implies \frac{1}{2c}\frac{\partial U}{\partial \xi}(\xi,\xi) - \frac{1}{2c}\frac{\partial U}{\partial \eta}(\xi,\xi) = 0 \end{aligned}$$

Unfortunately, these conditions are difficult to impose directly on the expression (11) without an explicit expression for $U(\xi,\eta)$.

So let us think of (11) in a different way. First off, the way one actually computes

$$-\frac{1}{4c^2}\int\int F(\xi,\eta)\,\partial\eta\partial\xi$$

is that one finds (or looks up) and anti-partial derivative of $F(\xi,\eta)$ with respect to η, and finds the anti-derivative of that expression with respect to ξ. We could write this procedure as

$$-\frac{1}{4c^2}\int^{\xi}\left(\int^{\eta} F(\xi',\eta')\,d\eta'\right)d\xi'$$

where ξ' and η' are “dummy variables of integration"; making it look half-way like a definite integral (an integral with definite limits). In fact, if we add lower endpoints of integration

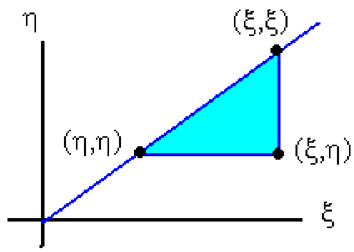
$$-\frac{1}{4c^2}\int_{\xi_0}^{\xi}\left(\int_{\eta_0}^{\eta} F(\xi',\eta')\,\partial\eta'\right)\partial\xi'$$

the only affect on the result is to add some "constants of integrations" which will could be absorbed into the definition of the functions $C_1(\xi)$ and $C_2(\eta)$ in (11).

So let us consider the particular solution where $\xi_0=\eta$ and $\eta_0=\xi'$. So that

$$\begin{aligned} U(\xi,\eta) &= -\frac{1}{4c^2}\int_{\eta}^{\zeta}\left(\int_{\xi'}^{\eta} F(\xi',\eta')\,d\eta'\right)d\xi' \\ &= \frac{1}{4c^2}\int_{\eta}^{\zeta}\left(\int_{\eta}^{\xi'} F(\xi',\eta')\,d\eta'\right)d\xi' \end{aligned} \tag{12}$$

where in the last equation we have simply reversed the endpoints of the integration over η'. The reason for doing so is we can now reinterprete the integral as an integral over a particular triangle in the (ξ',η')-plane: assuming $\xi>\eta$ (which is approprite for $t>0$) , the region will look like



and can be described

$$\triangle = \left\{ [\xi', \eta'] \in \mathbb{R}^2 \mid \eta \le \xi' \le \xi \quad , \quad \eta \le \eta' \le \xi' \right\}$$

As such that integral of $F(\xi', \eta')$ over $\triangle$ can be computed via the following iterated integral

$$\int_\triangle F(\xi', \eta')\, dA = \int_\eta^\xi \left(\int_\eta^{\xi'} F(\xi', \eta')\, d\eta' \right) d\xi'$$

Because the PDE for $\Phi(\xi, \eta)$ was obtained from the original PDE (1) via a change of variables, it follows that

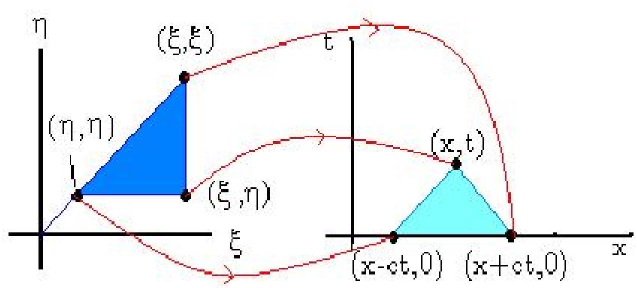
$$u_p(x,t) = U(\xi(x,t), \eta(x,t))$$

will automatically satisfy (1). What we shall show next is that, when $U(\xi, \eta)$ is defined by (14), the corresponding solution $u_p(x,t)$ of (1) automatically satisfies

$$\begin{aligned} u_p(x,0) &= 0 \quad , \\ u_t(x,0) &= 0 \quad . \end{aligned}$$

To see this, we express the integral over the triangle $\triangle$, in terms of our original coordinates x and t. The figure below shows hows the triangular region $\triangle$ in the (η, ξ) plane gets mapped over into the (x,t) plane when we we make a change of variables

$$\begin{matrix} \xi' = x' + ct' \\ \eta' = x' - ct' \end{matrix} \qquad \longleftrightarrow \qquad \begin{matrix} x' = \frac{1}{2}(\xi' + \eta') \\ t' = \frac{1}{2c}(\xi' - \eta') \end{matrix}$$



Using the usual change of variables formula for integrals

$$\frac{1}{4c^2} \int_\triangle F(\xi, \eta)\, d\xi' d\eta' = \frac{1}{4c^2} \int_0^t \int_{x-c(t-t')}^{x+c(t-t')} f(x', t') \left| \frac{\partial(\xi, \eta)}{\partial(x,t)} \right| dx' dt'$$

Here

$$\left| \frac{\partial(\xi, \eta)}{\partial(x,t)} \right| = \left| \det \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial t} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial t} \end{pmatrix} \right| = \left| \det \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix} \right| = |-2c| = 2c$$

is the Jacobian of the transformation.

Having successfully converted back to (x,t)-coordinates, we conclude

$$u_p(x,t) = U(\xi(x,t),\eta(x,t)) = \frac{1}{2c}\int_0^t \int_{x-c(t-t')}^{x+c\left(t-t'\right)} f(x',t')\,dx'dt'$$

will be a particular solution of the inhomogeneous wave equation

$$u_{tt} - c^2 u_{xx} = f(x,t)$$

Let's see what initial conditions it satisfies. We have

$$u_p(x,0) = \frac{1}{2c}\int_0^0 \int_x^x f(x',t')\,dx'dt' = 0$$

because we are, effectively, integrating over a single point in the xt-plane. It is also easy to check that

$$\frac{\partial}{\partial t}\left(\frac{1}{2c}\int_0^t \int_{x-c(t-t')}^{x+c\left(t-t'\right)'} f(x',t')\,dx'dt'\right)\Bigg|_{t=0} = 0$$

(for essentially the same reason) and so

$$u_{p,t}(x,0) = 0 \quad .$$

Thus, we can take

$$u_p(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct'}^{x+ct'} f(x',t')\,dx'dt'$$

as the unique solution to (8a)–(8c).

In summary, the general solution to the inhomogenous wave equation with inhomogeneous Cauchy boundary conditions:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f(x,t) \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}$$

is given by

$$u(x,t) = u_p(x,t) + u_o(x,t)$$

where

$$u_p(x,t) := \frac{1}{2c}\int_0^t \int_{x-c(t-t')}^{x+c\left(t-t'\right)} f(x,t)\,dxdt$$

is unique solution to the inhomogeneous wave equation with homogeneous boundary conditions and

$$u_o(x,t) := \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')\,dx'$$

is the unique solution to the homogeneous wave equation with inhomogeneous boundary conditions.