LECTURE 7

The Wave Equation

A wave equation (in $1+1$ dimensions) is a partial differential equation of the form

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} = f(x,t) \quad . \tag{1}$$

Such equations crop up in a variety of physical contexts; vibrating strings, electrical circuits, electromagnetism, and in general whereever some sort of oscillitory motion takes place. The function $f(x,t)$ is referred to as the driving term. Typically, it represents some external force function applied to an oscillitory system.

Now the homogeneous wave equation

$$\phi_{tt} - c^2\phi_{xx} = 0 \tag{2}$$

is rather exceptional for a PDE; because it is quite simple to write down its *general* solution.

Set

$$\begin{array}{rclcrcl} \zeta & = & x+ct & \qquad & t & = & \frac{\zeta+\eta}{2} \\ \eta & = & x-ct & & x & = & \frac{\zeta-\eta}{2c} \end{array} \tag{3}$$

and write

$$\phi(x,t) = \Phi(\zeta\,(x,t)\,,\eta\,(x,t)).$$

The circumstance that ϕ satisfies (1) forces Φ to satisfy a certain PDE. To find the PDE for Φ we apply the chain rule for partial differentiation:

$$\phi_t = \frac{\partial\zeta}{\partial t}\frac{\partial\Phi}{\partial\zeta} + \frac{\partial\eta}{\partial t}\frac{\partial\Phi}{\partial\eta} = c\Phi_\zeta - c\Phi_\eta \tag{4a}$$

$$\phi_x = \frac{\partial\zeta}{\partial x}\frac{\partial\Phi}{\partial\zeta} + \frac{\partial\eta}{\partial x}\frac{\partial\Phi}{\partial\nu} = \Phi_\zeta + \Phi_\eta \tag{4b}$$

These identities actually hold for any function ϕ related to Φ by the change of coordinates (3). In fact, equations (4a) and (4b) are really only about the way the differential operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ transform under the change of coordinates. It is common to write this "change of differential operators" as

$$\begin{array}{ccc} \dfrac{\partial}{\partial t} & \rightarrow & c\dfrac{\partial}{\partial\zeta} - c\dfrac{\partial}{\partial\eta} \\ \dfrac{\partial}{\partial x} & \rightarrow & \dfrac{\partial}{\partial\zeta} + \dfrac{\partial}{\partial\eta} \end{array}$$

To find expressions for the second order differential operators $\frac{\partial^2}{\partial t^2}$ and $\frac{\partial^2}{\partial x^2}$, we can write

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} &= \frac{\partial}{\partial t}\frac{\partial}{\partial t} = \left(c\frac{\partial}{\partial \zeta} - c\frac{\partial}{\partial \eta}\right)\left(c\frac{\partial}{\partial \zeta} - c\frac{\partial}{\partial \eta}\right) = c^2\frac{\partial^2}{\partial \zeta^2} - c^2\frac{\partial^2}{\partial \zeta \partial \eta} - c^2\frac{\partial^2}{\partial \eta \partial \zeta} + c^2\frac{\partial^2}{\partial \eta^2} \\
&= c^2\left(\frac{\partial^2}{\partial \zeta^2} - 2\frac{\partial^2}{\partial \zeta \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right) \\
\frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \eta}\right) = \frac{\partial^2}{\partial \zeta^2} + \frac{\partial^2}{\partial \zeta \partial \eta} + \frac{\partial^2}{\partial \eta \partial \zeta} + \frac{\partial^2}{\partial \eta^2} \\
&= \left(\frac{\partial^2}{\partial \zeta^2} + 2\frac{\partial^2}{\partial \zeta \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\phi_{tt} - c^2\phi_{xx} &= c^2\left(\Phi_{\zeta\zeta} - 2\Phi_{\zeta\eta} + \Phi_{\eta\eta}\right) - c^2\left(\Phi_{\zeta\zeta} - 2\Phi_{\zeta\eta} + \Phi_{\eta\eta}\right) \\
&= -4c^2\Phi_{\zeta\eta} \quad .
\end{aligned}$$

So if $\phi(x,t) = \Phi(\zeta(x,t), \eta(x,t))$ obeys

$$\phi_{tt} - c^2\phi_{xx} = 0$$

the function $\Phi(\zeta,\eta)$ must satisfy

$$4c^2\Phi_{\zeta\eta} = 0 \quad \Rightarrow \quad \Phi_{\xi\eta} = 0$$

But the general solution of

$$\Phi_{\zeta\eta} = 0$$

is

$$\Phi(\zeta,\eta) = \alpha(\zeta) + \beta(\eta) \quad .$$

Thus, the general solution of (3) is

$$\phi(x,t) = \alpha(\zeta(x,t)) + \beta(\eta(x,t)) = \alpha(x+ct) + \beta(x-ct) \quad . \tag{5}$$

That is to say, the general solution of the wave equation is the sum of an arbitrary function of $x+ct$ and an arbitrary function of $x-ct$.

1. The Wave Equation with Cauchy Boundary Conditions

We'll now look for solutions of the following boundary value problem:

$$\phi_{tt} - c^2\phi_{xx} = 0 \tag{6a}$$
$$\phi(x,0) = h(x) \tag{6b}$$
$$\phi_t(x,0) = p(x) \quad . \tag{6c}$$

Such a PDE/BVP describes (infinitely long) vibrating string with a certain initial configuration:

- the function $h(x)$ prescribes the displacement of the portion of the string at position x at time $t=0$;
- the function $p(x)$ prescribes the initial velocity of the portion of the string at position x at time $t=0$.

Such initial boundary conditions are called Cauchy boundary conditions.

With a general solution in hand, let us now try to satisfy the boundary conditions in (6b) and (6c). All we need to do is find functions $\alpha(x+ct)$ and $\beta(x-ct)$ satisfying

$$\alpha(x) + \beta(x) = f(x) \tag{7a}$$
$$c\alpha'(x) - c\beta'(x) = p(x) \quad . \tag{7b}$$

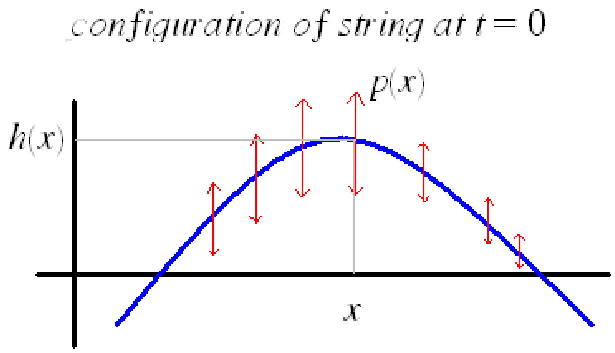


FIGURE 1

Integrating both sides of (7b) with respect to x yields (after adding in an arbitrary constant of integration K) yields

$$\begin{aligned} \alpha(x) + \beta(x) &= f(x) && \text{(8a)} \\ c\alpha(x) - c\beta(x) &= P(x) + K \quad . && \text{(8b)} \end{aligned}$$

where

$$P(x) := \int_0^x p(\zeta) d\zeta \quad .$$

Solving (8a) for $\alpha(x)$ we get

(9) $$\alpha(x) = f(x) - \beta(x) \quad .$$

Inserting this expression for $\alpha(x)$ into (8b) yields

$$c\left(f(x) - \beta(x)\right) - c\beta(x) = P(x) + K$$

or

(10) $$\beta(x) = \frac{1}{2} f(x) - \frac{1}{2c} P(x) - \frac{1}{2c} K \quad .$$

Inserting (10) into (9) yields

(11) $$\alpha(x) = \frac{1}{2} f(x) + \frac{1}{2c} P(x) + \frac{1}{2c} K \quad .$$

Finally, we insert (10) and (11) into (6) to obtain

$$\begin{aligned} \phi(x,t) &= \alpha\,(x + ct) + \beta\,(x - ct) \\ &= \frac{1}{2} f(x+ct) + \frac{1}{2c} P(x+ct) + \frac{1}{2c} K + \frac{1}{2} f(x-ct) - \frac{1}{2c} P(x-ct) - \frac{1}{2c} K \\ &= \frac{1}{2}\left[f\,(x+ct) + f\,(x-ct)\right] + \frac{1}{2c}\left[P\,(x+ct) - P\,(x-ct)\right] \\ &= \frac{1}{2}\left[f\,(x+ct) + f\,(x-ct)\right] + \frac{1}{2c}\left[\int_0^{x+ct} p(\zeta) d\zeta - \int_0^{x-ct} p(\zeta) d\zeta\right] \\ &= \frac{1}{2}\left[f\,(x+ct) + f\,(x-ct)\right] + \frac{1}{2c}\int_0^{x+ct} p(\zeta) d\zeta + \frac{1}{2c}\int_{x-ct}^{0} p(\zeta) d\zeta \end{aligned}$$

where in the last step we simply made a change of variables $\zeta \to -\zeta$ in the second integral. This allows to combine the two integrals to get

(12) $$\boxed{\phi(x,t) = \tfrac{1}{2}\left[f\,(x+ct) + f\,(x-ct)\right] + \tfrac{1}{2c}\int_{x-ct}^{x+ct} p(\zeta)\, d\zeta}$$

as the solution of (6a) – (cc).

2. Interpretation of solutions

To better understand the nature of the solution (12) to the PDE/BVP (6), we consider a pair of special cases.

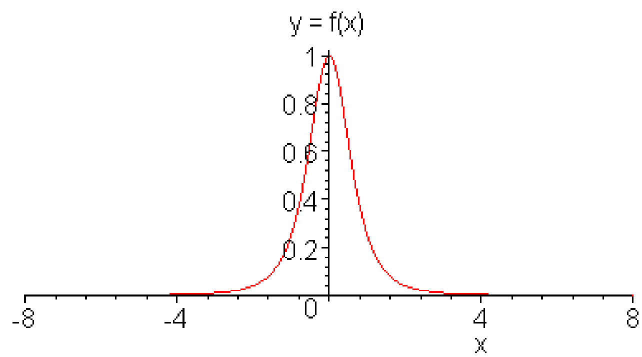
Case 1. $\phi(x,0) = f(x)$, $\phi_t(x,0) = 0$.

In this case, we have

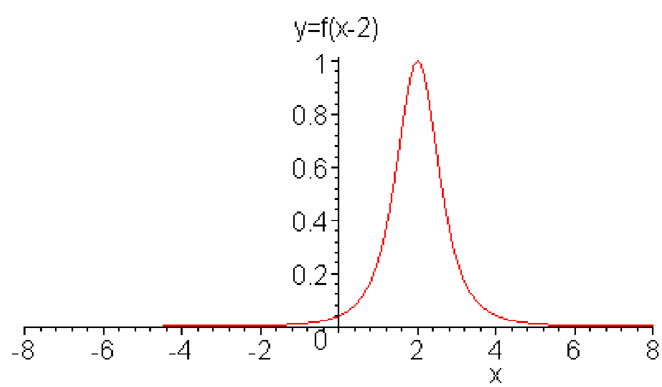
$$\phi(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) \quad .$$

If we think of $\phi(x,t)$ as representing the vertical displacement of an infinite horizontal string at the point x at time t, then the function $f(x)$ corresponds to an initial displacement; e.g., a plucking of the string at $t = 0$. The disturbance then propagates along the string in both directions maintaining the same shape as the initial displacement (once the two components of the disturbance separate).

To help you see this, consider the graph of a function $f(x)$ with a local maximum at $x = 0$.



Now consider the graph of the function $f(x-2)$



Notice how the local maximum moved from $x = 0$ off to the right to $x = 2$ while the graph has maintained the same shape. More generally, if we had plotted $f(x-ct)$ the original plot be shifted off to the *right* a distance ct. **Or** if we had plotted $f(x+ct)$ the original plot would be shifted off to the *left* a distance ct.

Case 2. $\phi(x,t) = 0$, $\phi_t(x,0) = p(x)$.

In this case, we have

$$(*) \qquad \phi(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} p(\tau)d\tau \quad .$$

Thus, the displacement of the string at point x at time t is given by the integral of $p(x)$ between the points $x-ct$ and $x+ct$.

Suppose for definiteness, that at time $t=0$ there was a "disturbance" confined to a small interval about 0, say $[-a,a]$. The function $p(x)$ that prescribes this a disturbance would then only be non-zero between $x=-a$ and $x=a$. Let us consider the solution (*) at a time point x and a time t such that

$$x-ct>a$$

The integral (*) should be proportional to area under the graph of $p(\tau)$ between $\tau=x-ct$ and $\tau=x+ct$ (assuming $p(x)$ is a positive function). But since we are assuming $p(\tau)=0$ for τ outside the interval $[-a,a]$, the area under the graph of $p(\tau)$ over an interval with no intersection with $[-a,a]$ will be zero. We conclude

$$\left.\begin{array}{ll} p(\tau)=0 & \text{if } |\tau|>a \\ x-ct>a & \end{array}\right\} \quad \Longrightarrow \quad \phi(x,t)=0$$

And in fact the first instant t when $\phi(x,t)$ might be non-zero will occur when

$$x-ct=a$$

that is, when

$$x-a=ct$$

In other words, the solutions at a point x will be zero until a time t such that x can be reached from a by travelling at the speed c.

3. Uniqueness of Solutions of Wave Equation with Cauchy Boundary Conditions

3.1. The energy functional. Consider the homogeneous wave equation representing a string of length L with fixed endpoints and whose intial transverse displacement at the point x is given by h(x) and whose initial transverse velocity at the point x is given by $p(x)$. The PDE/BVP corresponding to this system is

$$\begin{array}{rcl} \phi_{tt}-c^2\phi_{xx} & = & 0 \\ \phi(0,t) & = & 0 \\ \phi(L,t) & = & 0 \\ \phi(x,0) & = & h(x) \\ \phi_t(x,0) & = & p(x) \end{array} \tag{13}$$

Let $\phi(x,t)$ be any solution of (13) and set

$$I_\phi(t)=\frac{1}{2}\int_0^L\left[\frac{1}{c^2}\left(\phi_t(x,t)\right)^2+\left(\phi_x(x,t)\right)^2\right]dx \quad .$$

This function is related to the *total energy* of the string. The $\frac{1}{c^2}\left(\phi_t\right)^2$ part of the integrand is basically a kinetic energy term (analogous to $\frac{1}{2}mv^2$ in the Newtonian mechanics) and the $\left(\phi_x\right)^t$ is the contribution of the potential energy (as whenever $\left(\phi_x\right)^2\neq 0$, the tension in the string is increasing). We then have

$$\begin{array}{rcl} \frac{d}{dt}I_\phi(t) & = & \int_0^L\left[\frac{1}{c^2}\phi_t(x,t)\phi_{tt}(x,t)+\phi_x(x,t)\phi_{xt}(x,t)\right]dx \\ & = & \int_0^L\left[\frac{1}{c^2}\phi_t(x,t)\left(c^2\phi_{xx}(x,t)\right)+\phi_x(x,t)\phi_{xt}(x,t)\right]dx \\ & = & \int_0^L\phi_t(x,t)\frac{\partial}{\partial x}\phi_x(x,t)dx+\int_0^L\phi_x(x,t)\phi_{xt}(x,t)dx \\ & = & \phi_t(x,t)\phi_x(x,t)|_0^L-\int_0^L\phi_{tx}(x,t)\phi_x(x,t)dx+\int_0^L\phi_x(x,t)\phi_{xt}(x,t)dx \\ & = & 0 \end{array} \tag{14}$$

(To reach the fourth line, we integrated the first integral on the third line by parts. The first term on the fourth line vanishes since the boundary conditions $\phi(0,t)=0$, $\phi(L,t)=0$ imply

$$\phi_t(0,t)=0=\phi_t(L,t) \quad , \quad \forall\, t \quad .$$

The two remaining terms cancel one another.)

3.2. Uniqueness of Solutions. Now consider the following very general **inhomogeneneous** PDE/BVP

$$\begin{aligned} \Phi_{tt} - c^2\Phi_{xx} &= f(x,t) \\ \Phi(0,t) &= \alpha(t) \\ \Phi(L,t) &= \beta(t) \\ \Phi(x,0) &= H(x) \\ \Phi_t(x,0) &= P(x) \end{aligned} \tag{15}$$

and suppose that $\Phi_1(x,t)$ and $\Phi_2(x,t)$ are two solutions to the inhomgeneous problem (15). Then

$$\phi(x,t) = \Phi_1(x,t) - \Phi_2(x,t)$$

satisfies

$$\begin{aligned} \phi_{tt} - c^2\phi_{xx} &= 0 \\ \phi(0,t) &= 0 \\ \phi(L,t) &= 0 \\ \phi(x,0) &= 0 \\ \phi_t(x,0) &= 0 \end{aligned}$$

Therefore, in light of (14). we must have

$$0 = \frac{d}{dt} I_\phi(t) = \frac{d}{dt}\left[\frac{1}{2}\int_0^L \left[\frac{1}{c^2}\left(\phi_t(x,t)\right)^2 + \left(\phi_x(x,t)\right)^2\right] dx\right] \quad .$$

It follows that

$$I_\phi(t) = const$$

In fact, the initial conditions $\phi_t(x,0) = 0$, $\phi(x,0) = 0$, imply that

$$\phi_t(x,0) = \phi_x(x,0) = 0 \qquad \Rightarrow \qquad I_\phi(0) = 0 \quad .$$

Thus, we have

$$0 = \frac{1}{2}\int_0^L \left[\frac{1}{c^2}\left(\phi_t(x,t\right)^2 + \left(\phi_x(x,t\right)^2\right] dx \quad .$$

Note the integrand is the sum of two squares and so it is therefore a non-negative function of x. But if $f(x)$ is continuous and non-negative for all $x \in [0,L]$, then

$$\int_0^L f(x)\,dx = 0 \quad \Longrightarrow \quad f(x) = 0 \quad \forall\, x \in [0,L]$$

So we must have

$$\frac{1}{c^2}\left(\phi_t(x,t\right)^2 + \left(\phi_x(x,t\right)^2 = 0 \qquad \forall\, x \in [0,L]$$

But again, observing that the left hand side of the above equation is a sum of squares, we can conclude that

$$\begin{aligned} \phi_t(x,t) &= 0 \qquad \forall\, x \in [0,L] \\ \phi_x(x,t) &= 0 \qquad \forall\, x \in [0,L] \end{aligned}$$

and for all t. But this then implies

$$\phi(x,t) = \Phi_1(x,t) - F_2(x,t) = constant \quad .$$

But since $\Phi_1 = \Phi_2$ on the boundary, we must have this *constant* equal to zero. Hence,

$$\Phi_1(x,t) = \Phi_2(x,t)$$

and so the solution to (15), if it exists, is unique. □